ORDERED SEMIGROUPS

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1. Introduction. In this paper order will always mean linear or total order, and, unless otherwise stated, the composition of any semigroup will be denoted by $+$. A semigroup S is an *ordered semigroup* (notation o.s.) if S is an ordered set and for all a, b, c in S

$$a < b \text{ implies } a + c < b + c \text{ and } c + a < c + b.$$

If in addition $a + a > a$ for all a in S, then we call S a *positive ordered semigroup* (notation pos. o.s.). In particular an o.s. S is cancellative, and hence if e is an idempotent element of S, then e is the identity for S. Moreover, for a, b, c in S and n a positive integer we have the following rules

$$a > b \leftrightarrow a + c > b + c \leftrightarrow c + a > c + b.$$
$$a > b \leftrightarrow na > nb.$$
$$a > b \text{ and } c > d \rightarrow a + c > b + d.$$

Let Γ be an ordered set, and for each $\gamma \in \Gamma$ let S_γ be an o.s. such that $S_\alpha \cap S_\beta = \square$ (the null set) if $\alpha \neq \beta$. Consider $a \in S_\alpha$ and $b \in S_\beta$ where $\alpha \leq \beta$. Define $a < b$ if $\alpha < \beta$ or $\alpha = \beta$ and $a < b$ in S_α. Define $a + b = b + a = b$ if $\alpha < \beta$ and use the addition in S_α if $\alpha = \beta$. Then $Q = \bigcup_{\gamma \in \Gamma} S_\gamma$ is an ordered set and a semigroup—*the ordinal sum of the* S_γ. The S_γ are the *components* of Q.

In section 3 we give a necessary and sufficient condition for a semigroup S to be the ordinal sum of pos. o.s. (Theorem 3-1). We also show that if S is a pos. o.s., then there exists a rather natural o-homomorphism of S onto an ordinal sum of pos. o.s. each of which is o-isomorphic to a semigroup of positive real numbers. Cheheta [2] and Vinogradov [9] use an example of Malcev to show that an o.s. cannot necessarily be embedded in a group. Ore [8] has shown that if every pair of elements in a semigroup S has a common right multiple, then S can be embedded in a group $G = \{a - b : a, b \in S\}$. G is called

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the *difference group* of S. We show that if S is an o.s., then the order of S can be extended to an order of G in one and only one way. In section 5 we show that the order type of the set of all convex normal subgroups of G is determined by S.

2. Embedding theorems. Throughout this section S will denote an o.s.

THEOREM 2-1. *Suppose that S satisfies: $(*)$ for each pair a, b in S there exists a pair x, y in S such that $a+x=b+y$. Then there exists an o-group G such that $G=\{a-b : a, b\in S\}$ and $a-b$ is positive in G if and only if $a>b$ in S. Moreover, if H is an o-group that contains S as an ordered subsemigroup and is generated by S, then there exists an o-isomorphism π of G onto H such that $s\pi=s$ for all $s\in S$. We call G the difference group of S.*

This theorem is a corollary of a result of Ore [8] for integral domains. We outline the construction of an o-group G' that is o-isomorphic to G. Let $T=S\times S$ and define that $(a, b)\sim(c, d)$ if there exist x, y in S such that $a+x=c+y$ and $b+x=d+y$. Then $\sim$ is an equivalence relation. Denote the equivalence class containing (a, b) by $[a, b]$, and define that $[a, b]+[c, d]=[a+x, d+y]$ where $b+x=c+y$. Then the set G' of all equivalence classes is a group, $[a, a]$ is the identity, $[b, a]$ is the inverse of $[a, b]$, and the mapping τ of s upon $[s+x, x]$ is an isomorphism of S into G'.

$[a, b]=[a+x, x]-[b+x, x]=a\tau-b\tau$. Thus there is at most one way of extending the order of S to an ordering of G'. Namely, define that $[a, b]$ is positive in G' if $a>b$ in S. Let $\mathscr{P}$ be the set of all positive elements in G'. If $[a, a]\neq[b, c]$, then $b>c$ or $b<c$ in S, and hence $[b, c]\in\mathscr{P}$ or $-[b, c]=[c, b]\in\mathscr{P}$. If $[a. b]$ and $[c, d]$ belong to $\mathscr{P}$, then $a>b$ and $c>d$, and $[a, b]+[c, d]=[a+x, d+y]$ where $b+x=c+y$. Thus $a+x>b+x=c+y>d+y$, and hence $[a, b]+[c, d]\in\mathscr{P}$. If $[a, b]\in\mathscr{P}$ and $[c, d]\in G'$, then $X=[d, c]+[a, b]+[c, d]=[d, c]+[a+x, d+y]=[d+u, d+y+v]$ where $b+x=c+y$ and $c+u=a+x+v$. To show that $X\in\mathscr{P}$ it suffices to show that $u>y+v$. Pick r and s in S such that $u+r=y+s$. Then $a+x+v+r=c+u+r=c+y+s=b+x+s$. If $v+r\geq s$, then $a+x+v+r>b+x+s$ because $a>b$. Thus $v+r<s$, and hence $y+v+r<y+s=u+r$. Therefore $y+v<u$.

