Can. J. Math., Vol. XXVII, No. 1, 1975, pp. 138–145

THE OSCILLATION OF FOURTH ORDER LINEAR DIFFERENTIAL OPERATORS

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Define the self-adjoint operator

$$L_4 y = (r y'')'' - (q y')' + py$$

where $r(x) > 0$ on $(0, \infty)$ and q and p are real-valued. The coefficient q is assumed to be differentiable on $(0, \infty)$ and r is assumed to be twice differentiable on $(0, \infty)$.

The oscillatory behavior of L_4 as well as the general even order operator has been considered by Leighton and Nehari [**5**], Glazman [**2**], Reid [**7**], Hinton [**3**], Barrett [**1**], Hunt and Namboodiri [**4**], Schneider [**8**], and Lewis [**6**].

The operator L_4 is said to be *oscillatory on* $(0, \infty)$ if for every $c > 0$ there are numbers a and b and a function y, $y \not\equiv 0$, such that $b > a > c$, $L_4 y = 0$, and

$$(1) \quad y(a) = y'(a) = 0 = y'(b) = y(b).$$

Otherwise, L_4 is said to be *nonoscillatory on* $(0, \infty)$.

Given $a > 0$, define $\mathscr{D}(b)$ for all $b > a$ to be the set of all real-valued functions y with the following properties:

(i) y and y' are absolutely continuous on $[a, b]$,

(ii) y'' is essentially bounded on $[a, b]$, and

(iii) (1) holds.

THEOREM 1. *The following two statements are equivalent.*

(i) *The operator L_4 is nonoscillatory on* $(0, \infty)$.

(ii) *There exist $a > 0$ such that for all $b > a$, $y \not\equiv 0$ and $y \in \mathscr{D}(b)$ implies that*

$$\int_a^b r(y'')^2 + q(y')^2 + py^2 > 0.$$

The reader should consult Reid's paper [**7**] for a proof of Theorem 1. The left side of the above inequality is referred to as the *quadratic functional for* L_4 and is denoted by $I_b(y)$.

A consequence of Theorem 1 is the fact that if for all $a > 0$ we can find a $b > a$ and a function $y \in \mathscr{D}(b)$ such that $y \not\equiv 0$ and

$$(2) \quad \int_a^b r(y'')^2 + q(y')^2 + py^2 \leqq 0,$$

Received July 23, 1973 and in revised form, March 25, 1974.

then L_4 is oscillatory on $(0, \infty)$. This method of proof was utilized independently by Glazman [**2**] and Hinton [**3**]. All of the theorems which follow also utilize this method of proof.

Hunt and Namboodiri [**4**] showed that if

$$\int^{\infty} x^{2(n-1)} p = -\infty, \quad p \leqq 0, \quad \text{and} \int^{\infty} r^{-1} = \infty,$$

then $(-1)^n (r\, y^{(n)})^{(n)} + py$ is oscillatory on $(0, \infty)$. The following theorem shows that for $n = 2$ and r bounded the sign restriction on p is not necessary.

We shall adopt the notation that $f_+(x) = \max\{0, f(x)\}$.

THEOREM 2. *If* $r(x) \leqq M$,

$$\int^{\infty} q_+ < \infty, \quad \textit{and} \int^{\infty} x^2 p = -\infty,$$

then L_4 *is oscillatory on* $(0, \infty)$.

Proof. Given $a > 0$, define $\omega(x)$ to be the third degree polynomial satisfying $\omega(0) = 0 = \omega'(0)$, $\omega(1) = a + 1$, and $\omega'(1) = 1$. Define $y(x)$ as follows:

$$y(x) = \begin{cases} 0, & x < a \\ \omega(x-a), & a \leqq x < a+1 \\ x, & a+1 \leqq x < b_1 \\ -(x - b_1 - 1)^2/2 + b_1 + 1/2, & b_1 \leqq x < b_1 + 2 = b_2 \\ -x + b_2 + b_1, & b_2 \leqq x < b_3 \\ (b-x)^2/2, & b_3 \leqq x < b_3 + 1 = b \\ 0, & b \leqq x. \end{cases}$$

Given b_1, b_3 can be chosen so that y and y' will be continuous at b_3. Consequently, y will be an element of $\mathscr{D}(b)$.

Since $r(x) \leqq M$, it follows that

$$\int_a^b r(y'')^2 \leqq M\left[\int_0^1 (\omega'')^2 + 3\right].$$

Also,

$$\int_a^b [q(y')^2 + py^2] \leqq \int_a^{a+1} [q(y')^2 + py^2] + \int_{a+1}^{b_1} x^2 p + \int_{b_1}^{b} py^2 + \int_{a+1}^{\infty} q_+$$

since $(y')^2 \leqq 1$ on $[a+1, b]$.

