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RESEARCH ARTICLE

Small-amplitude periodic travelling waves in dimer Fermi–Pasta–Ulam–Tsingou lattices without symmetry

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Abstract
We prove the existence of small-amplitude periodic travelling waves in dimer Fermi–Pasta–Ulam–Tsingou (FPUT) lattices without assumptions of physical symmetry. Such lattices are infinite, one-dimensional chains of coupled particles in which the particle masses and/or the potentials of the coupling springs can alternate. Previously, periodic travelling waves were constructed in a variety of limiting regimes for the symmetric mass and spring dimers, in which only one kind of material data alternates. The new results discussed here remove the symmetry assumptions by exploiting the gradient structure and translation invariance of the travelling wave problem. Together, these features eliminate certain solvability conditions that symmetry would otherwise manage and facilitate a bifurcation argument involving a two-dimensional kernel and cokernel.

1. Introduction

1.1. The travelling wave problem

A dimer Fermi–Pasta–Ulam–Tsingou (FPUT) lattice is a chain of infinitely many particles coupled to their nearest neighbours by springs, with motion restricted to the horizontal direction, and with at least one of the following material heterogeneities: either the particle masses alternate, or the spring potentials alternate, or both alternate. A dimer with alternating particles and identical springs is called a mass dimer; one with alternating springs and identical masses is a spring dimer. See Figure 1. Dimers are among the simplest nontrivial generalizations of the classical monatomic FPUT lattice, in which all of the particles have the same mass and all of the springs have the same potential [12, 21, 36, 39]. These lattices, and their many variants and generalizations, are prototypical models of wave dynamics in granular media [9, 10].

Figure 1(a) and (b) suggests that the mass and spring dimers possess certain physical 'symmetries' that a 'general' dimer, in which both masses and springs alternate, does not. We sketch such a general dimer, along with some notation for future use, in Figure 2. Physically, the mass dimer is the same when 'reflected' about a particle, as is the spring dimer when reflected about a spring. Such symmetries manifest themselves mathematically in a variety of useful ways, as we elaborate in Section 5, and these manifestations have been key to multiple prior analyses of mass and spring dimer dynamics. Here we consider the general dimer, and one of the main novelties of our techniques is that we do not rely at all on symmetry.

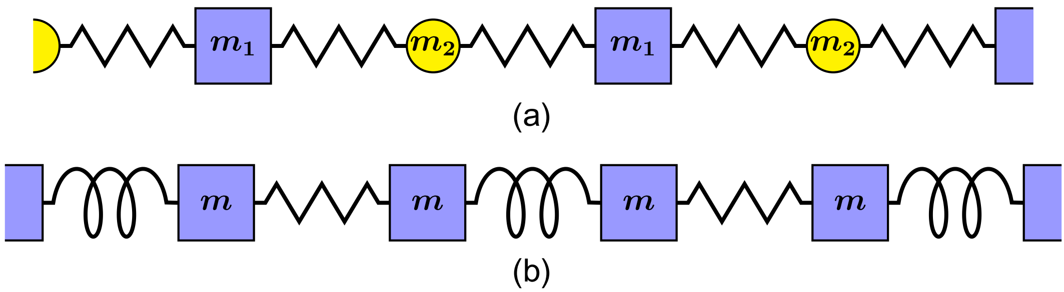

Figure 1. The symmetric mass and spring dimers. (a) A mass dimer with alternating masses m_1 *and* m_2 *and identical springs. (b) A spring dimer with alternating springs and identical masses* m.

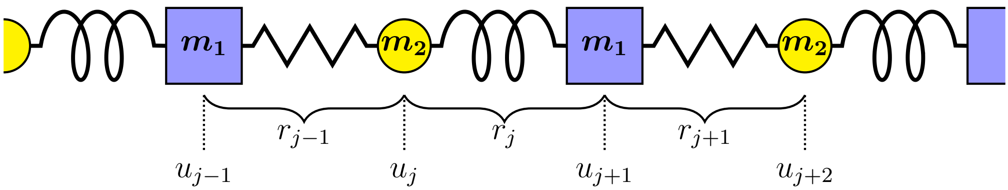

Figure 2. A general dimer with alternating masses and springs.

Specifically, we construct nontrivial periodic travelling waves for general dimers with wave speed greater than a certain critical threshold called the lattice's 'speed of sound' – that is, supersonic periodic travelling waves. We state our precise results below in Theorem 1.1 and discuss the connection of these travelling waves to the long wave problem in dimers, and related problems, in Section 1.2.

