M. Koike
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ON McKAY'S CONJECTURE

MASAO KOIKE

Let $\eta(z)$ be Dedekind's η-function. For any set of integer $g = (k_1, \cdots, k_s)$, $k_1 \geq k_2 \geq \cdots \geq k_s \geq 1$, put $\eta_g(z) = \prod_{i=1}^{s} \eta(k_i z)$. In this paper, we shall prove McKay's conjecture which gives some combinatorial conditions about k_i on which $\eta_g(z)$ is a primitive cusp form. As to McKay's conjecture, we refer [5].

To state our result precisely, we introduce some notation. For every positive integer N, put

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z) \,|\, c \equiv 0 \bmod N \right\}.$$

Let k be a positive integer and let ε be a Dirichlet character mod N such that $\varepsilon(-1) = (-1)^k$. We denote by $S_k(N, \varepsilon)$ (resp. $S_k^0(N, \varepsilon)$) the space of the cusp forms (resp. new forms) of type (k, ε) on $\Gamma_0(N)$. We call $f(z) = \sum_{n=1}^{\infty} a_n e(nz)$ in $S_k^0(N, \varepsilon)$ primitive cusp form if it is a common eigenfunction of all the Hecke operators and $a_1 = 1$ where $e(z) = e^{2\pi i z}$. Then it is well-known that $S_k^0(N, \varepsilon)$ has a basis whose elements are all primitive cusp forms.

McKay conjectured

THEOREM. *Let $\eta_g(z) = \prod_{i=1}^{s} \eta(k_i z)$ be as above. The following statements* (a) *and* (b) *are equivalent.*

(a) *$\eta_g(z)$ is a primitive cusp form.*

(b) *$g = (k_1, \cdots, k_s)$ satisfies the conditions* (1)~(4);

(1) *k_1 is divisible by k_i for all $1 \leq i \leq s$.*

(2) *Put $N = k_1 k_s$, then $N/k_i = k_{s+1-i}$ for all $1 \leq i \leq s$.*

(3) $\sum_{i=1}^{s} k_i = 24$.

(4) *s is even.*

In these cases, $\eta_g(z)$ is a primitive cusp form in $S_{s/2}^0(k_1 k_s, \varepsilon)$ for some Dirichlet character ε mod $k_1 k_s$.

Proof of Theorem. First we shall show that (b) implies (a). In [5], they already proved this result, but, for the completeness of the paper, we indicate the outline of the proof. We denote by $g = t_1^{n_1} \cdots t_j^{n_j}$, $t_1 > \cdots > t_j \geq 1$ and $0 < n_1, \cdots, n_j \in \boldsymbol{Z}$, the set of integers $g = (t_1, \cdots, t_1, t_2, \cdots, t_2, \cdots, t_j, \cdots, t_j)$ where t_i is contained n_i-times in g. For each $g = t_1^{n_1} \cdots t_j^{n_j}$, put

$$s = s(g) = \sum_{i=1}^{j} n_i, \quad k = k(g) = \frac{1}{2}s(g) \quad \text{and} \quad N = N(g) = t_1 t_j \,.$$

Then it is easily seen that all g which satisfy (1)~(4) are given by the following:

Table 1

$s(g)$	g
2	$23\cdot1$, $22\cdot2$, $21\cdot3$, $20\cdot4$, $18\cdot6$, $16\cdot8$, 12^2
4	$15\cdot5\cdot3\cdot1$, $14\cdot7\cdot2\cdot1$, $12\cdot6\cdot4\cdot2$, $11^2\cdot1^2$, $10^2\cdot2^2$, $9^2\cdot3^2$, $8^2\cdot4^2$, 6^4
6	$8^2\cdot4\cdot2\cdot1^2$, $7^3\cdot1^3$, $6^3\cdot2^3$, 4^6
8	$6^2\cdot3^2\cdot2^2\cdot1^2$, $5^4\cdot1^4$, $4^4\cdot2^4$, 3^8
10	$4^4\cdot2^2\cdot1^4$
12	$3^6\cdot1^6$, 2^{12}
16	$2^8\cdot1^8$
24	1^{24}