Since

$$\int^{\infty} x^2 p = -\infty,$$

there is a number c such that $t \geqq c$ implies that

$$\int_a^{a+1} [q(y')^2 + py^2] + M\left[\int_0^1 (\omega'')^2 + 3\right] + \int_{a+1}^{\infty} q_+ + \int_{a+1}^{t} x^2 p < 0.$$

https://doi.org/10.4153/CJM-1975-017-4 Published online by Cambridge University Press

We will next show that

$$\int_{b_1}^{b} py^2 < 0.$$

Then, we will have that inequality (2) holds, and the proof will be complete.
Let

$$Q(t) = \int_{c}^{t} x^2 p \, dx$$

and b_1 be its last zero on $[a, \infty)$. Integration by parts yields the inequality

$$\int_{b_1}^{b} py^2 \, dt = -\int_{b_1}^{b} Q \left(\frac{2y}{t^4}\right) (ty' - y) \, dt$$

$$< -\int_{b_1}^{b_1+1} Q \left(\frac{2y}{t^3}\right) (ty' - y) \, dt$$

since for $t \in [b_1 + 1, b]$, $Q(t) \leqq 0$, $y'(t) \leqq 0$, and $y(t) \geqq 0$. For $t \in [b_1, b_1 + 1]$ calculations show that

$$ty'(t) - y(t) = (b_1^2 - t^2)/2 \leqq 0.$$

Therefore,

$$\int_{b_1}^{b} py^2 < 0 \qquad \text{since } y(t) \geqq 0 \text{ and } Q(t) \leqq 0 \text{ on } [b_1, b_1 + 1].$$

The next theorem considers L_4 when p is integrable. In this case we define

$$P_1(x) = \int_{x}^{\infty} p.$$

THEOREM 3. *If*

$$-\infty < \int^{\infty} p < \infty, \textit{ then } r(x) \leqq M, \, q \leqq 0, \textit{ and}$$

$$\int^{\infty} x^{\alpha} P_1 = -\infty \textit{ for } 0 \leqq \alpha \leqq 1, \textit{ then then } L_4 \textit{ is oscillatory on } (0, \infty).$$

Proof. Given $a > 0$, let $\omega(x)$ be the third degree polynomial which satisfies the conditions

$$\omega(0) = 0 = \omega'(0), \; \omega(1) = (a + 1)^{\beta}, \text{ and } \omega'(1) = \beta(a + 1)^{\beta - 1}$$

where $\beta = (\alpha + 1)/2$. Define $y(x)$ as follows:

$$y(x) = \begin{cases} 0, & x \leqq a \\ \omega(x-a), & a < x \leqq a+1 \\ x^{\beta}, & a+1 < x \leqq b_1 \\ -(x-b_2)^2/2 + b_1^{\beta} + (b_1 - b_2)^2/2, & b_1 < x \leqq b_2 \\ y(b_2), & b_2 < x \leqq b_3 \\ y(b+a-x), & b_3 < x \leqq b \\ 0, & b < x. \end{cases}$$

Given b_1 we choose b_2 so that y' is continuous at b_1, and given b_3 we pick b so that $b - b_3 = b_2 - a$. Hence, for $b_1 \geqq a + 1$ and $b_3 \geqq b_2$ we will have that $y \in \mathscr{D}(b)$.

Calculations show that since $b_2 - b_1 \leqq 1$ then

$$\int_a^b r(y'')^2 \leqq 2M\left[\int_0^1 (\omega'')^2 + \beta^2(\beta-1)^2 \int_a^\infty (x^{\beta-2})^2 dx + 1\right].$$

By integrating by parts, we obtain the equality

$$\int_a^b py^2 = 2\int_a^b yy'P_1.$$

Since

$$\int^\infty x^\alpha P_1 = -\infty \text{ and } \alpha = 2\beta - 1,$$

there is a number c such that $t \geqq c$ implies that

$$1 + \int_a^b r(y'')^2 + 2\int_a^{a+1} yy'P_1 + 2\beta\int_{a+1}^t x^{2\beta-1}P_1 < 0.$$

Let

$$Q(t) = \int_c^t x^\alpha P_1$$

and b_1 be the last zero of $Q(t)$. Then,

$$2\int_{b_1}^{b_2} yy'P_1 = -2\int_{b_1}^{b_2} Q(t)\left(\frac{yy'}{t^\alpha}\right)' dt$$

$$< -2\int_{b_1}^{b_2} Q(t)\frac{[(y')^2 - y]}{t^\alpha} dt < 0$$

since $y \geqq 1$, $0 \leqq y' \leqq 1$, $y'' = -1$, and $Q(t) \leqq 0$ on $[b_1, b_2]$.