Here is our problem. Index the particles by integers $j \in \mathbb{Z}$ and let u_j be the displacement of the jth particle from its equilibrium position, let m_j be the mass of the jth particle, and let $\mathcal{V}_j$ be the potential of the spring connecting the jth and $(j+1)$st particles. To ensure a dimer structure, we assume

$$m_{j+2} = m_j \qquad \text{and} \qquad \mathcal{V}_{j+2} = \mathcal{V}_j$$

for all j. More precisely, after nondimensionalization [15, Sec. 1.3, App. F.5], we take

$$m_j = \begin{cases} 1, \ j \text{ is odd} \\ m, \ j \text{ is even} \end{cases} \qquad \text{and} \qquad \mathcal{V}_j'(r) = \begin{cases} r + r^2 + \mathcal{O}(r^3), \ j \text{ is odd} \\ \kappa r + \beta r^2 + \mathcal{O}(r^3), \ j \text{ is even.} \end{cases} \tag{1.1}$$

The minimum regularity required for our proofs is that $\mathcal{V}_j \in \mathcal{C}^7(\mathbb{R})$ for precise technical reasons detailed in Appendix A.5, but for broader applications to FPUT travelling wave problems, we may as well assume $\mathcal{V}_j \in \mathcal{C}^\infty(\mathbb{R})$. Our methods require that the heterogeneity appear at the linear level, so we will always assume

$$\frac{1}{m} > 1 \qquad \text{or} \qquad \kappa > 1. \tag{1.2}$$

We emphasize that m and κ, and indeed all of the material data, are fixed throughout our analysis and that virtually all operators, quantities and thresholds depend on at least these quantities; we do not indicate such dependence in our notation. Additionally, beyond the regularity requirements on $\mathcal{V}_j$, the nonlinear terms, even the quadratic ones, play no important role.

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Newton's second law requires that the displacements u_j satisfy

$$m_j\ddot{u}_j = \mathcal{V}_j'(u_{j+1} - u_j) - \mathcal{V}_{j-1}'(u_j - u_{j-1}). \tag{1.3}$$

Under the travelling wave ansatz

$$u_j(t) = \begin{cases} p_1(j - ct), & j \text{ is odd} \\ p_2(j - ct), & j \text{ is even}, \end{cases} \qquad \mathbf{p}(X) := \begin{pmatrix} p_1(X) \\ p_2(X) \end{pmatrix}, \ X = j - ct, \tag{1.4}$$

these equations of motion become the advance-delay problem

$$\begin{cases} c^2p_1'' = \mathcal{V}_1'(S^1p_2 - p_1) - \mathcal{V}_2'(p_1 - S^{-1}p_2) \\ c^2mp_2'' = \mathcal{V}_2'(S^1p_1 - p_2) - \mathcal{V}_1'(p_2 - S^{-1}p_1). \end{cases} \tag{1.5}$$

Here, for $\theta \in \mathbb{R}$, S^θ is the shift operator

$$(S^\theta p)(X) := p(X + \theta).$$

The following is our main result for (1.5). We use the notation for periodic Sobolev spaces developed in Appendix A.2.

Theorem 1.1. *Suppose that the lattice's material data m_j and $\mathcal{V}_j$ satisfy the dimer condition (1.1) and the linear heterogeneity condition (1.2). Let the wave speed* c *in the ansatz (1.4) satisfy $|c| > c_\star$, where the lattice's 'speed of sound' $c_\star$ is defined in (2.8). Then there exists $a_c > 0$ such that for $|a| \le a_c$, there is a travelling wave solution $\mathbf{p}_c^a$ to (1.5) of the form*

$$\mathbf{p}_c^a(X) = \boldsymbol{\phi}_c^a(\omega_c^a X), \qquad \boldsymbol{\phi}_c^a(x) = a\boldsymbol{\nu}_1^c(x) + a^2\boldsymbol{\psi}_c^a(x). \tag{1.6}$$

The smooth, 2π-periodic profile term $\boldsymbol{\phi}_c^a$ and the frequency $\omega_c^a \in \mathbb{R}$ have the following properties.