Finally suppose that H is an o-group that is generated by S. Let $[a, b]\pi'=a-b$ for all $[a, b]\in G'$. If $[a, b]=[c, d]$, then $a+x=c+y$ and $b+x=d+y$.

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Thus $a-b=a+x-x-b=c+y-y-d=c-d$, and hence π' is single valued. $([a, b]+[c, d])\pi'=[a+x, d+y]\pi'=a+x-y-d$, where $b+x=c+y$. Thus $x-y=-b+c$ and $a+x-y-d=a-b+c-d=[a, b]\pi'+[c, d]\pi'$. If $0=[a, b]\pi'=a-b$, then $[a, b]$ is the identity of G'. If $[a, b]\in\mathscr{P}$, then $a>b$ in S and hence in H. Thus $[a, b]\pi'=a-b$ is positive in H. $S\leq G'\pi'\leq H$ and, since H is generated by S, $G'\pi'=H$. Therefore π' is an o-isomorphism of G' onto H. This completes the proof of the theorem.

Corollary I. *S satisfies $(*)$ if and only if S can be embedded in an o-group $G=\{a-b : a, b\in S\}$.*

For suppose that $G=\{a-b : a, b\in S\}$ and that a and b belong to S. Then $-a+b\in G$ and hence $-a+b=x-y$ for some $x, y\in S$. Thus $b+y=a+x$.

Corollary II. *Suppose that S satisfies $(*)$ and let G be the difference group of S. Then for a, b, c in S*

(a) *$a-b=c-d$ if and only if there exist x, y in S such that $a+x=c+y$ and $b+x=d+y$.*

(b) *$a-b+c-d=a+x-(d+y)$ for all x, y in S such that $b+x=c+y$.*

(c) *$a-b>c-d$ if and only if there exist x, y in S such that $a+x>c+y$ and $b+x=d+y$.*

The equivalence of (i) and (ii) in the following corollary is well known and has been proven by Tamari, Alimov, and Nakada ([4] p. 309).

Corollary III. *For a commutative semigroup A the following are equivalent.*

(i) *A can be embedded in an o-group.*

(ii) *A can be ordered.*

(iii) *A satisfies the cancellation law, and $na=nb$ implies that $a=b$, for all a, b in A and all positive integers n.*

Proof. Clearly (i) implies (ii), and since any commutative o.s. satisfies $(*)$, (ii) implies (i). An easy argument shows that (ii) implies (iii). Finally assume that A is cancellative, and let $G=\{a-b : a, b\in A\}$ be the difference group of A. If $x=a-b\in G$ and $nx=0$, then $0=nx=na-nb$, and hence $na=nb$. Thus by (iii) $a=b$, and $0=a-b=x$. Therefore (iii) implies that the difference group G of A exists and is abelian and torsion free. But this means that G can be ordered (see for example [7]).

Suppose that A is a cancellative commutative semigroup with identity 0. Then if A can be ordered, it is torsion free, but the converse is false. For consider the semigroup $B = N \oplus N$, where N is the additive semigroup of non-negative integers. For (a, b) and (c, d) in B define that $(a, b) \sim (c, d)$ if $a \equiv c \bmod 2$, $b \equiv d \bmod 2$ and $a+b = c+d$. Then it is easy to show that $\sim$ is a congruence relation. Let $[a, b]$ be the congruence class that contains (a, b). $B/\sim\, = \{[a, b] : a, b \in N\} = \{[2n, 0], [2n+1, 0], [0, 2n+1]$ and $[2n+1, 1]$ for all $n \in N\}$ is a commutative semigroup with identity $[0, 0]$. It is easy to show that $B/\sim$ satisfies the cancellation law and is torsion free, but $2[1, 1] = 2[0, 2]$ and $[1, 1] \neq [0, 2]$. Thus (iii) of the last corollary is not satisfied, and hence $B/\sim$ cannot be ordered.

Let $P = \{x \in S : x + x > x\}$ and $N = \{x \in S : x + x < x\}$. The following five propositions are easy to verify (or see [1] for proofs).

1) $P = \{x \in S : x + s > s \text{ for all } s \in S\} = \{x \in S : s + x > s \text{ for all } s \in S\}$.

2) $N = \{x \in S : x + s < s \text{ for all } s \in S\} = \{x \in S : s + x < s \text{ for all } s \in S\}$.

3) P and N are subsemigroups of S.

4) $N < P$. That is, $n < p$ for all $n \in N$ and all $p \in P$.

5) If S does not have an identity, then $S = N \cup P$ and an identity 0 can be adjoined to S so that $T = S \cup \{0\}$ is a semigroup. Moreover, the order of S can be extended to an order of T in one and only one way, namely $N < 0 < P$. If we adjoin an identity to a pos. o.s we shall call the result a pos. o.s. *with zero*. An o.s. S is *naturally ordered* if for all a, b in S

(R) $a > b$ implies $a = b + x$ for some x in S, and

(L) $a > b$ implies $a = x + b$ for some x in S.

Note that a pos. o.s. P satisfies (R) if and only if $b + P = \{a \in P : a > b\}$ for all b in P.

