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ON QUADRATIC FUNCTIONALS

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In this note a general solution of the problem of the characterisation of quadratic functionals posed by Vukman is given.

THEOREM. *Let A be a complex $*$-algebra with identity e and let X be a vector space which is also a unitary left A-module. Suppose there exists a mapping $Q\colon X \longrightarrow A$ with the properties*

(i) $Q(x+y)+Q(x-y)=2Q(x)+2Q(y)$ *for all pairs* $x,y \in X$, *and*
(ii) $Q(ax)=aQ(x)a^*$ *for all* $x \in X$ *and all* $a \in A$.

Under these conditions the mapping $B(\cdot,\cdot)\colon X \times X \longrightarrow A$ defined by the relation

$$B(x,y)=(1/4)(Q(x+y)-Q(x-y))+(i/4)(Q(x+iy)-Q(x-iy))$$

satisfies the following:

(1) $B(\cdot,\cdot)$ *is additive in both arguments;*

(2) $\begin{matrix} B(ax,y)=aB(x,y) \\ B(x,ay)=B(x,y)a^* \end{matrix}$, *for all pairs* $x,y \in X$ *and all* $a \in A$;

(3) $Q(x)=B(x,x)$ *for all* $x \in X$.

REMARK: A functional $Q\colon X \longrightarrow A$ which satisfies (i) and (ii) is called an A-quadratic functional and a mapping $B\colon X \times X \longrightarrow A$ for which conditions (1) and (2) are fulfilled is called an A-sesquilinear functional. If A is the complex number field then this result reduces to Kurepa's extension of the Jordan-Neumann theorem which characterises pre-Hilbert space among all normed spaces.([**3**])

PROOF: As in the proof of Kurepa's result (see [**3**], [**5**] and also [**6**]) one can prove that the function $W(\cdot,\cdot)$ defined by relation $W(x,y)=Q(x+y)-Q(x-y)$ is additive in both variables. Therefore the same is true for the functional B. A short computation shows that $Q(x)=B(x,x)$ for all $x \in X$. Hence it remains to prove (2). For this purpose we define a new functional $S\colon A\times A \longrightarrow A$ by $S(a,b)=aB(x,y)b^*-B(ax,by)$

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where x and y are fixed vectors. From the fact that B is biadditive it follows that the functional S is also biadditive. Using (ii) one can easily obtain

$$S(ca, cb) = cS(a, b)c^*, \qquad a, b, c \in A.$$

A short computation yields $S(ia, b) = iS(a, b)$ and $S(a, ib) = -iS(a, b)$. For any four elements $a, b, c, d \in A$ we have that $S(ab, ac) + S(ab, dc) + S(db, ac) + S(db, dc)$ $= S((a + d)b, (a + d)c) = (a + d)S(b, c)(a^* + d^*) = aS(b, c)a^* + dS(b, c)a^* + aS(b, c)d^*$ $+ dS(b, c)d^*$. This yields $S(ab, dc) + S(db, ac) = dS(b, c)a^* + aS(b, c)d^*$. Replacing d and c by e we get

$$S(ab, e) + S(b, a) = S(b, e)a^* + aS(b, e). \tag{4}$$

Let us put the element ia instead of a. We obtain

$$iS(ab, e) - iS(b, a) = -iS(b, e)a^* + iaS(b, e). \tag{5}$$

Comparing (4) and (5) we see that $S(ab, e) = aS(b, e)$ and $S(b, a) = S(b, e)a^*$. Replacing b by e by using the relation $S(e, e) = 0$ we complete the proof. ∎

This result was proved in [4] and [8] under the stronger assumption that A is a Banach $*$-algebra (see also [6] and [7]) using the fact that such algebras have enough invertible elements. It should be mentioned that in the proof of the present general result an idea similar to those of Davison [1] was used.

References

[1] T.M.K. Davison, 'Jordan derivations and quasi-bilinear forms', *Comm. Algebra* **121** (1984), 23–32.

[2] S. Kurepa, 'The Cauchy functional equation and scalar product in vector spaces', *Glasnik Math. Fiz.-Astr.* **19** (1964), 23–36.

[3] S. Kurepa, 'Quadratic and sesquilinear functionals', *Glas. Mat. Fiz.-Astr.* **20** (1965), 79–92.

[4] P. Šemrl, 'On quadratic and sesquilinear functionals', *Aequationes Math.* **31** (1986), 184–190.

[5] P. Vrbová, 'Quadratic and bilinear forms', *Časopis Pěst. Mat.* **98** (1973), 159–161.

[6] J. Vukman, 'A result concerning additive functions in hermitian Banach $*$-algebras and an application', *Proc. Amer. Math. Soc.* **91** (1984), 367–372.

[7] J. Vukman, 'Some results concerning the Cauchy functional equation in certain Banach algebras', *Bull. Austral. Math. Soc.* **31** (1985), 137–144.

[8] J. Vukman, 'Some functional equations in Banach algebras and an application', *Proc. Amer. Math. Soc.* (to appear).

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