# GYROSTATS IN FREE ROTATION

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**Abstract.** We consider the problem of the attitude dynamics of a gyrostat under no external torques and constant internal spins. We introduce coordinates to represent the orbits of constant angular momentum as a flow on a sphere. This new representation shows that the problem is a particular case in the class of dynamical systems defined by a Hamiltonian that is a polynomial of at most degree two in a base of the Lie algebra $so(3)$.

## 1. Introduction

A gyrostat $\mathcal{G}$ is a mechanical system made of a rigid body $\mathcal{P}$ called the *platform* and other bodies $\mathcal{R}$ called the *rotors*, connected to the platform and in such a way that the motion of the rotors does not modify the distribution of mass of the gyrostat $\mathcal{G}$. Due to this double spinning, the gyrostat on the one hand and the rotors on the other, the gyrostat is also known with the name of *dual-spin* body.

One of the first applications of this model is found in the work of Volterra (1922) to study the rotation of the Earth. Later on, the problem has been studied mainly from a theoretical interest and a wide list of references may be found in Leimanis' textbook (Leimanis, 1965). In the last years, the gyrostat model has attracted the attention of aerospace engineers and it is used for controlling the attitude dynamics of spacecrafts and for stabilizing their rotations. See, for instance, Cochran *et al.* (1982), Hall and Rand (1994), Hall (1995) and also Hughes (1986) for further references. Recently, Tong *et al.* (1995) and also Chiang (1995) have reexamined this problem.

In fact, the last two papers just quoted motivated us in presenting this note. Mainly due to the classical procedure followed, both fail in describing the actual problem. Indeed, Tong *et al.* (1995) used a Mercator map based

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on Andoyer variables (derived from spherical trigonometry!) for picturing the phase portrait. As we show here, the phase flow in this problem takes place on a sphere, and a sphere cannot be projected on a Mercator map without creating artificial singularities. Unfortunately, it so happens that the singularities brought by the projection in the Mercator map of Tong *et al.* correspond precisely to two equilibria of the gyrostat. They belong to the dynamical system, yet they cannot appear on the Mercator map. As for Chiang, we remark that, by integrating the Euler equations for an axially symmetric gyrostat in terms of elliptic functions, he discovers three types of motion, namely, circulations, librations and asymptotic motions. He then reasons by analogy to conclude that the addition of a rotor along an inertial axis of the platform creates a dynamical system that is strictly equivalent to a general Eulerian rotor, we mean a triaxial rigid body rotating freely about a fixed point. Should he have drawn the phase flow on a sphere, he would have realized that, at the difference of the triaxial rigid body (that has six equilibria, two unstable joined by four heteroclinic orbits and four stable), a symmetric gyrostat with only one rotor, when it is not degenerate, has at most four equilibria (three stable and one unstable in conformity with Poincaré's Index Theorem). We shall be more precise in a note that we are preparing for publication; there we show how the phase flow evolves through an alternance of pitchforks and oyster bifurcations.

The equations of the motion for the gyrostat (when the relative angular moments of the rotors are known functions of the time) are usually processed in two steps: a differential system (I) of order three to account for the evolution of the angular momentum $\boldsymbol{G}$ and another differential system (II) to determine the attitude of the rotor in space. Usually the unknowns in system (I) are taken to be the components of the angular velocity $\boldsymbol{\omega}$, but they lead to complicated visualizations as in the model of Poinsot (1851). We prefer to operate with the components of $\boldsymbol{G}$ in the inertial frame, because the phase flow determined by (I) then occurs on a sphere of constant radius (Hughes, 1986, p. 113).

## 2. Hamiltonian of the Gyrostat

Let us assume that the gyrostat has a fixed point $O$, that we will identify with the center of mass of the gyrostat and centered on it there are two orthonormal reference frames:

- $\mathcal{S}$, the space frame $O\boldsymbol{s}_1\boldsymbol{s}_2\boldsymbol{s}_3$, fixed in the space.
- $\mathcal{B}$, the body frame $O\boldsymbol{b}_1\boldsymbol{b}_2\boldsymbol{b}_3$, fixed in the platform.

Since by definition the motion of the rotors does not alter the distribution of mass of the gyrostat, there is a constant inertia tensor associated to $\mathcal{G}$

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and we may assume that the body frame is precisely the frame of principal axes of inertia of the gyrostat.