THEOREM 2-2. *If S satisfies (R), then S satisfies ($*$) and hence S is an ordered subsemigroup of its difference group G. If S is naturally ordered, then S contains the semigroup of all positive elements of G. A pos. o.s. P is the semigroup of all positive elements of an o-group if and only if P is naturally ordered.*

Proof. Consider a, b in S. If $a > b$, then $a = b + x$ for some x in S. Thus $a + b = b + (x + b)$. Similarly if $a \leq b$, then $a + u = b + v$ for some u, v in S. Therefore S satisfies ($*$). Suppose that S is naturally ordered, and consider a

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positive element y in the difference group G of S. $y = a - b$, where $a, b \in S$ and $a > b$. Thus $a = x + b$ for some $x \in S$, and hence $y = a - b = x \in S$.

Finally suppose that P is a naturally pos. o.s. and let $\mathscr{P}$ be the semigroup of all positive elements of the difference group G of P. Then we have shown that $P \supseteq \mathscr{P}$. If $p \in P$, then $p + p > p$ in P and hence $p = p + p - p > 0$ in G. Therefore $P \subseteq \mathscr{P}$.

LEMMA 2-1. *Let a, b, c be elements of S. If $a + b < b + a$, then $a + nb < nb + a$ and $na + nb \leq n(a + b) < n(b + a) \leq nb + na$ for all positive integers n, where the equalities hold if and only if $n = 1$.*

This follows by a simple induction argument or see [6] for a proof.

COROLLARY. *If p and q are positive integers and $pa = qb$, then $a + b = b + a$.*

For if $a + b < b + a$, then $(p + 1)a = a + pa = a + qb < qb + a = pa + a = (p + 1)a$, a contradiction.

Note that Lemma 2-1 and its corollary are true for an ordinal sum of o.s. For if $a + b < b + a$, then a and b belong to the same component. In [6] the following theorem (which we use later) is proven.

THEOREM 2-3. *For an o.s. S the following are equivalent.* (i) *There exists an o-isomorphism of S into a subsemigroup of the* (*naturally ordered*) *additive group R of real numbers.* (ii) *For each pair $a < b$ in S, there exist positive integers m and n such that $ma < (m + 1)b$ and $(n + 1)a < nb$.*

THEOREM 2-4. *Suppose that the center $Z = \{z \in S : z + s = s + z \text{ for all } s \in S\}$ of S is not empty. Then there exists o.s. T such that*

1) *S is an ordered subsemigroup of T,*

2) *T contains the difference group G of Z and T is generated by S and G,*

3) *If T' is an o.s. that satisfies* 1) *and* 2), *then there exists a unique o-isomorphism π of T onto T' such that $s\pi = s$ for all $s \in S$.*

We outline a proof, leaving out the straightforward computations. Let $Q = S \times Z$ and for (a, b) and (c, d) in Q define that

$$(a, b) + (c, d) = (a + c, b + d) \text{ and}$$
$$(a, b) \sim (c, d) \text{ if } a + d = b + c.$$

Then Q is a semigroup, and $\sim$ is a congruence relation. As usual, denote the equivalence class containing (a, b) by $[a, b]$. For $[a, b]$ and $[c, d]$ in $Q/\sim$

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define that $[a, b] > [c, d]$ if $a+d > b+c$. Then $(Q/\sim, +, >)$ is an o.s. and the mapping τ of $a \in S$ upon $[a+z, z]$, where z is a fixed element in Z is an o-isomorphism of S into $Q/\sim$. $G' = \{[a, b] : a, b \in Z\}$ is the center of $Q/\sim$ and the difference group of $Z\tau$. Clearly $Q/\sim$ is generated $S\tau$ and G'. Thus there exists an o-semigroup T that satisfies 1) and 2). Moreover G is the center of T. Finally suppose that T and T' are o.s. that satisfy 1) and 2), and consider $t \in T$. $t = s+g = s+z_1-z_2$, where $s \in S$, $g \in G$ and $z_1, z_2 \in Z$. Define $t\sigma = [s+z_1, z_2]$. Then σ is on o-isomorphism of T onto $Q/\sim$. Similarly we define an o-isomorphism σ' of T' onto $Q/\sim$, and then $\pi = \sigma\sigma'^{-1}$ is the desired o-isomorphism of T onto T'.

3. Positive ordered semigroups

THEOREM 3-1. *A semigroup S is an ordinal sum of pos. o.s. if and only if*

(I) *S is an ordered set, and for all a, b, c in S,*

(II) *if $a < b$, then $a+c \leq b+c$ and $c+a \leq c+b$,*

(III) *$a+a > a$,*

(IV) *if $a+b = a+c$, then $b = c$ or $a+b = a$, and if $b+a = c+a$, then $b = c$ or $b+a = a$.*

Proof. It is easy to verify that an ordinal sum of pos. o.s. satisfies these four conditions. Conversely assume that S is a semigroup that satisfies (I), (II), (III) and (IV). Then S satisfies (III′) $a+b \geq \max\{a, b\} \leq b+a$ for all a, b in S. For if $a+b < a$, then $a+2b \leq a+b$. If $a+2b < a+b$, then $2b < b$, but this contradicts (III). If $a+2b = a+b$, then by (IV) $2b = b$ or $a+b = a$, a contradiction. Therefore $a+b \geq a$, and by a similar argument $a+b \geq b$.