Since $|P_1| \to 0$ as $x \to \infty$, we can pick b_3 so that

$$2(b_2 - a)|P_1(x)|y(b_2) \max_{x\in[a,b_2]} |y'| \leqq 1$$

for all $x \geqq b_3$. Hence,

$$2\int_{b_2}^{b} yy'P_1 \leqq 2\int_{b_3}^{b} |yy' \cdot P_1| \leqq 1.$$

Therefore, inequality (2) holds for y, and the proof is complete.

Most theorems concerning the oscillation of L_4 place the burden of making the quadratic functional negative for some $y \in \mathscr{D}(b)$ upon either p or q, but not both. The next two theorems investigate how the combined negativity of p and q can cause L_4 to be oscillatory.

THEOREM 4. *If r is bounded, q is bounded above, p is non-positive, and*

$$\int^{\infty} (x^2p + q) = -\infty,$$

then L_4 is oscillatory on $(0, \infty)$.

Proof. Given $a > 0$, for $\beta = 1$ choose $\omega(x)$ as in Theorem 3. Define $y(x)$ as follows:

$$y(x) = \begin{cases} 0, & x < a \\ \omega(x-a), & a \leqq x < a+1 \\ x, & a+1 \leqq x < b_1 \\ -(x-b_2)^2/2 + b_1 + 1/2, & b_1 \leqq x < b_1 + 1 = b_2 \\ -(x-b_2)^2/2 + b_1 + 1/2, & b_2 \leqq x \leqq b_3 \\ -(b_3-b_2)(x-b_3) + y(b_3), & b_3 < x < b_4 \\ (x-b)^2/2, & b_4 \leqq x < b \\ 0, & b \leqq x. \end{cases}$$

Given b_3 it is clear that b_4 and b can be chosen to insure the continuity of y and y'. This will require that $b - b_4 = b_3 - b_2$.

Since $0 < r(x) \leqq M$ and if we require that $b_3 - b_2 \leqq 1$, then calculations show that

$$\int_a^b r(y'')^2 \leqq M\left[\int_0^1 (\omega'')^2 + 3\right].$$

Since $p \leqq 0$, $q \leqq N$ for some constant N, and $(y')^2 \leqq 1$ on $[b_1, b_2]$, then

$$\int_a^b [q(y')^2 + py^2] \leqq \int_a^{a+1} q(y')^2 + \int_{a+1}^{b_1} (q + x^2p) + N + \int_{b_2}^{b} q(y')^2.$$

Since

$$\int_{a+1}^{\infty} (q + x^2p) \to -\infty \quad \text{as} \quad t \to \infty$$

there is a constant b_1 such that $t \geqq b_1$ implies that

$$\int_a^b r(y'')^2 + \int_a^{a+1} q(y')^2 + \int_{a+1}^{t} (q + x^2p) + N + 1 < 0.$$

We need only to show that $b_3 - b_2$ can be made so small that

$$\int_{b_2}^{b} q(y')^2 \leqq 1.$$

First, note that by considering the graph of y on $[b_2, b]$ we know that

$$b - b_2 \leqq 2 + b_4 - b_3 \leqq 2 + y(b_2)/(b_3 - b_2).$$

Consequently,

$$\int_{b_2}^{b} q(y')^2 \leqq N(b_3 - b_2)^2(b - b_2) \leqq N(b_3 - b_2)(2(b_3 - b_2) + y(b_2)).$$

Therefore, we can choose b_3 so that

$$\int_{b_2}^{b} q(y')^2 \leqq 1,$$

and the proof is complete.

Theorem 4 shows that for $M > 0$, $(My'')'' - (qy')' + py$ is oscillatory on $(0, \infty)$ where

$$q(x) = A \sin Bx \quad \text{and} \quad p(x) = -q_+(x)/x^2$$

or

$$q(x) = C > 0 \quad \text{and} \quad p(x) = -(Cx + 1)/x^3.$$

THEOREM 5. *If* $r(x) \leqq M x^\alpha$ *for* $\alpha < 1$, $p \leqq 0$, $q \leqq 0$, *and*

$$\liminf_{x \to \infty} x^{1-\alpha} \int_x^\infty (q + t^2 p) < -4M,$$

then L_4 *is oscillatory on* $(0, \infty)$.

