

A COINCIDENCE THEOREM IN TOPOLOGICAL VECTOR SPACES

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In this paper we prove a coincidence theorem in not necessarily locally convex topological vector spaces, which contains, as a special case, a coincidence theorem proved by Felix Browder. As an application, a result about the existence of maximal elements is obtained.

1. Introduction

In the recent time many papers are devoted to the fixed point theory in not necessarily locally convex topological vector spaces [3]-[13].

There are many important topological vector spaces which are not locally convex, as for example $L^p(0 < p < 1)$, $S(0,1)$ (the space of all equivalence classes of measurable functions on $[0,1]$), $H^p(0 < p < 1)$. So, it is of interest to find fixed point theorems for mappings defined on such spaces. The book [7] contains the most important results from the fixed point theory in topological vector spaces. In this paper we prove some generalizations of Proposition 2 and Theorem 1 from Browder's paper [1] to topological vector spaces which are not necessarily locally convex.

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First, we shall give some notations and definitions.

Let E be a topological vector space and $M \subseteq E$. It is assumed that all topological spaces in this paper are Hausdorff.

By \mathcal{Z}_{CO}^M we shall denote the family of all nonempty, convex subsets of M . Let F be another topological vector space, C a nonempty subset of E and T a multivalued mapping from C into F . The mapping T is said to be upper semicontinuous if for each neighbourhood V of zero in F and each point $x_0 \in C$, there exists a neighbourhood U of zero in E such that:

$$T(x) \subseteq T(x_0) + V, \text{ for all } x \in (x_0 + U) \cap C.$$

In [5] we introduced the following definition.

DEFINITION 1. Let E be a topological vector space, $K \subseteq E$ and U the fundamental system of neighbourhoods of zero in E . The set K is said to be of Zima's type if for every $V \in U$ there exists $U \in U$ such that:

$$co(U \cap (K - K)) \subseteq V \quad (co \text{ is the convex hull}).$$

Remark. In [6] we proved the following result: If K is a convex subset of Zima's type of a Hausdorff topological vector space E then:

(1) $A \subseteq K$, A is precompact $\Rightarrow co A$ is precompact.

By (1) we proved in [6] a generalization of Sadovskii's fixed point theorem.

Let us give an example of a subset of Zima's type.

DEFINITION 2. Let E be a vector space over \mathbb{R} and $\|\cdot\|^*$: $E \rightarrow [0, \infty)$ so that the following conditions are satisfied:

1. For every $x \in E$, $\|x\|^* = \|-x\|^*$ and $\|x\|^* \leq 0 \Leftrightarrow x = 0$.

2. For every $x, y \in E$: $\|x + y\|^* \leq \|x\|^* + \|y\|^*$.

3. If $\|x_n - x_0\|^* \rightarrow 0$ ($x_n \in E$, $n \in \mathbb{N} \cup \{0\}$) and $t_n \in \mathbb{R}$ ($n \in \mathbb{N}$),

$t_n \rightarrow t_0$, when $n \rightarrow \infty$, then $\|t_n x_n - t_0 x_0\|^* \rightarrow 0$, $n \rightarrow \infty$.

Then $\|\cdot\|^*$ is said to be a paranorm and the pair $(E, \|\cdot\|^*)$ a paranormed space.

If $(E, || \cdot ||^*)$ is a paranormed space then E is a topological vector space in which the fundamental system of neighbourhoods of zero is given by the family $V = \{V_r\}_{r>0}$, where:

$$V_r = \{x \mid ||x||^* < r\}.$$

Let K be a nonempty subset of E where $(E, || \cdot ||^*)$ is a paranormed space. The following condition is introduced in [16]. Suppose that there exists $C(K) > 0$ so that:

$$(2) \quad ||tx||^* \leq C(K)t||x||^*, \text{ for every } t \in [0,1] \text{ and every } x \in K-K.$$

It is easy to see that from (2) we obtain:

$$co(U_{\frac{r}{C(K)}} \cap (K-K)) \subseteq U_r, \text{ for every } r > 0.$$

Let $S(0,1)$ be the space of finite measurable functions (classes) on the interval $[0,1]$ with the metric $d(\hat{x}, \hat{y}), \hat{x}, \hat{y} \in S(0,1)$ defined by:

$$d(\hat{x}, \hat{y}) = \int_0^1 \frac{|x(t)-y(t)|}{1+|x(t)-y(t)|} \mu(dt), \quad \begin{matrix} \{x(t)\} \in \hat{x} \\ \{y(t)\} \in \hat{y} \end{matrix}.$$