For a, b in S we define that $a \sim b$ if $a+b > \max\{a, b\} < b+a$. Clearly $\sim$ is symmetric, and by (III) it is reflexive. Suppose that $a \sim b$ and $b \sim c$. Then $c+b > b$, and thus $a+c+b \geq a+b$. If $a+c+b = a+b$, then by (IV) $c+b = b$ or $a+b = a$. Then $b \nsim c$ or $a \nsim b$, a contradiction. Thus $a+c+b > a+b$, and hence $a+c > a$. By symmetry it follows that $a \sim c$, and hence $\sim$ is an equivalence relation.

Let $\bar{a} = \{b \in A : b \sim a\}$, and consider b, c in $\bar{a}$. We show that $a+b+c > \max\{a, b+c\}$. By symmetry it follows that $b+c \in \bar{a}$, and hence that $\bar{a}$ is a semigroup. If $a+b+c < a+b$, then $b+c < b$, and hence $b \nsim c$. Thus $a+b+c \geq a+b > a$. If $a+b+c < b+c$, then $a+b < b$, and hence $a \nsim b$. If $a+b+c$

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$= b+c$, then by (IV) $a+b=b$ or $b+c=c$, and hence $a+b$ or $b \dotplus c$. Therefore $a+b+c > b+c$.

We next show that $\bar{a}$ is a pos. o.s. Consider x, y, z in $\bar{a}$. If $x<y$, then $x+z<y+z$. For otherwise $x+z=y+z$, and thus $x=y$ or $x+z=z$, a contradiction. By symmetry if $x<y$, then $z+x<z+y$. Thus $\bar{a}$ is an o.s., and since $\bar{a}$ satisfies (III) it is a pos. o.s.

In order to prove that S is the ordinal sum of the semigroups $\bar{a}$ it suffices to show that if $a<b$ and $\bar{a} \neq \bar{b}$, then $a+b=b$ and $\bar{a}<\bar{b}$. $a+b \leq b$ because $\bar{a} \neq \bar{b}$, and by (III′) $a+b \geq b$. Pick $a' \in \bar{a}$ and $b' \in \bar{b}$. $a'+a+b=a'+b$. Hence by (IV) $a'+a=a'$ or $a'+b=b$. But $a'+a>a$ because $a' \sim a$. If $b' \leq a'$, then $b'+b \leq a'+b=b$, and hence $b' \dotplus b$. Therefore $a'<b'$, and hence $\bar{a}<\bar{b}$.

For the rest of this section we investigate pos. o.s. The information obtained will then apply to semigroups that satisfy the four properties of Theorem 3-1. *For the remainder of this section let P denote a* pos. o.s.

LEMMA 3-1. *For all a, b in P and all positive integers m, $(m+1)a+(m+1)b$ is greater than $ma+mb$ and $mb+ma$.*

Proof. $(m+1)a>ma$ and $(m+1)b>mb$. Thus $(m+1)a+(m+1)b>ma+mb$. Suppose that $a \geq b$. If $a \geq mb$, then $(m+1)a+(m+1)b>(m+1)a=a+ma \geq mb+ma$. If $a<mb$, then since $mb<(m+1)b \leq (m+1)a$, there exists a positive integer n such that $na<(m+1)b \leq (n+1)a$. Thus $(m+1)a+(m+1)b>(m+1)a+na=(n+1)a+ma>mb+ma$. By an entirely similar argument if $a<b$, then $(m+1)a+(m+1)b>mb+ma$.

LEMMA 3-2. *For all a, b in P and all positive integers m:*

(i) $(m+1)(a+b)$ *is greater than* $m(a+b)$ and $m(b+a)$.

(ii) $(m+1)a+(m+1)b$ *is greater than* $m(a+b)$ *and* $m(b+a)$.

(iii) $(m+1)(a+b)$ *is greater than* $mb+ma$ *and* $ma+mb$.

Proof. (i) $(m+1)(a+b)=m(a+b)+a+b>m(a+b)$ and $(m+1)(a+b)=a+m(b+a)+b>m(b+a)+b>m(b+a)$. (ii) If $a+b \geq b+a$, then by Lemma 2-1, $(m+1)a+(m+1)b \geq (m+1)(a+b)$, and by (i) $(m+1)(a+b)>m(a+b)$ and $m(b+a)$. If $a+b<b+a$, then by Lemma 3-1, $(m+1)a+(m+1)b>mb+ma$, and by Lemma 2-1, $mb+ma \geq m(b+a)>m(a+b)$. (iii) If $b+a \geq a+b$, then by Lemma 2-1, $(m+1)(a+b) \geq (m+1)a+(m+1)b$, and by Lemma 3-1, $(m+1)a+(m+1)b>ma+mb$ and $mb+ma$. If $a+b$

$> b+a$, then by Lemma 2-1, $(m+1)(a+b) > (m+1)(b+a) > (m+1)b + (m+1)a$, and by Lemma 3-1, $(m+1)b+(m+1)a > mb+ma$ and $ma+mb$.

Remark. Lemmas 3-1 and 3-2 remain true if P is an ordinal sum of pos. o.s. In fact, the given proofs apply.

For a and b in P we define that $a\sigma b$ if $(m+1)a > mb$ and $(m+1)b > ma$ for all positive integers m.