It is also easily seen that all g which satisfy (2)~(4) but not (1) are given by the following:

Table 2

$s(g)$	g
2	$19\cdot5$, $17\cdot7$, $15\cdot9$, $14\cdot10$, $13\cdot11$
4	$10\cdot6\cdot5\cdot3$, $7^2\cdot5^2$
6	$5^3\cdot3^3$

For each g in Table 1, we can prove that the corresponding $\eta_g(z)$ is a cusp form in $S_k(N, \varepsilon_g)$ for some Dirichlet character ε_g mod N by applying Theorem 1 in [3]. We should remark that, in [3], they considered only the case when $s(g)$ is divisible by 4, but their method can be applied to

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the case when $s(g)$ is even. When $s(g)$ is divisible by 4, ε_g are seen to be trivial. When $s(g)=2$, ε_g are given in Proposition 2. In remaining cases, ε_g are given in Table 3:

Table 3

g	$8^2\cdot4\cdot2\cdot1^2$	$7^3\cdot1^3$	$6^3\cdot2^3$	4^6	$4^4\cdot2^2\cdot1^4$
cond. of ε_g	8	7	3	4	4
$\varepsilon_g(d)$	$\left(\frac{2}{\lvert d\rvert}\right)(-1)^{(d-1)/2}$	$\left(\frac{d}{7}\right)$	$\left(\frac{d}{3}\right)$	$(-1)^{(d-1)/2}$	$(-1)^{(d-1)/2}$

Proposition 1. *For each g in Table 1 such $s(g)\geq 4$, we have*

$$\dim_C S_{s(g)/2}(N_g, \varepsilon_g) = \dim_C S^0_{s(g)/2}(N_g, \varepsilon_g) = 1\,.$$

Proof. The proof is done by direct calculations of dimensions via Hijikata's trace formula [2]. Q.E.D.

Hence, for each g in Table 1 such that $s(g)\geq 4$, $\eta_g(z)$ is proved to be a primitive cusp form.

Proposition 2. *For each g in Table 1 such that $s(g)=2$, $\eta_g(z)$ is an element in $S_1(N_g, \varepsilon_g)$ which is obtained from a L-function of certain quadratic field $\boldsymbol{Q}(\sqrt{d_g})$ with certain ideal character χ_g of conductor f_g: In these cases, the ideal class groups are all cyclic, therefore, L-functions with characters χ_g depend only on orders of χ_g:*

g	N_g	d_g	f_g	$\varepsilon_g(d)$	order of χ_g
$23\cdot1$	23	-22	1	$\left(\frac{d}{23}\right)$	3
$22\cdot2$	44	-11	2	$\left(\frac{d}{11}\right)$	3
$21\cdot3$	63	-7	3	$\left(\frac{d}{7}\right)$	4
$20\cdot4$	80	5	$4\mathfrak{p}_\infty$	$\left(\frac{5}{\lvert d\rvert}\right)(-1)^{(d-1)/2}$	2
$18\cdot6$	108	-3	6	$\left(\frac{d}{3}\right)$	3
$16\cdot8$	128	-8	4	$\left(\frac{2}{\lvert d\rvert}\right)(-1)^{(d-1)/2}$	4
12^2	144	-4	6	$(-1)^{(d-1)/2}$	4

Proof. The proof is done by direct calculations of these Fourier coefficients and by showing that these coincide to each other to some extent which depends on g. Q.E.D.

We shall show that (a) implies (b).