The space $S(0,1)$ is also a paranormed space with the paranorm:

$$||x||^* = \int_0^1 \frac{|x(t)|}{1+|x(t)|} \mu(dt), \quad \hat{x} \in S(0,1).$$

Let $M > 0$ and K_M be the subset of $S(0,1)$ defined by:

$$K_M = \{\hat{x}, \hat{x} \in S(0,1), |x(t)| \leq M, t \in [0,1] \text{ for some } \{x(t)\} \in \hat{x}\}.$$

We proved in [6] that $C(K_M)$ from (2) is in this case $1 + 2M$.

2. A Coincidence Theorem for Multivalued mappings.

The following lemma is a generalization of Proposition 2 from Browder's paper [1].

LEMMA. Let E and F be topological vector spaces, C a compact subset of E , $T:C \rightarrow Z_{CO}^F$ an upper semicontinuous mapping such that for a given convex subset A of F :

$$T(x) \cap A \neq \emptyset \text{ for every } x \in C$$

and $T(C)$ be of Zima's type. Let V_0 be a neighbourhood of zero in F and U_0 a neighbourhood of zero in E . Then there exists a continuous singlevalued mapping $f : C \rightarrow \text{co } M, M \subseteq A, \text{card } M < \infty$ such that for each $x \in C$, there exists $u \in C$ such that $x \in u + U_0$ and $f(x) \in T(u) + V_0$.

Proof. Let \tilde{V}_0 be a neighbourhood of zero in F such that $\text{co}(\tilde{V}_0 \cap (T(C) - T(C))) \subseteq V_0$. Since the set $T(C)$ is of Zima's type such a neighbourhood exists. Further, the mapping T is upper semicontinuous and so there exists, for each $x \in C$, a neighbourhood U_x of zero in E , such that:

$$(3) \quad v \in x + U_x \Rightarrow T(v) \subseteq T(x) + \tilde{V}_0.$$

We shall suppose that $U_x \subseteq U_0$, for every $x \in C$. Let V_x be an open symmetric neighbourhood of zero in E such that $V_x + V_x \subseteq U_x$. From the compactness of the set C it follows that there exists a finite open covering : $\{x_j + V_{x_j}\}_{j=1}^m$ of the set C .

$$\text{Let } W = \bigcap_{j=1}^m V_{x_j} \text{ and } \{v_1, v_2, \dots, v_n\} \subseteq C \text{ so that:}$$

$$C \subseteq \bigcup_{s=1}^n \{v_s + W\}.$$

Let $\{h_1, h_2, \dots, h_n\}$ be a partition of unity for the covering $\{v_s + W\}_{s=1}^n$ and $y_s \in T(v_s) \cap A \neq \emptyset$, for every $s \in \{1, 2, \dots, n\}$. As in Browder's paper [1], let us define the mapping f in the following way:

$$f(x) = \sum_{s=1}^n h_s(x) y_s, \text{ for every } x \in C.$$

It remains to be proved that the mapping f satisfies all the conditions which are given in the Lemma. Suppose that $x \in C$ and let $j \in \{1, 2, \dots, m\}$ be such that $x \in x_j + V_{x_j}$. If $s \in \{1, 2, \dots, n\}$ is such that $h_s(x) \neq 0$

then $x \in v_x + W \subseteq v_s + V_{x_j}$. Since V_{x_j} is symmetric we obtain

$$v_s \in x + V_{x_j} \subseteq x_j + V_{x_j} + V_{x_j} \subseteq x_j + U_{x_j}.$$