The attitude of $\mathcal{B}$ in $\mathcal{S}$ results in three rotations by means of the Euler angles.

The nutation angle $\vartheta$ ($0 \leq \vartheta \leq \pi$) is defined by the dot product $\cos\vartheta = \boldsymbol{b}_3 \cdot \boldsymbol{s}_3$.. The vector $\boldsymbol{l}$, the direction of the intersection of the space plane $(\boldsymbol{s}_1, \boldsymbol{s}_2)$ with the body plane $(\boldsymbol{b}_1, \boldsymbol{b}_2)$, is obtained by $\boldsymbol{l} = \boldsymbol{s}_3 \times \boldsymbol{b}_3/\sin\vartheta$. This vector is related with the axes $(\boldsymbol{s}_1, \boldsymbol{s}_2)$ by

$$\boldsymbol{l} = \boldsymbol{s}_1 \cos\phi + \boldsymbol{s}_2 \sin\phi, \qquad 0 \leq \phi < 2\pi,$$

where the angle $\phi$, usually known as the precession angle, is the longitude of the node $\boldsymbol{l}$ reckoned from the space axis $\boldsymbol{s}_1$. By denoting $\psi$ (with $0 \leq \psi < 2\pi$) the longitude of the body vector $\boldsymbol{b}_1$ reckoned from the node $\boldsymbol{l}$, this vector is also expressed as the combination

$$\boldsymbol{l} = \boldsymbol{b}_1 \cos\psi - \boldsymbol{b}_2 \sin\psi.$$

By means of the composite rotation $R = R(\phi, \boldsymbol{s}_3) \circ R(\vartheta, \boldsymbol{l}) \circ R(\psi, \boldsymbol{b}_3)$ (see (Deprit and Elipe, 1993) for details), the space frame $\mathcal{S}$ is mapped onto the body frame $\mathcal{B}$. Making use of the differential of $R$,

$$dR = \boldsymbol{s}_3\, d\phi + \boldsymbol{l}\, d\vartheta + \boldsymbol{b}_3\, d\psi, \tag{1}$$

and of the total angular momentum

$$\boldsymbol{G} = G_1\boldsymbol{s}_1 + G_2\boldsymbol{s}_2 + G_3\boldsymbol{s}_3 = g_1\boldsymbol{b}_1 + g_2\boldsymbol{b}_2 + g_3\boldsymbol{b}_3,$$

it is possible to overlay a symplectic structure through the requirements that the conjugate moments $(\Phi, \Theta, \Psi)$ of the Euler angles satisfy the Cartan 1-form

$$\boldsymbol{G} \cdot d\boldsymbol{R} = \Phi\, d\phi + \Theta\, d\vartheta + \Psi\, d\psi.$$

Taking (1) into account, we see that the conjugate moments to the Eulerian angles are the projections of the total angular vector $\boldsymbol{G}$ onto the non orthonormal basis $\boldsymbol{s}_3, \boldsymbol{l}, \boldsymbol{b}_3$, that is, that

$$\Phi = \boldsymbol{G} \cdot \boldsymbol{s}_3, \qquad \Theta = \boldsymbol{G} \cdot \boldsymbol{l}, \qquad \Psi = \boldsymbol{G} \cdot \boldsymbol{b}_3.$$

Hence, by inversion, there results

$$\begin{aligned} g_1 &= \left(\frac{\Phi - \Psi\cos\vartheta}{\sin\vartheta}\right)\sin\psi + \Theta\cos\psi, \\ g_2 &= \left(\frac{\Phi - \Psi\cos\vartheta}{\sin\vartheta}\right)\cos\psi - \Theta\sin\psi, \\ g_3 &= \Psi \end{aligned} \tag{2}$$

for the components in the body frame and

$$G_1 = \Theta \cos\phi + \left(\frac{\Psi - \Phi\cos\vartheta}{\sin\vartheta}\right)\sin\phi,$$
$$G_2 = \Theta \sin\phi - \left(\frac{\Psi - \Phi\cos\vartheta}{\sin\vartheta}\right)\cos\phi, \tag{3}$$
$$G_3 = \Phi$$

in the space frame.