1) *σ is a congruence relation.* For clearly σ is symmetric and $a\sigma a$ because $(m+1)a > ma$. If $a\sigma b$ and $b\sigma c$, then $(m+2)a > (m+1)b > mc$ and $(m+2)c > (m+1)b > ma$ for all m. Let $m=2n$, then $2(n+1)a > 2nc$ and $2(n+1)c > 2na$. Hence $(n+1)a > nc$ and $(n+1)c > na$, and $a\sigma c$. Finally suppose that $a\sigma b$. By Lemma 3-2, $(m+3)(a+c) > (m+2)a+(m+2)c > (m+1)b+(m+1)c > m(b+c)$ for all m. Let $m=3m$, then $3(n+1)(a+c) > 3n(b+c)$. Thus $(n+1)(a+c) > n(b+c)$ and similarly $(n+1)(b+c) > n(a+c)$ for all n. Therefore $(a+c)\sigma(b+c)$.

2) *The semigroup P/σ is commutative.* For by (i) of Lemma 3-2, $(m+1)(a+b) > m(b+a)$ and $(m+1)(b+a) > m(a+b)$ for all m. Therefore $(a+b)\sigma(b+a)$.

For the remainder of this section we shall denote the elements of P by a, b, c and the elements of P/σ by A, B, C. Moreover, m, n, p, q will always denote positive integers. If ρ is a congruence relation over a semigroup S, then ρ^* will always denote the natural homomorphism of S onto S/ρ. P/σ is an ordinal sum of pos. o.s., and this can be shown by verifying that P/σ satisfies the four properties of Theorem 3-1. But we wish to show something stronger. Namely, that P/σ is an ordinal sum of pos. o.s. each of which is a subsemigroup of positive reals.

3) *If $a > b$, then $a\sigma^* = b\sigma^*$ or $x > y$ for all x in $a\sigma^*$ and y in $b\sigma^*$.* For suppose that there exists an x in $a\sigma^*$ and y in $b\sigma^*$ such that $y \geq x$. $(m+2)x > (m+1)a > (m+1)b > my$. Now let $m=2n$ and cancel. Then $(n+1)x > ny$ for all n and also $(n+1)y \geq (n+1)x > nx$ for all n. Thus $x\sigma y$, and $a\sigma^* = x\sigma^* = y\sigma^* = b\sigma^*$. For $a\sigma^*$ and $b\sigma^*$ in P/σ we define that $a\sigma^* < b\sigma^*$ if $a\sigma^* \neq b\sigma^*$ and $a < b$ in P. Then by (3) this definition is independent of the choice of representatives a and b.

LEMMA 3-3. (i) *P/σ is an ordered set and $A < B$ implies that $A+C$*

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$\leq B+C$ *for all* A, B, C *in* P/σ. (ii) $A<A+A$. (iii) *If* $A<B$, *then* $nA<nB$.

Proof. (i) If $a\sigma^* < b\sigma^*$ and $b\sigma^* < c\sigma^*$, then $a<b$ and $b<c$. Hence $a<c$ and $a\sigma^* \leq c\sigma^*$. If $a\sigma^* = c\sigma^*$, then $a \in c\sigma^*$, but then $a>b$, a contradiction. Thus $a\sigma^* < b\sigma^*$. If $a\sigma^* \neq b\sigma^*$, then $a<b$ or $b<a$, and so $a\sigma^* < b\sigma^*$ or $b\sigma^* < a\sigma^*$. (ii) Clearly $A \leq 2A$. Suppose that $2A = A = a\sigma^*$. Then $a\sigma 2a$, and hence $(m+1)a > (2m)a$ for all m. In particular for $m=1$, $2a>2a$, a contradiction. Thus $A < A+A$. (iii) Clearly $nA \leq nB$. Suppose that $nA = nB$ where $a\sigma^* = A$ and $b\sigma^* = B$. Then $na\sigma nb$, and so $(m+1)na > mnb$ and $(m+1)nb > mna$ for all m. But then $(m+1)a > mb$ and $(m+1)b > ma$. Thus $a\sigma b$, and hence $A = a\sigma^* = b\sigma^* = B$, a contradiction.

For A and B in P/σ we define that $A\tau B$ if there exist positive integers m and n such that $mA > B$ and $nB > A$.

4) τ *is an equivalence relation.* For clearly τ is symmetric and by (ii) of Lemma 3-3, $2A > A$. Thus $A\tau A$. If $A\tau B$ and $B\tau C$, then $nA > B$, $pB > A$, $mB > C$ and $qC > B$ for some positive integers m, n, p, q. By (iii) of Lemma 3-3, $mnA > mB > C$ and $pqC > pB > A$. Therefore $A\tau C$.

Let $A\tau^*$ be the equivalence class that contains A. We shall show later that τ is a congruence relation, and so τ^* is the natural homomorphism of P/σ onto $(P/\sigma)/\tau$.

