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ON MODULAR REPRESENTATION ALGEBRAS AND A CLASS OF MATRIX ALGEBRAS

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Abstract

Let G be a cyclic group of prime order p and K a field of characteristic p. The set of classes of isomorphic indecomposable (K, G)-modules forms a basis over the complex field for an algebra $\mathcal{A}_p$ (Green, 1962) with addition and multiplication being derived from direct sum and tensor product operations.

Algebras $\mathcal{A}_n$ with similar properties can be defined for all $n \geqslant 2$. Each such algebra is isomorphic to a matrix algebra $\mathfrak{M}_n$ of $n \times n$ matrices with complex entries and standard operations. The characters of elements of $\mathcal{A}_n$ are the eigenvalues of the corresponding matrices in $\mathfrak{M}_n$.

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1. Introduction

Let G be a cyclic group of prime order p and K a field of characteristic p. A G-module is a (K, G)-module with the elements of G acting as right operators: there exist exactly p distinct isomorphism classes of indecomposable G-modules, with K-dimension $1, \ldots, p$. (For further details, see Green, 1962 or Renaud, 1979.)

Choose representatives $V_1, \ldots, V_p$ from the classes with V_i having K-dimension i. The modular representation algebra $\mathcal{A}_p$ has basis $\{V_1, \ldots, V_p\}$ over the complex field, with products defined by

$$V_r \times V_s = \sum_{i=1}^{p} a_{irs} V_i$$

where a_{irs} is the number of modules isomorphic to V_i in the direct sum decomposition of $V_r \otimes_K V_s$.

This commutative product is expressible by the formula (Renaud, 1979):
For $1 \leqslant r \leqslant s \leqslant p$,

$$V_r \times V_s = \sum_{i=1}^{c} V_{s-r+2i-1} + (r-c)V_p$$

where

$$c = \begin{cases} r & \text{if } r+s \leqslant p, \\ p-s & \text{if } r+s \geqslant p. \end{cases}$$

$\mathcal{A}_p$ may be regarded as generated by V_2 with relation $V_r = V_2 \times V_{r-1} - V_{r-2}$ for $2 < r \leqslant p$, restricted by $V_2 \times V_p = 2V_p$, and hence elements in $\mathcal{A}_p$ are polynomials in V_2.

This class of algebras can be extended in a natural way: for all integers $n \geqslant 2$ let $\mathcal{A}_n$ be the algebra with identity V_1 and generator V_2, defining relation for V_r and restriction as above, with p replaced by n. Note that this is not in general a representation algebra of a group: the V_i are abstract elements, not modules.

We wish to show $\mathcal{A}_n$ is isomorphic to a particular matrix algebra $\mathfrak{M}_n$.

2. The matrix algebra

Let $\mathfrak{M}_n$ be the algebra generated by the $n \times n$ matrix W_2^n which has entries 1 on the sub- and super-diagonals, 1 at each end of the main diagonal, and 0 elsewhere. We wish to show $\mathfrak{M}_n$ is isomorphic to $\mathcal{A}_n$.

Let W_1^n be the unit $n \times n$ matrix. Let $W_r^n = W_2^n \times W_{r-1}^n - W_{r-2}^n$. Clearly necessary and sufficient conditions for the isomorphism to hold are that the restriction $W_2^n \times W_n^n = 2W_n^n$ hold, and that the W_r^n be linearly independent. To show this, and also to describe the W_r^n in general, we have

PROPOSITION. *W_r^n is the $n \times n$ matrix (a_{ij}) with*

$$\begin{aligned} a_{ij} = 1 \quad & \textit{if (i) } i+j-1 \leqslant r \\ & \textit{or (ii) } 2n-(i+j-1) \leqslant r \\ & \textit{or (iii) } |i-j|-1 \equiv r \pmod 2 \textit{ and } |i-j| < r, \\ a_{ij} = 0 \quad & \textit{otherwise}. \end{aligned}$$

PROOF. The proposition clearly holds for $r = 1$ and $r = 2$. Examination of the pattern in diagrams of $W_k^n \times W_2^n$ versus those of W_{k+1}^n and W_{k-1}^n is then sufficient.

COROLLARY. (i) *W_n^n is the $n \times n$ matrix with every entry* 1. *Hence $W_2^n \times W_n^n = 2W_n^n$.*

(ii) *Examination of the first row of the W_r^n shows they are linearly independent. Hence $\mathcal{M}_n$ is isomorphic to $\mathcal{C}_n$.*

3. The characters of $\mathcal{C}_n$

A character of $\mathcal{C}_n$ is a non-trivial homomorphism, ϕ: $\mathcal{C}_n \to \mathbf{C}$ where $\mathbf{C}$ is the complex field. Green (1962) derived the characters for $\mathcal{C}_p$: the general case is similar.