It is just a matter of computing partial derivatives to check that the Poisson brackets satisfy the identities

$$\{g_1\,;\,g_2\} = -g_3, \quad \{g_2\,;\,g_3\} = -g_1, \quad \{g_3\,;\,g_1\} = -g_2,$$
$$\{G_1\,;\,G_2\} = G_3, \quad \{G_2\,;\,G_3\} = G_1, \quad \{G_3\,;\,G_1\} = G_2. \tag{4}$$

On account of (1), the angular velocity $\boldsymbol{\omega}$ of the frame $\mathcal{B}$ with respect to the frame $\mathcal{S}$ is

$$\boldsymbol{\omega} = \dot\phi\,\boldsymbol{s}_3 + \dot\vartheta\,\boldsymbol{l} + \dot\psi\,\boldsymbol{b}_3;$$

while in the body frame its expression is

$$\boldsymbol{\omega} = \omega_1\boldsymbol{b}_1 + \omega_2\boldsymbol{b}_2 + \omega_3\boldsymbol{b}_3.$$

By writing the vectors $\boldsymbol{b}_i$ in terms of the non orthogonal basis $\boldsymbol{s}_3$, $\boldsymbol{l}$, $\boldsymbol{b}_3$, one gets easily that

$$\omega_1 = \dot\phi\sin\vartheta\sin\psi + \dot\vartheta\cos\psi,$$
$$\omega_2 = \dot\phi\sin\vartheta\cos\psi - \dot\vartheta\sin\psi, \tag{5}$$
$$\omega_3 = \dot\phi\cos\vartheta + \dot\psi.$$

Now, we proceed to compute the kinetic energy of the gyrostat. Although this physical quantity must be calculated in the inertial frame, we compute it in the body frame, because (as we will see) the resulting expression is invariant to rotations. Let $\boldsymbol{x} = x_1\boldsymbol{b}_1 + x_2\boldsymbol{b}_2 + x_3\boldsymbol{b}_3$ be the position vector of a particle $P$ of the gyrostat with mass $d\,m$; its absolute velocity is $d\boldsymbol{x}/dt = \boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{x}$, where $\boldsymbol{v} = \dot x_1\boldsymbol{b}_1 + \dot x_2\boldsymbol{b}_2 + \dot x_3\boldsymbol{b}_3$. If the particle belongs to the platform (recall that it is a rigid body), then $\boldsymbol{v} = 0$. The kinetic energy of the gyrostat is obtained by computing the volume quadrature

$$\begin{aligned} T &= \frac{1}{2}\int_{\mathcal{G}} (\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{x})^2\,d\,m \\ &= \frac{1}{2}\int_{\mathcal{G}} (\boldsymbol{\omega}\times\boldsymbol{x})^2\,d\,m + \boldsymbol{\omega}\cdot\int_{\mathcal{R}} (\boldsymbol{x}\times\boldsymbol{v})\,d\,m + \frac{1}{2}\int_{\mathcal{R}} \boldsymbol{v}^2\,d\,m \\ &= \frac{1}{2}\boldsymbol{\omega}\cdot\mathbb{I}\,\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\boldsymbol{f} + T_{\mathcal{R}}, \end{aligned} \tag{6}$$

where $\mathbb{I}$ is the diagonal tensor of inertia of the gyrostat $\mathcal{G}$ while $\boldsymbol{f} = f_1\boldsymbol{b}_1 + f_2\boldsymbol{b}_2 + f_3\boldsymbol{b}_3$ is the angular moment of the rotors and $T_{\mathcal{R}}$ is the kinetic energy of the rotors in their relative motion.

The Hamiltonian is the Legendre transformation with respect to the velocities of the Lagrangian function. Let us call the abbreviation $\boldsymbol{q} = (\phi, \theta, \psi)$ to denote the set of Eulerian angles; the kinetic energy of the gyrostat (6) is made of the addition of a pure quadratic term $(\frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}\,\boldsymbol{\omega})$ in the velocities $\dot{\boldsymbol{q}}$, plus a linear part $(\boldsymbol{\omega} \cdot \boldsymbol{f})$ in the velocities $\dot{\boldsymbol{q}}$ (for $\boldsymbol{f}$ does not depend on the Euler angles), plus a function of the time $(T_{\mathcal{R}}(t))$. By virtue of Euler's theorem for homogeneous functions, the Legendre transformation of the Lagrangian (with a potential $V(\boldsymbol{q})$) is

$$\mathcal{L}_L : \dot{\boldsymbol{q}} \longrightarrow \nabla_{\dot{\boldsymbol{q}}}(T - V) \cdot \dot{\boldsymbol{q}} - (T - V) = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}\,\boldsymbol{\omega} - T_{\mathcal{R}} + V,$$

and since the relative kinetic energy is a function of $t$ only, the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}\,\boldsymbol{\omega} + V(\boldsymbol{q}).$$