5) *If* $A<B$ *and* $A\tau^* \neq B\tau^*$, *then* $A\tau^* < B\tau^*$ *and* $A+B=B$. For suppose that there exist X in $A\tau^*$ and Y in $B\tau^*$ such that $X \geq Y$. Then $nX \geq nY > B$ and $mB \geq mA > X$ for some m and n. Thus $X\tau B$, and hence $A\tau^* = X\tau^* = B\tau^*$. $A = a\sigma^*$ and $B = b\sigma^*$. Since $a+b>b$, $(m+1)(a+b) > m(a+b) > mb$ for all m. Thus it suffices to show that $(m+1)b > m(a+b)$ for all m. Now $nA < B$ for all n, for otherwise $A \in B\tau^*$. Thus $na < b$ for all n. $(n+2)b = b+(n+1)b > (n+1)a + (n+1)b > n(a+b)$. Now let $n = 2m$ and cancel to get $(m+1)b > m(a+b)$. Thus $A+B=B$.

6) *If* $A<B$ *and* $A\tau C$, *then* $A+C<B+C$. For $A = a\sigma^*$, $B = b\sigma^*$, $C = c\sigma^*$, $a<b$ and $(n+1)a < nb$ for some positive integer n. By Lemma 3-3, $A+C \leq B+C$. Suppose (by way of contradiction) that $A+C = B+C$. Then $(m+3)a + (m+3)c > (m+2)(a+c) > (m+1)(b+c) > mb+mc$. Therefore $(m+3)a + 3c > mb$ for all m. Since $A\tau C$, there exists an integer h such that $hA > C$ and $3hA > 3C$ by Lemma 3-3. Let $k = 3h$, then $ka > 3c$. $(k+3)nb$

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$> (k+3)(n+1)a = [(k+3)n+3]a + ka > [(k+3)n+3]a + 3c$. Now let $m = (k+3)n$. Then $mb > (m+3)a + c$, a contradiction.

THEOREM 3-2. *For each A in P/σ, $A\tau^*$ is an ordered subsemigroup of P/σ that is o-isomorphic to an additive semigroup of positive real numbers. P/σ is an ordinal sum of the* pos. o.s. $A\tau^*$.

Proof. Consider B, C in $A\tau^*$. $A = a\sigma^*$, $B = b\sigma^*$ and $C = c\sigma^*$, where a, b, $c \in P$. There exist positive integers m, n, r, s such that $mB > A$, $nA > B$, $rC > A$ and $sA > C$. Thus $mb > a$, $na > b$, $rc > a$ and $sa > c$. Let $q = \max\{m, r\}$. Then $qb > a$ and $qc > a$. Thus $(q+1)(b+c) > qb + qc > 2a > a$, and by Lemma 3-3, $(q+2)(B+C) > (q+1)(B+C) \geq A$. Let $t = \max\{n, s\}$. Then $ta > b$ and $ta > c$. Thus $2ta > b + c$ and $(2t+1)A > 2tA \geq B + C$. Therefore $B + C \in A\tau^*$, and so $A\tau^*$ is a semigroup. By Lemma 3-3, $A\tau^*$ is ordered, and thus by (6) $A\tau^*$ is an o.s. In order to prove that A^* is o-isomorphic to a semigroup of positive real numbers, it suffices by Theorem 2-3 to show that if X, $Y \in A\tau^*$ and $X < Y$, then there exist positive integers m and n such that $(m+1)X < mY$ and $nX < (n+1)Y$.

$X = x\sigma^*$ and $Y = y\sigma^*$ for some x and y in P. Since $X < Y$, $nX < nY < (n+1)Y$ for all n. Hence $nx < (n+1)y$ for all n. Suppose (by way of contradiction) that $(m+1)X \geq mY$ for all m. If for some m, $(m+1)X = mY$, then $(m+2)X = (m+1)X + X < mY + Y = (m+1)Y$. Therefore $(m+1)X > mY$ for all m. Thus $(m+1)x > my$ and $mx < (m+1)y$ for all m. Therefore $X = Y$, a contradiction. Thus by Theorem 2-3 there exists an isomorphism π of $A\tau^*$ into the additive group of reals. But for $B \in A\tau^*$, $B < 2B$. Hence $B\pi < 2(B\pi)$. Therefore $B\pi$ is a positive real number. It follows at once from (4) and (5) that P/σ is the ordinal sum of the $A\tau^*$.

COROLLARY. *τ is a congruence relation on P/σ.*

Proof. Consider X, Y, Z in P/σ, and assume that $X\tau Y$. If $Z\tau X$, then since $X\tau^*$ is a semigroup $X + Z$ and $Y + Z$ belong to $X\tau^*$. Thus $(X+Z)\tau(Y+Z)$. Suppose that $X\tau^* \neq Z\tau^*$. If $Z < X$, then $Z\tau^* < X\tau^* = Y\tau^*$. Thus by (5) $X + Z = X$ and $Y + Z = Y$. If $X < Z$, then $Y\tau^* = X\tau^* < Z\tau^*$. Thus by (5) $X + Z = Z = Y + Z$. In either case $(X+Z)\tau(Y+Z)$.