(i) *The leading order term $\boldsymbol{\nu}_1^c$ has an exact formula given below by (2.12).*
(ii) *The remainder term $\boldsymbol{\psi}_c^a$ is orthogonal to $\boldsymbol{\nu}_1^c$ and uniformly bounded in* a *in the sense that*

$$\langle \boldsymbol{\nu}_1^c, \boldsymbol{\psi}_c^a \rangle_{L^2_{\text{per}}} = 0 \qquad \text{and} \qquad \sup_{|a| \le a_c} \|\boldsymbol{\psi}_c^a\|_{H^r_{\text{per}}} < \infty, \ r \ge 0,$$

where the periodic Sobolev spaces L^2_{per} and H^r_{per} are defined in Appendix A.2.
(iii) *The frequency ω_c^a has the expansion*

$$\omega_c^a = \omega_c + a\xi_c^a,$$

where $\omega_c > 0$ is the lattice's 'critical frequency', as developed in Theorem 2.1, and

$$\sup_{|a| \le a_c} |\xi_c^a| < \infty.$$

These solutions are locally unique up to shifts and translations in the following sense. If $\mathbf{p}(X) = \boldsymbol{\phi}(\omega X)$ solves (1.5) with $\|\boldsymbol{\phi}\|_{H^2_{\text{per}}}$ and $|\omega - \omega_c|$ both sufficiently small, then there exist α, $\theta \in \mathbb{R}$ and $|a| \le a_c$ such that $\boldsymbol{\phi}(x) = \alpha\boldsymbol{\nu}_0 + \boldsymbol{\phi}_c^a(x + \theta)$, where $\boldsymbol{\nu}_0$ has the exact formula given by (2.11).

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We approach this theorem from multiple points of view. Specifically, Sections 3 and 4 give proofs inspired by the techniques of Wright and Scheel [40] for constructing asymmetric solitary wave solutions to a system of coupled KdV equations; the lack of symmetry in their problem manifests itself mathematically in a complication very close to ours, as we discuss below in Remark 3.4. The proof of local uniqueness up to translations follows from Corollary 3.3, and the proof of local uniqueness up to shifts appears in Section 3.5. Section 5 gives proofs in the special cases of mass and spring dimers when symmetry is present; this offers fresh perspectives on the prior results from [16, 20]. And Section 6 develops precise quantitative estimates for the solution components from Theorem 1.1 relative to the wave speed c in the special case that $|c|$ is close to the speed of sound $c_\star$ (rather than just greater than $c_\star$ as in the theorem); we have excluded these estimates from the theorem above, as they are extremely technical. In particular, the exact, but general, hypotheses of Theorem 6.2 subsume all prior constructions of dimer periodics into one quantitative result. For brevity, Theorem 1.1 does not contain our results in the long wave scaling, which we discuss instead in Section 6.2.

Remark 1.2. *The majority of travelling wave results (periodic or not) for lattices are stated in relative displacement coordinates: $r_j = u_{j+1} - u_j$. See Figure 2. We find it more convenient to work in the original equilibrium displacement coordinates u_j, from which relative displacement results can easily be obtained (though the converse is not necessarily true).*

We finally state the actual periodic travelling wave problem that we solve to prove Theorem 1.1; the following notation was not strictly necessary above, but all of our subsequent work depends on it. Since we are interested in periodic travelling waves, we adjust the original travelling wave ansatz (1.4) by decoupling the profile and frequency via the additional ansatz

$$\mathbf{p}(X) = \boldsymbol{\phi}(\omega X), \qquad \boldsymbol{\phi} := \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

The new profiles ϕ_1 and ϕ_2 are now 2π-periodic and $\omega \in \mathbb{R}$. We emphasize that the parameter ω is now the wavenumber and does not denote a dispersion relation; this parameter ω will serve as the crucial bifurcation parameter in our analysis.

The travelling wave equations (1.5) then become

$$\begin{cases} c^2\omega^2\phi_1'' = \mathcal{V}_1'(S^\omega\phi_2 - \phi_1) - \mathcal{V}_2'(\phi_1 - S^{-\omega}\phi_2) \\ mc^2\omega^2\phi_2'' = \mathcal{V}_2'(S^\omega\phi_1 - \phi_2) - \mathcal{V}_1'(\phi_2 - S^{-\omega}\phi_1). \end{cases} \tag{1.7}$$