Since our paper deals exclusively with a gyrostat in free rotation $(V = 0)$, the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}\,\boldsymbol{\omega}, \tag{7}$$

Rather than expressing $\mathcal{H}$ in terms of the canonical coordinates and moments, we will express it in terms of the total angular moment that, as we observed in the Introduction, plays a critical role in our work.
N.B. The independent variable $t$ does not appear explicitly on the Hamiltonian (7), hence the kinetic energy of the gyrostat $\mathcal{G}$ considered as a rigid body (7) is preserved along the motion, whereas the total energy $T$ is not.

## 3. Motion of the Angular Momentum

The angular momentum vector of the gyrostat in the body frame is

$$\begin{aligned}
\boldsymbol{G} &= \int_{\mathcal{G}} (\boldsymbol{x} \times (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}))\, dm \\
&= \int_{\mathcal{P}} (\boldsymbol{x} \times (\boldsymbol{\omega} \times \boldsymbol{x}))\, dm + \int_{\mathcal{R}} (\boldsymbol{x} \times (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{x}))\, dm \\
&= \int_{\mathcal{G}} (\boldsymbol{x} \times (\boldsymbol{\omega} \times \boldsymbol{x}))\, dm + \int_{\mathcal{R}} (\boldsymbol{x} \times \boldsymbol{v})\, dm \\
&= \mathbb{I}\,\boldsymbol{\omega} + \boldsymbol{f}.
\end{aligned}$$

Let us denote by $a_j$ the inverse of the principal moment of inertia $I_j$, that is $a_j = 1/I_j$. Assume moreover that $a_1 \geq a_2 \geq a_3 > 0$. With these conventions, there is a diagonal tensor $\boldsymbol{A}$, such that the angular velocity is $\boldsymbol{\omega} = \boldsymbol{A}(\boldsymbol{G} - \boldsymbol{f})$. The Hamiltonian (7) in these notations is

$$\mathcal{H} = \frac{1}{2}(\boldsymbol{G} - \boldsymbol{f}) \cdot \boldsymbol{A}(\boldsymbol{G} - \boldsymbol{f}). \tag{8}$$

From here on, we shall assume that the rotor moment is constant ($f_i =$ constant, $i = 1, 2, 3$). Hence, expanding the expression (8), and after dropping the constant terms ($\sum a_i f_i^2$), we find the following expression for the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(a_1 g_1^2 + a_2 g_2^2 + a_3 g_3^2) - (a_1 g_1 f_1 + a_2 g_2 f_2 + a_3 g_3 f_3). \tag{9}$$

The Poisson structure (4) gives rise to the equations of the motion

$$\begin{aligned}
\dot{g}_1 &= \{g_1\,;\,\mathcal{H}\} = (a_3 - a_2)g_2 g_3 + a_2 f_2 g_3 - a_3 f_3 g_2, \\
\dot{g}_2 &= \{g_2\,;\,\mathcal{H}\} = (a_1 - a_3)g_1 g_3 + a_3 f_3 g_1 - a_1 f_1 g_3, \\
\dot{g}_3 &= \{g_3\,;\,\mathcal{H}\} = (a_2 - a_1)g_1 g_2 + a_1 f_1 g_2 - a_2 f_2 g_1.
\end{aligned} \tag{10}$$

From this system, one can easily prove that the norm of the angular momentum vector $\boldsymbol{G}$ is an integral

$$\|\boldsymbol{G}\|^2 = g_1^2 + g_2^2 + g_3^2 = G^2 = \text{constant}; \tag{11}$$

the history of the rotation of $\boldsymbol{G}$ in the body frame is represented as a curve on the $S^2$ sphere of constant radius $G$.