Then (3) implies that $T(v_s) \subseteq T(x_j) + \tilde{V}_0$ and since $y_s \in T(v_s)$ ($s \in \{1, 2, \dots, n\}$) we obtain:

$$h_s(x) \neq 0 \Rightarrow y_s \in T(x_j) + \tilde{V}_0.$$

Let us prove that $f(x) \in T(x_j) + V_0$. From $y_s \in T(x_j) + \tilde{V}_0$ it follows that there exists $z_s \in T(x_j)$ and $u_s \in \tilde{V}_0$ such that $y_s = z_s + u_s$. Further, since $y_s \in T(v_s)$ and $z_s \in T(x_j)$ it follows that $u_s \in \tilde{V}_0 \cap (T(C) - T(C))$. From the definition of the mapping f we obtain, for every $x \in C$:

$$\begin{aligned} f(x) &= \sum_{s: h_s(x) \neq 0} h_s(x) y_s = \sum_{s: h_s(x) \neq 0} h_s(x) (z_s + u_s) = \\ &= \sum_{s: h_s(x) \neq 0} h_s(x) z_s + \sum_{s: h_s(x) \neq 0} h_s(x) u_s. \end{aligned}$$

Since $T(x_j)$ is convex, from the relation $z_s \in T(x_j)$, for every s such that $h_s(x) \neq 0$, we obtain:

$$(4) \quad \sum_{s: h_s(x) \neq 0} h_s(x) z_s \in T(x_j).$$

On the other hand, from $u_s \in \tilde{V}_0 \cap (T(C) - T(C))$, for every s such that $h_s(x) \neq 0$, we obtain:

$$(5) \quad \sum_{s: h_s(x) \neq 0} h_s(x) u_s \in \text{co}(\tilde{V}_0 \cap (T(C) - T(C))).$$

From (4) and (5) we have that:

$$f(x) \in T(x_j) + \text{co}(\tilde{V}_0 \cap (T(C) - T(C))) \subseteq T(x_j) + V_0$$

which implies for $u = x_j$ that the condition for f in the Lemma is satisfied.

Remark. Since every subset of a locally convex space is of Zima's type from the Lemma, Proposition 2 follows from [1].

THEOREM. Let C be a compact, convex subset of a topological vector space E , C_1 a compact, convex subset of a topological vector space F , $T: C \rightarrow 2_{CO}^{C_1}$ an upper semicontinuous mapping, $S: C_1 \rightarrow 2_{CO}^C$ an upper semicontinuous mapping such that the sets $T(C)$, $S(C_1)$ are of Zima's type. Then there exist $x_0 \in C$ and $y_0 \in T(x_0)$ so that $x_0 \in S(y_0)$.

Proof. The proof is similar to the proof of Theorem 1, from [1]. By $G(T)$ and $G(S)$ we shall denote graphs of T and S respectively. Let U be any neighbourhood of zero in E and V any neighbourhood of zero in F . Let us prove that:

$$(6) \quad G(T) \cap (G(S^{-1}) + (U \times V)) \neq \emptyset.$$

Let \tilde{U} be a neighbourhood of zero in E such that $\tilde{U} + \tilde{U} \subseteq U$, and \tilde{V} a neighbourhood of zero in F such that $\tilde{V} + \tilde{V} \subseteq V$. From the Lemma it follows that there exist a continuous mapping f_1 of C into C_1 such that:

(i) For each $x \in C$ there exists $(u, v) \in G(T)$ such that $x \in u + \tilde{U}$ and $f_1(x) \in v + \tilde{V}$, $f_1(C) \subseteq \text{Lin}(M)$, $\text{card } M < \infty (M \subseteq C_1)$ and a continuous mapping $f_2: C_1 \rightarrow C$ such that:

(ii) For each $y \in C_1$, there exists $(v_1, u_1) \in G(S)$ such that $y \in v_1 + \tilde{V}$ and $f_2(y) \in u_1 + \tilde{U}$, $f_2(C_1) \subseteq \text{Lin}(P)$, $\text{card } P < \infty (P \subseteq C)$.

Since $f_2 f_1: C \rightarrow C$ is a continuous mapping such that $f_2 f_1(C)$ is a subset of the linear hull of a finite subset of C , from Brouwer's fixed point theorem it follows that there exists $x_0 \in C$ so that $f_2 f_1(x_0) = x_0$.

Let u, v, u_1 and v_1 are chosen from (i) and (ii) for $x = x_0$ and $y = y_0 = f_1(x_0)$. Then $x_0 \in u + \tilde{U}$ and $x_0 = f_2(y_0) \in u_1 + \tilde{U}$ and if we suppose that \tilde{U} is symmetric we obtain $u - u_1 \in \tilde{U} + \tilde{U} \subseteq U$. Similarly

$v - v_1 \in \tilde{V} + \tilde{V} \subseteq V$, if we suppose that \tilde{V} is symmetric. Hence $(u, v) \in (u_1, v_1) + U \times V$ which implies (6).