LEMMA. *For $g = t_1^{n_1} \cdots t_j^{n_j}$ and $h = r_1^{m_1} \cdots r_l^{m_l}$, we suppose that*

$$\eta_g(z) = c \cdot \eta_h(z)$$

where c is a non-zero constant. Then we have

$$j = l\,, \quad t_i = r_i \quad \text{and} \quad n_i = m_i \quad \text{for all } 1 \le i \le j\,.$$

Proof. It is sufficient to prove the following: let $t_1 > \cdots > t_j \geq 1$ and $n_1, \cdots, n_j$ for all $1 \leq i \leq j$ be integers. Suppose that $\prod_{i=1}^{j} \eta(t_i z)^{n_i} =$ const $\neq 0$. Then $n_i = 0$ for all $1 \leq i \leq j$. Put $\varphi(x) = \prod_{n=1}^{\infty}(1 - x^n)$. The above condition implies $\prod_{i=1}^{j-1} \varphi(x^{t_i})^{n_i} = (\text{const}) \cdot \varphi(x^{t_j})^{-n_j}$. Suppose $n_j \neq 0$. Then the right hand side contains the term x^{t_j} with a non-zero coefficient, but the left hand side can not contain such term. This is a contradiction. Q.E.D.

Suppose that $\eta_g(z) = \prod_{i=1}^{j} \eta(t_i z)^{n_i} = \sum_{n=1}^{\infty} a_n e(nz)$ is a primitive cusp form in $S_k^0(N, \varepsilon)$. Since $a_1 = 1$, we have $\sum_{i=1}^{j} t_i n_i = 24$: The condition (3) is proved.

To prove the conditions (1) and (2), we have to consider W_Q-operator (see [1]). Let $N = Q \cdot Q'$ where Q and Q' are prime to each other. Let $W_Q = \begin{pmatrix} Qx, & y \\ Nu, & Qv \end{pmatrix}$ such that $x, y, u, v \in \boldsymbol{Z}$ and $\det W_Q = Q$. Put $d_i = (t_i, Q)$ for any i. Then, by some calculation, we see that

$$(1) \qquad \eta_g(z) \,|\, W_Q = (\text{const}) \cdot Q^{(1/4)s} \prod_i d_i^{-(1/2)n_i} \eta\left(\frac{Qt_i}{d_i^2} z\right)^{n_i}.$$

It is also well-known (see [1]) that

$$(2) \qquad \eta_g(z) | W_N = (\text{const}) \cdot \bar{\eta}_g(z)\,,$$

where $\bar{\eta}_g(z) = \sum_{n=1}^{\infty} \bar{a}_n e(nz)$, $\bar{a}_n =$ the complex conjugate of a_n. In our case, it is clear that $\bar{\eta}_g(z) = \eta_g(z)$. Therefore from (1) and (2), we have

$$\eta_g(z) = (\text{const}) \cdot N^{s/4} \prod_i t_i^{-n_i/2} \prod_i \eta\left(\frac{N}{t_i} z\right)^{n_i}.$$

Hence, by Lemma, it follows that

$$\frac{N}{t_i} = t_{j+1-i} \quad \text{and} \quad n_i = n_{j+1-i} \quad \text{for all } 1 \leq i \leq j .$$

Thus the condition (2) is proved. Especially, we have

$$\eta_g(z) | W_N = (-i)^{s/2} \eta_g(z) .$$

To prove the condition (1), we should note that $\eta_g(z) | W_Q$ is known to be a constant times of some other primitive cusp form. Hence, from (1), we have

$$\sum_i \frac{Qt_i}{d_i^2} n_i = 24 . \tag{3}$$

It is easily seen that each g in Table 1 satisfies (3) and each g in Table 2 does not saisfy (3). Hence the condition (1) is proved. This completes the proof of Theorem. Q.E.D.

Remark 1. In [4], Mason reported some connection between the sporadic simple group M_{24} and $\eta_g(z)$ for some g in Table 1.

Remark 2. It will be interesting to consider quotients of products of η-functions and to find what kind of conditions of n_i would give primitive cusp forms. There are several examples for such cases:

$$g = 4^{-2} \cdot 8^8 \cdot 16^{-2}, \qquad 2^{-4} \cdot 4^{16} \cdot 8^{-4} .$$

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Department of Mathematics
Faculty of Science
Nagoya University
Chikusa-ku, Nagoya 464
Japan