We compress (1.7) in the form

$$\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) = 0, \tag{1.8}$$

where

$$\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) := c^2\omega^2 M\boldsymbol{\phi}'' + \begin{pmatrix} \mathcal{V}_2'(\phi_1 - S^{-\omega}\phi_2) - \mathcal{V}_1'(S^\omega\phi_2 - \phi_1) \\ \mathcal{V}_1'(\phi_2 - S^{-\omega}\phi_1) - \mathcal{V}_2'(S^\omega\phi_1 - \phi_2) \end{pmatrix} \tag{1.9}$$

and

$$M := \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix}. \tag{1.10}$$

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The primary challenge that we confront is that the linearization $D_{\boldsymbol{\phi}}\boldsymbol{\Phi}_c(0, \omega_c)$ has a three-dimensional kernel and cokernel. Translation invariance allows us to reduce both dimensions to two (and by restricting the domain to the right subspace, we could lower the dimension of the kernel even further), but, in the absence of symmetry, we cannot get below that. Specifically, we must contend with a two-dimensional cokernel, which adds two solvability conditions to our problem without giving us extra variables to help meet them. We now discuss the broader relevance of this periodic travelling wave problem and, in the process, why this dimension counting is so important.

1.2. Motivation, context and connections other FPUT travelling wave problems

Our primary motivation in constructing these particular periodics is the long wave problem for dimers. This limit looks for travelling waves whose profiles are close to a suitably scaled sech^2-type solution of a KdV equation that acts as the 'continuum limit' for the lattice. More precisely, one posits $\mathbf{p}(X) = \epsilon^2\mathbf{h}(\epsilon X)$ and $c^2 = c_\star^2 + \epsilon^2$, with $c_\star$ given by (2.8) and $\epsilon > 0$ small. The solutions from Theorem 1.1 are not strictly long-wave solutions, as they lack this scaling and are valid for all wave speeds $|c| > c_\star$. We discuss this further in Section 6.2. The relevance of this ansatz is that in a 'polyatomic' FPUT lattice, for which the material data repeats with some arbitrary period, long wave-scaled solutions to certain KdV equations (whose coefficients depend on the lattice's material data) are very good approximations to solutions to the equations of motion over very long time scales [8, 25, 38].

Faver and Wright constructed long wave solutions for the mass dimer [20] and Faver treated the spring dimer [16]. Faver and Hupkes produced a different development of mass and spring dimer nanopterons via spatial dynamics in [19] and obtained results for equilibrium displacement coordinates as we do; Deng and Sun [13] performed a related spatial dynamics analysis to yield similar results. These dimer travelling waves were not solitary waves, as Friesecke and Pego found for the monatomic lattice [23], but rather nanopterons [7]: the superposition of a leading-order localized (here, sech^2-type) term, a higher-order localized remainder, and a high-frequency periodic 'ripple' of amplitude small beyond all algebraic orders of the long wave parameter. Both constructions relied on lattice symmetries in two critical steps to adapt functional analytic techniques from Beale's work on capillary gravity water waves [5] and its later deployment by Amick and Toland [3] for a model fourth-order KdV equation.

Firstly, as mentioned above, the periodics in [16, 20] were constructed with a modified 'bifurcation from a simple eigenvalue' argument in the style of Crandall and Rabinowitz and Zeidler [11, 41]. We adapt further this bifurcation analysis in our arguments, and our preferred reference is [31, Thm. 1.5.1]. Symmetry permitted the restriction of the travelling wave problem $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) = 0$ from (1.8) and (1.9) to function spaces on which the linearization $D_{\boldsymbol{\phi}}\boldsymbol{\Phi}_c(0, \omega)$ at $\boldsymbol{\phi} = 0$ and $\omega = \omega_c$, with ω_c as the 'critical frequency' from Theorem 2.1, had a one-dimensional kernel and cokernel. This was the key to the modified bifurcation from a simple eigenvalue argument.

Up to a useful linear change of coordinates that diagonalized the travelling wave problem and the long wave scaling, the long wave periodics in [16, 20] have the same structure as ours from Theorem 1.1. However, the main technical accomplishment of our results here is that we manage a *two*-dimensional kernel *and* cokernel in the absence of symmetry via other inherent properties of the lattice – namely, the special 'orthogonality condition' that $\langle \boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega), \boldsymbol{\phi}' \rangle_{L^2_{\text{per}}} = 0$, proved in Corollary 3.2 and Lemma 4.1. While there certainly exist other results on bifurcations with two-dimensional kernels, their hypotheses are inappropriate for our problem. For example, [32] and [34] assume certain 'nondegeneracy' conditions on what for us would be the second derivative $D^2_{\boldsymbol{\phi}\boldsymbol{\phi}}\boldsymbol{\Phi}_c(0, \omega_c)$, over which we expect to have no control (beyond its existence), while [1] and [4] assume some (non)resonance conditions among their critical frequencies. Indeed, the two-dimensional kernel is far less of a problem than the two-dimensional *cokernel*, as we cannot make the latter smaller by restricting the domain to a better-behaved subspace.