There is a natural 1-1 order preserving correspondence between the congruence relations of P/σ and the congruence relations of P that contain σ, Therefore τ can also be considered as a congruence relation on P, where $a\tau b$ if there exist positive integers m and n such that $ma > b$ and $nb > a$. Consider X and Y in P/τ. $X = x\tau^*$ and $Y = y\tau^*$ for some x and y in P. We define that $X < Y$ if $X \neq Y$ and $x < y$ in P. Then P/τ is an ordered set and τ^* is an o-homomorphism of P onto P/τ. Denote the addition in P/τ by $[+]$. Then since $X + Y \subseteq \max[X, Y]$ in P, $X[+]Y = \max[X, Y]$ in P/τ. X is a subsemigroup of P and X/σ is o-isomorphic to a subsemigroup of the positive reals. Thus in Clifford's terminology [3], P/τ is a semilattice and P is a semilattice of the semigroups $X \in P/\tau$. In particular, $P - X$ is a subsemigroup of P and the number of components $A\sigma^*$ of P/σ is equal to the number of elements in P/τ which we shall denote by $|P/\tau|$.

A subsemigroup C of P is *convex* if $a \in P$, $c \in C$ and $a < c$ imply that $a \in C$. It is easy to show that the set $\mathscr{S}$ of all convex subsemigroups of P is ordered by inclusion, and that if A and B are convex subsemigroups of P and $A \supset B$, then $A \setminus B$ is a semigroup. Moreover if A covers B, and $a \in A \setminus B$, then $a\tau^* = A \setminus B$. For each $a \in P$ let $P^a = \{x \in P : x\tau^* \leq a\tau^*\}$. Then P^a is a convex subsemigroup of P and if C is a convex subsemigroup of P, then $C = \bigcup_{a \in C} P^a$. Thus the order type of $\mathscr{S}$ is completely determined by P/τ.

Let G be an o-group and let Γ be the set of all pairs of convex subgroups G^γ, G_γ of G such that G^γ covers G_γ. Define that $(G_\alpha, G^\alpha) < (G_\beta, G^\beta)$ if $G^\alpha \leq G_\beta$. Then Γ is ordered, and the order type of Γ is the *rank* of G.

THEOREM 3-3. *If P is a naturally pos. o.s., then the rank of the difference group G of P equals the order type of P/τ.*

For by Theorem 2-2, P is the semigroup of all positive elements of G, and a convex subgroup of G is determined by its set of positive elements. Thus if $(G_\gamma, G^\gamma) \in \Gamma$, then $G_\gamma \cap P$ and $G^\gamma \cap P$ are convex subsemigroups of P and $G^\gamma \cap P$ covers $G_\gamma \cap P$. Moreover $(G^\gamma \cap P) \setminus (G_\gamma \cap P) = a\tau^*$, where $a \in (G^\gamma \cap P) \setminus (G_\gamma \cap P)$.

Remark. If P is a commutative naturally pos. o.s. and the components $A\tau^*$ of P/σ are d-closed, then the c-closure C of the difference group G of P is uniquely determined by P/σ. For C is isomorphic to the Hahn group $H(\Gamma, R_\gamma)$, where Γ is an ordered set with order type equal to the rank of G and the R_γ are isomorphic to the components G^γ/G_γ of G (see [5] for these concepts). But Γ is determined by P/σ and the components of G are just the difference

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groups of the components of P/σ.

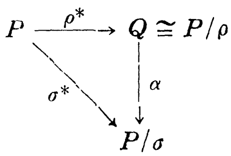
Let P be a positive o.s. that satisfies $(*)$ and let G be the difference group of P. It should be made clear that there is virtually no relationship between the order type of P/τ and the rank of G, even if G is abelian. For example let $G = R \oplus R \oplus R$, where R is the additive group of real numbers. Define (a, b, c) in G positive if $c > 0$ or $c = 0$ and $b > 0$ or $c = b = 0$ and $a > 0$. Let $P = \{(a, b, c) \in G : c > 0\}$. Then G is the difference group of P, $|P/\tau| = 1$, and the rank of G is 3. By generalizing this example it is easy to see that for $|P/\tau| = 1$ the rank of G can be any given order type. But we shall show (Theorem 5-1) that P does determine the order type of the set of all convex normal subgroups of G.

4. Relationships between P and P/σ. Throughout this section let P be a pos. o.s. A semigroup Q is a *t-semigroup* if Q is an ordered set and $ma < (m+1)a$ for all a in Q and all positive integers m.

LEMMA 4-1. *Let ρ^* be an o-homomorphism of P onto a t-semigroup Q. For a and b in P define $a\rho b$ if $a\rho^* = b\rho^*$. Then ρ is a congruence relation on P and $\rho \subseteq \sigma$.*

Proof. If $a\rho b$, then $a\rho^* = b\rho^*$. Hence $(m+1)(a\rho^*) > m(b\rho^*)$ and $(m+1)\cdot(b\rho^*) > m(a\rho^*)$. Thus since ρ^* is an o-homomorphism, $(m+1)a > mb$ and $(m+1)b > ma$ for all m. Therefore $a\sigma b$.

Now consider $q \in Q$. $q = a\rho^*$ for some $a \in P$. Define $q\alpha = a\sigma^*$. Then by the usual arguments α is an o-homomorphism of Q onto P/σ such that $p\rho^*\alpha = p\sigma^*$ for all $p \in P$. We have the following diagram and theorem.