COROLLARY 1. *Let $C = C_1$, $E = F$ and $S = Id_C$ in the Theorem. Then T has a fixed point.*

Remark. If in the Theorem we suppose, instead of the assumption that $S(C_1)$ is of Zima's type, that $S(\overline{co}T(C))$ is of Zima's type, it is obvious that the Theorem remains valid, since we can take $C_1' = \overline{co}T(C)$. If F is complete and C_1 is of Zima's type then from the Remark after Definition 1 it follows that it is enough to suppose that C_1 is closed and convex.

If $S = Id_C$ in [3] is proved a fixed point theorem if $T(C)$ is of Zima's type.

COROLLARY 2. *Let C be a compact convex subset of topological vector space E , C_1 a nonempty subset of topological vector space F , $T: C \rightarrow 2_{co}^{C_1}$ and $S: C_1 \rightarrow 2_{co}^C$ an upper semicontinuous mapping such that $\overline{co} S(T(C))$ is of Zima's type. If for every $x \in C$ there exists $y \in C_1$ such that $x \in \text{int}T^{-1}(y)$ then there exists $x_0 \in C$ and $y_0 \in T(x_0)$ such that $x_0 \in S(y_0)$.*

Proof. As in [14] it follows that there exists a continuous mapping $f: C \rightarrow C_1$ such that $f(x) \in T(x)$, for every $x \in C$. For the sake of completeness we include the proof. Since C is compact, from $C = \bigcup_{y \in C_1} \text{int}T^{-1}(y)$, it follows that there exists a finite set $\{y_1, y_2, \dots,$

$y_n\}$ from C_1 such that $C = \bigcup_{i=1}^n T^{-1}(y_i)$. Let $\{f_1, f_2, \dots, f_n\}$ be a

partition of unity subordinated to the covering $\{\text{int}T^{-1}(y_i)\}_{i=1}^n$ and let

$f: C \rightarrow F$ be defined in the following way:

$$f(x) = \sum_{i=1}^n f_i(x)y_i, \quad x \in C.$$

If $f_i(x) \neq 0$ then $x \in \text{int}T^{-1}(y_i) \subseteq T^{-1}(y_i)$ and so we have the implication: $f_i(x) \neq 0 \Rightarrow y_i \in T(x)$.

Hence, f is a continuous mapping from C into C_1 such that $f(x) \in T(x)$, for every $x \in C$, and let $R(x) = S(f(x))$, for every $x \in C$. Then R is an upper semicontinuous mapping from C into 2_{co}^C and $R(C) \subseteq S(T(C))$ which implies that $\overline{\text{co}}R(C)$ is of Zima's type. From this it follows that there exists $x_0 \in C$ such that $x_0 \in R(x_0) = S(f(x_0))$. If we take that $y_0 = f(x_0)$ the proof is complete.

As an application, we can prove a Proposition about the existence of a maximal element in the sense given below [14].

DEFINITION 3. Let X be a subset of a topological vector space E and for every $x \in X$, Tx is a subset of X (may be empty). A point $x_0 \in X$ is said to be a maximal element of T if $T(x_0) = \emptyset$.

PROPOSITION. Let C be a compact convex subset of a topological vector space E , C_1 a nonempty convex subset of a topological vector space F , for every $x \in C$, $Tx \subseteq C_1$ and S an upper semicontinuous mapping of C_1 into 2_{co}^C such that $Tx \neq \emptyset$ implies: $x \notin S(\text{co } Tx)$ and there exists $y_x \in C_1$ such that $x \in \text{int}T^{-1}(y_x)$.

If $\overline{\text{co}}S(C_1)$ is of Zima's type there exists $x_0 \in C$ such that $Tx_0 = \emptyset$.

Proof. Suppose that $Tx \neq \emptyset$, for every $x \in C$. Then $\text{co } Tx \neq \emptyset$ and the mappings S and $G: x \rightarrow \text{co } Tx$ satisfy all the conditions of Corollary 2. From this we obtain that there exists $x_0 \in C$ such that $x_0 \in S(G(x_0)) = S(\text{co } Tx_0)$ which is a contradiction. So there exists $x_0 \in C$ such that $Tx_0 = \emptyset$.

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