The equations (10) admit two integrals, the kinetic energy (9) and the norm of the total angular momentum (11), therefore it is integrable. The phase space of (10) may be regarded as a foliation of invariant manifolds

$$S^2(G) = \{(g_1, g_2, g_3) \,|\, g_1^2 + g_2^2 + g_3^2 = G^2\}.$$

By using the angular momentum $\boldsymbol{G}$ instead of the angular velocity $\boldsymbol{\omega}$, the geometric model depicting the rotations of $\boldsymbol{G}$ is a sphere with constant radius. Most importantly, unlike Poinsot's ellipsoids, the underlying model is independent of the ellipsoid of inertia.

## 4. Parametric Classification

The differential system (10) belongs to a general class of Hamiltonian systems, the one of the type

$$\mathcal{H} = \mathcal{T}_2 + \mathcal{T}_1, \quad \text{with} \quad \mathcal{T}_2 = \frac{1}{2}\sum_{1\leq i,j\leq 3} A_{ij}\,\xi_i \xi_j \quad \text{and} \quad \mathcal{T}_1 = \frac{1}{2}\sum_{1\leq i\leq 3} B\,\xi_i.$$

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The unknowns $\xi$ have the Poisson structure

$$\{\xi_i\,;\,\xi_j\} = -\sum_{1<k<3} \epsilon_{i,j,k}\,\xi_k,$$

where $\epsilon_{i,j,k}$ stands for the Levi-Civita symbol. The six coefficients $A_{ij}$ and the three coefficients $B_i$ are parameters independent of the variables $\xi$. The class depends on 9 parameters, but it is possible to reduce it to 5 standard classes; this is done by rotations in the phase space $(\xi_1, \xi_2, \xi_3)$. As the table below shows, one can roughly divide the classes by the number of parameters they contain.

| Case | parameters | Hamiltonian | Reference |
|---|---|---|---|
| A | 1 | $\mathcal{H} = \frac{1}{2}\xi_1^2 + R\xi_2$ | [a] |
| B.a | 2 | $\mathcal{H} = \frac{1}{2}\xi_1^2 + Q\xi_1 + R\xi_2$ | [b] |
| B.b | | $\mathcal{H} = \frac{1}{2}\xi_1^2 + \frac{1}{2}P\xi_2^2 + Q\xi_1$ | [c] |
| C | 3 | $\mathcal{H} = \frac{1}{2}\xi_1^2 + \frac{1}{2}P\xi_2^2 + Q\xi_1 + R\xi_2$ | [d] |
| D | 4 | $\mathcal{H} = \frac{1}{2}\xi_1^2 + \frac{1}{2}P\xi_2^2 + Q\xi_1 + R\xi_2 + S\xi_3$ | [e] |

[a]Lanchares (1993)
[b]Lanchares and Elipe (1995a)
[c]Lanchares and Elipe (1995b)
[d]Lanchares *et al.* (1995)
[e]Frauendiener (1995)

Cases A, B.a, B.b and C were identified and analyzed extensively by Elipe and Lanchares. Case D is due to Frauendiener (1995) and is yet to be analyzed.

In fact, to each class corresponds a different type of gyrostat.

Case A.– Axially symmetric gyrostat ($a_1 = a_2 > a_3$) with one rotor spinning about anyone of the principal axes of inertia ($f_i \neq 0$ and $f_j = f_k = 0$). The gyrostat examined by Chiang belongs to his class.

Case B.a – Axially symmetric gyrostat ($a_1 = a_2 > a_3$) with three rotors spinning about each principal axis of inertia ($f_1, f_2, f_3 \neq 0$).

Case B.b – Asymmetric gyrostat ($a_1 > a_2 > a_3$) with one rotor spinning about any one of the principal axes of inertia ($f_i \neq 0,\ f_j = f_k = 0$). The gyrostat studied by Tong *et al.* falls into this case.

Case C – Asymmetric gyrostat ($a_1 > a_2 > a_3$) with two rotors spinning about any two of the principal axes of inertia ($f_i \neq 0$, $f_j \neq 0$ and $f_k = 0$).

Case D – Asymmetric gyrostat ($a_1 > a_2 > a_3$) with three rotors spinning about each principal axis of inertia ($f_1, f_2, f_3 \neq 0$).

The conclusion: a purely mathematical study of an abstract type of Hamiltonians has turned into a conceptual frame for analyzing a very concrete physical system like the gyrostat.

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