BULL. AUSTRAL. MATH. SOC. MOS 46L20, 20M20

VOL. 4 (1971), 201-203.

On the semigroup of all continuous linear mappings on a Banach space

Sadayuki Yamamuro

It is well-known that every ring automorphism of the ring of all linear transformations of a real vector space into itself is inner. We shall show that, if this ring is regarded as a semigroup with respect to composition and the dimension of the vector space is not less than 2 , every semigroup automorphism is inner.

Let E be a real Banach space and $L(E)$ be the set of all continuous linear mappings of E into itself. With pointwise addition and composition, $L(E)$ is a ring, and, with the usual upper bound norm, it is a Banach algebra. In the sequel, we assume that the dimension of E is not 1 .

M. Eidelheit [1] has shown that every algebraic automorphism of this ring is inner; in other words, if ϕ is a one-to-one mapping of $L(E)$ onto itself such that

(1) $$\phi(f+g) = \phi(f) + \phi(g) \quad \text{for all} \quad f, g \in L(E)$$

and

(2) $$\phi(fg) = \phi(f)\cdot\phi(g) \quad \text{for all} \quad f, g \in L(E) ,$$

then there exists an invertible $h \in L(E)$ such that

(3) $$\phi(f) = hfh^{-1} \quad \text{for every} \quad f \in L(E) .$$

Eidelheit also proved in the same paper that, if $L(E)$ is regarded as a semigroup with respect to the composition, then every *continuous* semigroup automorphism is inner. He has done so by showing that the continuity and the condition (2) imply (1).

Received 23 October 1970. The author wishes to express his hearty thanks to his friend, K.D. Magill, Jr, for a very useful suggestion.

In this note, we shall show that the continuity is not necessary, that is, every semigroup automorphism of $L(E)$ is a ring automorphism.

THEOREM. *Every automorphism of the semigroup $L(E)$ is inner.*

Proof. Eidelheit proved that, if ϕ is a semigroup automorphism, there exists a one-to-one mapping h of E onto itself such that the condition (3) is satisfied and, moreover,

(4) $\quad h(\xi x) = \mu(\xi)h(x)$ for any $x \in E$ and real number ξ ,

where μ is a one-to-one mapping of R (the set of all real numbers) onto R such that $\mu(1) = 1$ and $\mu(\xi\eta) = \mu(\xi)\cdot\mu(\eta)$ for any $\xi, \eta \in R$, and

(5) $h(x+y) = h(x) + h(y)$ if x and y are linearly independent.

(Then he used the continuity of ϕ to show that h is homogeneous and hence $h \in L(E)$.)

Now, we shall show that from (3), (4) and (5) it follows that h is additive.

Let x be an arbitrary non-zero element and take another element y such that x and y are linearly independent. For arbitrary $\xi, \eta \in R$, there exist continuous linear functionals $\overline{x}$ and $\overline{y}$ such that

$$\langle x, \overline{x}\rangle = \xi \ , \quad \langle y, \overline{y}\rangle = \eta \ , \quad \langle y, \overline{x}\rangle = 0 \ \text{ and } \ \langle x, \overline{y}\rangle = 0 \ ,$$

where, for instance, $\langle x, \overline{x}\rangle$ denotes the value of $\overline{x}$ at x . Then, put

$$f = x \otimes \overline{x} + x \otimes \overline{y} \ ,$$

where, for instance, $x \otimes \overline{x}$ is a one-dimensional mapping defined by

$$(x \otimes \overline{x})(z) = \langle z, \overline{x}\rangle x \quad \text{for every } z \in E \ .$$

Then, $f \in L(E)$, $f(x) = \xi x$ and $f(y) = \eta x$. Now, since $\phi(f) \in L(E)$, it follows from (5) that

$$\phi(f)h(x+y) = \phi(f)h(x) + \phi(f)h(y) \ .$$

On the other hand, it follows from (3) that $\phi(f)h = hf$. Therefore,

$$h\big((\xi+\eta)x\big) = hf(x+y) = hf(x) + hf(y) = h(\xi x) + h(\eta x) \ ,$$

and, by (4), we have

$$\mu(\xi+\eta) = \mu(\xi) + \mu(\eta) \ ,$$

which, together with (4), implies that $\mu(\xi) = \xi$ for every $\xi \in R$. Therefore, h is homogeneous.

Now, for arbitrary x and y, if these are linearly dependent, $y = \alpha x$ for some α, and hence

$$h(x+y) = h\big((1+\alpha)x\big) = (1+\alpha)h(x) = h(x) + \alpha h(x) = h(x) + h(y) .$$

This fact and (5) imply that h is additive. Therefore, from (3) it follows that ϕ is a ring automorphism.

REMARK 1. As Eidelheit mentioned, the theorem is not true for one-dimensional spaces. For instance, $\phi(\xi) = \xi^3$ is a semigroup automorphism of R which is not inner.

REMARK 2. As is easily seen, the same theorem holds for the multiplicative semigroup of all linear mappings of a real vector space whose dimension is not less than 2.

Reference

[1] M. Eidelheit, "On isomorphisms of rings of linear operators", *Studia Math.* 9 (1940), 97-105.

Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra, ACT.

https://doi.org/10.1017/S0004972700046463 Published online by Cambridge University Press