Proof. For $0 \leqq \mu < 1$, let $\omega_\mu(x)$ be the third degree polynomial with the following properties:

$$\omega_\mu(0) = 0 = \omega_\mu'(0) \quad \text{and} \quad \omega_\mu(1 - \mu) = 1 = \omega_\mu'(1 - \mu).$$

Also, let $\beta(x) = x^2/2$, and define $y(x)$ as follows:

$$y(x) = \begin{cases} 0, & x < \mu\rho \\ \omega_\mu((x - \mu\rho)/\rho), & \mu\rho \leqq x < \rho \\ x/\rho, & \rho \leqq x < R \\ -R \cdot \omega_\mu((1 - \mu)(2R - x)/R)/\rho(1 - \mu) & \\ \qquad + R/\rho + R/\rho(1 - \mu), & R \leqq x \leqq 2R \\ y(2R), & 2R < x < S \\ -\beta((x - S)/S) + y(2R), & S \leqq x < 2S \\ -(x - 2S)/S + y(2R) - 1/2, & 2S \leqq x < T \\ T \cdot \beta((x - 2T)/T)/S, & T \leqq x < 2T \\ 0, & 2T \leqq x. \end{cases}$$

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If $\rho > 0$ then calculations show that

$$\int_{\mu\rho}^{2T} r(y'')^2 \leqq M\rho^{\alpha-3}\left[\int_0^{1-\mu}(\omega_\mu'')^2 + 2^\alpha(1-\mu)\left(\int_0^{1-\mu}\frac{(\omega_\mu'')^2}{\rho^{\alpha-1}R^{1-\alpha}}\right) + 2^\alpha\rho^{3-\alpha}\frac{(S^{\alpha-1}+T^{\alpha-1})}{S^2}\right].$$

Calculations also show that

$$F(\mu) = \int_0^{1-\mu}(\omega_\mu'')^2 = \frac{4(\mu^2+\mu+1)}{(1-\mu)^3}.$$

Note that $F(\mu)$ is increasing on $[0, 1)$, and $F(0) = 4$.

Since

$$\liminf_{x\to\infty} x^{1-\alpha}\int_x^\infty (q+t^2p)\,dt < -4M,$$

there is a sequence $\langle\rho_k\rangle \to \infty$ as $k \to \infty$ and a number $\delta > 0$ such that

$$\lim_{k\to\infty} \rho_k^{1-\alpha}\int_{\rho_k}^\infty (q+t^2p)dt = -4M - \delta.$$

Since $F(0) = 4$ and F is continuous at 0, we can choose μ such that $0 < \mu < 1$ and $F(\mu) = 4 + \delta/6M$. For a given $a > 0$ there is a positive integer N such that $k \geqq N$ implies that

$$\rho_k^{1-\alpha}\int_{\rho_k}^\infty (q+t^2p)dt < -4M - \frac{3\delta}{4}$$

and $\mu\rho_k \geqq a$. Let $\rho = \rho_N$. Since

$$\lim_{x\to\infty} \rho^{1-\alpha}\int_\rho^x (q+t^2p)dt < -4M - \frac{3\delta}{4}$$

we can choose R so large that

$$\rho^{1-\alpha}\int_\rho^R (q+t^2p)dt < -4M - \frac{\delta}{2},$$

$R \geqq 2\rho$, and

$$2^\alpha(1-\mu)\int_0^{1-\mu}\frac{(\omega_\mu'')^2}{(\rho^{\alpha-1}R^{1-\alpha})} < \frac{\delta}{6M}.$$

Pick S so large that

$$2^\alpha\rho^{3-\alpha}(S^{\alpha-1}+T^{\alpha-1})/S^2 < \delta/6M$$

when $T \geqq 2S$. Choose $T \geqq 2S$ so that $y(x)$ is continuous at $x = T$, i.e., $T = S + 2S\,y(2R)/3$. Note that since $R \geqq 2\rho$ then $y(2R) \geqq 2$, and this

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implies that $T > 2S$. Consequently,

$$\int_{\mu\rho}^{2T} r(y'')^2 \leqq M\rho^{\alpha-3}\left(4 + \frac{\delta}{2M}\right).$$

Since p and q are not positive

$$\int_{\mu\rho}^{2T} r(y'')^2 + q(y')^2 + py^2 \leqq \rho^{\alpha-3}\left[4M + \frac{\delta}{2} + \rho^{1-\alpha}\int_{\rho}^{R}(q + t^2p)\right] < 0.$$

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