THEOREM 4-1. *P/σ is the smallest o-homomorphic image of P that is a t-semigroup. In particular, P/σ is the smallest o-homomorphic image of P that is an ordinal sum of pos. o.s.*

Remarks. (1) Let ρ be a congruence relation on P. Then P/ρ is a t-semigroup and ρ^* is an o-homomorphism if and only if for all $a, b \in P$: (A) If $a < b$, then $a\rho^* = b\rho^*$ or $x < y$ for all $x \in a\rho^*$ and $y \in b\rho^*$, and ma(NOT ρ)

$(m+1)a$ for all m. Thus σ is the join of all congruence relations that satisfy (A). (2) If $|P/\tau|=1$ and ρ is a congruence relation on P such that P/ρ is an ordinal sum of pos. o.s. and ρ^* is an o-homomorphism, then P/ρ is a pos. o.s..

5. Relationship between P and its quotient group G. Let P be a pos. o.s. and let $\Delta=\{\rho : \rho$ is a congruence relation on P, P/ρ is a pos. o.s. or a pos. o.s. with zero, and ρ^* is an o-homomorphism$\}$.

LEMMA 5-1. *Δ is ordered by inclusion.*

Proof. Consider α, $\beta\in\Delta$ and suppose (by way of contradiction) that there exist $a, b, c, d\in P$ such that $a\alpha b$, $a(\text{NOT } \beta)b$, $c(\text{NOT } \alpha)d$ and $c\beta d$. Case I. $a>b$ and $c>d$. Then $a\beta^*>b\beta^*$ and $c\alpha^*>d\alpha^*$. If $a+d\leq b+c$, then $a\beta^*+d\beta^*\leq b\beta^*+c\beta^*$ and $d\beta^*=c\beta^*$. Thus $a\beta^*\leq b\beta^*$, a contradiction. If $a+d>b+c$, then $a\alpha^*+d\alpha^*\geq b\alpha^*+c\alpha^*$ and $a\alpha^*=b\alpha^*$. Thus $d\alpha^*\geq c\alpha^*$, a contradiction. Similarly in the other three cases we get a contradiction.

For the remainder of this section we assume that P is a pos. o.s. which satisfies $(*)$. In particular, the results obtained are valid for commutative pos. o.s. Let G be the difference group of P and let π be an o-homomorphism of P into a pos. o.s. with zero. Then clearly $P\pi$ satisfies $(*)$. Let H be the difference group of $P\pi$ and for $g=a-b$ in G define $g\bar{\pi}=a\pi-b\pi$.

LEMMA 5-2. *$\bar{\pi}$ is the unique extension of π to an o-homomorphism of G onto H.*

Proof. If $a-b=c-d$, where $a, b, c, d\in P$, then by Corollary II of Theorem 2-1, there exist $x, y\in P$ such that $a+x=c+y$ and $b+x=d+y$. Thus $a\pi+x\pi=c\pi+y\pi$ and $b\pi+x\pi=d\pi+y\pi$, and so by applying this corollary again, $a\pi-b\pi=c\pi-d\pi$. Thus $\bar{\pi}$ is single valued. The lemma now follows by repeated use of Corollary II and straightforward computation.

It is well known and easy to verify that the kernel of any o-homomorphism of an o-group is a convex normal subgroup. Let $\mathscr{C}$ be the set of all convex normal subgroups of G except G itself. Then $\mathscr{C}$ is ordered with respect to inclusion.

THEOREM 5-1. *There exists a 1-1 order preserving mapping of Δ onto $\mathscr{C}$.*

Proof. For each $\rho\in\Delta$ let $\bar{\rho}$ be the unique extension of ρ^* to G (which is assured by Lemma 5-2), and let $\rho\eta=K(\bar{\rho})=\{x\in G : x\bar{\rho}=0\}$. We wish to show that η is the desired mapping. Since $\bar{\rho}$ is uniquely determined by ρ, η is single valued. Let $\alpha, \beta \in \Delta$ and $\alpha \subseteq \beta$. If $x \in K(\bar{\alpha})$, then $x = a - b$, where $a, b \in P$ and $0 = x\bar{\alpha} = (a-b)\bar{\alpha} = a\bar{\alpha} - b\bar{\alpha} = a\alpha^* - b\alpha^*$. Thus $a\alpha b$ and hence $a\beta b$. But then $0 = a\beta^* - b\beta^* = x\bar{\beta}$. Therefore $x \in K(\bar{\rho})$ and $\alpha\eta \subseteq \beta\eta$. If $\alpha \neq \beta$, then there exist $a, b \in P$ such that $a\beta b$ but not $a\alpha b$, but this means that $a - b \in K(\bar{\beta}) \setminus K(\bar{\alpha})$. Therefore η is $1-1$ and order preserving. Next consider $C \in \mathscr{C}$ and let N be the natural o-homomorphism of G onto G/C. Let ρ be the congruence relation induced on P by N ($a\rho b$ if and only if $aN = bN$). Define that $a\rho^* > b\rho^*$ if $a + C > b + C$. Then it follows by a straightforward computation that $\rho \in \Delta$ and $\rho\eta = C$. Therefore η is a $1-1$ orderpreserving mapping of Δ onto $\mathscr{C}$.

If $|P/\tau| = 1$ or equivalently if P/σ is o-isomorphic to a subsemigroup of positive reals, then $\sigma \in \Delta$ and $\Delta = \{\rho : \rho$ is a congruence relation on P, P/ρ is a pos. o.s. and ρ^* is an o-homomorphism$\}$.

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