Let x be indeterminate over a commutative, associative algebra with identity i. The Chebyshev polynomials S_k are defined by

$$S_0(x) = i, \qquad S_1(x) = x, \qquad S_k(x) = xS_{k-1}(x) - S_{k-2}(x) \quad \text{for } k \geq 2.$$

(For further details, see Abramovitz and Stegun (1972), Chapter 22.)

Now for $1 \leq r \leq n$, $V_r = S_{r-1}(V_2)$. Let ϕ: $\mathcal{C}_n \to \mathbf{C}$ be a character of $\mathcal{C}_n$: clearly $\phi(V_1) = 1$ and since $V_n \times (V_2 - 2V_1) = 0$, $\phi(V_n) = 0$ or $\phi(V_2) = 2$. The second case gives the dimension character

$$\delta(V_r) = r, \qquad 1 \leq r \leq n.$$

For the first case, let $\phi(V_2) = x$. Then we require $S_{n-1}(x) = 0$, and this has solutions $x_j = 2\cos(\pi j/n), j = 1,\ldots,n-1$.

Moreover, $S_{r-1}(2\cos\theta) = \sin r\theta/\sin\theta$, $r \geq 1$, and hence there are $n-1$ other characters,

$$\phi_j(V_r) = \frac{\sin(\pi jr/n)}{\sin(\pi j/n)}, \qquad 1 \leq r \leq n, 1 \leq j \leq n-1.$$

4. The eigenvalues of matrices in $\mathcal{M}_n$

The characteristic polynomial of W_2^n is derived from the equation

$$W_n^n(W_2^n - 2W_1^n) = 0;$$

thus the eigenvalues of W_2^n are the solutions of $S_{n-1}(\lambda)(\lambda - 2) = 0$, and these are just the characters at V_2. Hence the eigenvalues of W_r^n are the character values at V_r.

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REMARK. This gives rise to an eigenvalue result of some minor interest: let A be the $n \times n$ matrix symmetric about both diagonals, with

$$a_{ij} = (-1)^{i+1}\left\{x_j + \sum_{k=1}^{i-1} (-1)^k (x_{j-k} + x_{j+k})\right\}$$

for $1 \leqslant i \leqslant \frac{1}{2}(n+1)$, $i \leqslant j \leqslant n+1-i$, where $x_j \in \mathbf{C}$, $j = 1,\ldots,n$.

For example, with $n = 5$,

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_2 & x_1 - x_2 + x_3 & x_2 - x_3 + x_4 & x_3 - x_4 + x_5 & x_4 \\ x_3 & x_2 - x_3 + x_4 & x_1 - x_2 + x_3 - x_4 + x_5 & x_2 - x_3 + x_4 & x_3 \\ x_4 & x_3 - x_4 + x_5 & x_2 - x_3 + x_4 & x_1 - x_2 + x_3 & x_2 \\ x_5 & x_4 & x_3 & x_2 & x_1 \end{pmatrix}.$$

Now

$$A = x_n W_n^n + \sum_{i=1}^{n-1} (x_i - x_{i+1}) W_i^n$$

and hence the eigenvalues of A are

$$\lambda_0 = \sum_{i=1}^{n} x_i \quad \text{and} \quad \lambda_j = \sum_{i=1}^{n-1} (x_i - x_{i+1}) \frac{\sin(\pi ij/n)}{\sin(\pi j/n)}$$

for $j = 1,\ldots,n-1$.

5. The eigenvectors of W_r^n

The eigenvectors of W_r^n also have interesting properties. For W_2^n, the eigenvalues are $\lambda_j = 2\cos(\pi j/n)$, $j = 0,\ldots,n-1$. Let λ_j have a corresponding eigenvector $[y_1,\ldots,y_n]$. It is not difficult to show that we may choose $y_1 = 1$, and that for $1 < i \leqslant n$

$$\begin{aligned} y_i &= S_{i-1}(\lambda_j) - S_{i-2}(\lambda_j) \\ &= \cos(\pi ij/n) + \sin(\pi ij/n)\tan(\pi j/n). \end{aligned}$$

Moreover, it can be deduced that this set of eigenvectors is orthogonal.

REMARK. Any matrix similar to W_2^n will generate an algebra similar to $\mathfrak{M}_n$: one such with a pattern as clear as that of W_2^n is the $n \times n$ matrix (r_{ij}) with

$$r_{ij} = \begin{cases} 1 & \text{if } |i-j| = 1, \quad i+j < 2(n-1); \\ 2 & \text{if } i = j = n; \\ 0 & \text{otherwise.} \end{cases}$$

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Of course, it is trivially obvious that the matrix $\bigoplus_{i=0}^{n-1}[2\cos(\pi i/n)]$ also generates an algebra isomorphic to $\mathfrak{M}_n$.

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