The second use of symmetry in the full nanopteron constructions of [16, 20] was somewhat subtler and involved the actual need for periodics in the first place. The obstacle was that attempting to solve the travelling wave problem (1.5) by a merely localized perturbation from the sech^2-type continuum

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limit – and thereby construct solitary travelling waves in dimers – resulted in an overdetermined system with two unknowns (the two components of the localized perturbation) but four equations. These are the expected two components from (1.5) and a surprising 'solvability condition': the vanishing of the Fourier transform of a certain related operator at $\pm\omega_c$. Symmetry ensured that the vanishing at ω_c implied the vanishing at $-\omega_c$, reducing the overdetermined problem to only three equations, which were managed by adding a third variable via the periodic amplitude – which is exactly why we seek periodics in Theorem 1.1 that are parametrized in amplitude. While the full nanopteron problem in the general dimer without symmetry remains challenging and beyond the scope of our work here to address, the periodic solutions constructed here will be a fundamental component of the nanopteron ansatz for the general dimer's long wave problem.

Periodic travelling waves for lattices have been constructed in several other 'material limit' regimes in addition to the long wave limit. These include the small mass limit for mass dimers by Hoffman and Wright [27, Thm. 5.1], the equal mass limit for mass dimers by Faver and Hupkes [18, Prop. 3.3], and the small mass limit for the mass-in-mass variant of monatomic FPUT by Faver [14, Thm. 2]. Each of these limits views the heterogeneous lattice as a small material perturbation of a monatomic FPUT lattice, and the nanopteron is a small nonlocal perturbation of a monatomic solitary wave [23, 24]. While each limit has a nontrivially different solvability condition that makes the travelling wave problem overdetermined, all of the periodic constructions are fundamentally alike and can be deduced from Theorem 6.2.

In all of these nanopteron problems, it is not enough to have a family of periodic solutions whose amplitude can serve as an extra variable to close the overdetermined travelling wave problem. Additionally, one also needs exact, uniform, quantitative estimates on how these periodic solutions behave with respect to the overall problem's natural small parameter (the long wave problem ϵ, the small mass ratio, etc.) To that end, a result like Theorem 1.1, which does not uniformly depend on the wave speed c, is not enough. This is the motivation for the results in Section 6, which are too cumbersome to be included in Theorem 1.1.

These are not the only methods for producing periodic travelling waves for FPUT, and we give a brief, selected overview of others here for both monatomic lattices and dimers, in various limiting regimes and for various kinds of material data. Friesecke and Mikikits-Leitner [22] adapted the perturbative approach for monatomic solitary waves from [23] to prove the existence of long wave periodics in the monatomic lattice that were small perturbations of a KdV cnoidal profile. Pankov constructed periodics in the monatomic lattice using variational methods [36], as did Qin for mass dimers [37] with spring force given by the FPUT β-model, i.e., roughly of the form $\mathcal{V}'(r) = r + \mathcal{O}(r^3)$; while these proofs do not give information on amplitude, the constructed periodics do exist for arbitrary wavenumbers and frequencies. Herrmann constructed both solitary waves and periodic travelling waves in monatomic FPUT with convex spring potentials via variational methods [26]; these periodics can be constructed to have arbitrary mean value, which in turn determines the speed of the travelling wave. Iooss used spatial dynamics and centre manifold theory to capture all small travelling waves in monatomic FPUT, including solitary waves and periodics [29]. Betti and Pelinovsky used an implicit function theorem argument to produce periodics in mass dimers with Hertzian spring potentials [6], and we note with interest that their proofs also relied on symmetry to reduce the dimension of a key linearization's kernel. James also used implicit function theory to construct periodics parametrized by amplitude in monatomic FPUT with Hertzian potential [30]. Finally, we mention that Lombardi's spatial dynamics method for nanopterons under very general hypotheses includes the full development of periodics from that point of view [35], with the more stringent requirement that the spring potentials be real analytic.

1.3. Notation

We summarize several aspects of notation that we will use without further comment.

- If $\mathcal{X}$ is a vector space, then $\mathcal{I}_{\mathcal{X}}$ is the identity operator on $\mathcal{X}$.

- If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are normed spaces and $f\colon \mathcal{U} \subseteq \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is differentiable at some $(x_0, y_0) \in \mathcal{U}$, then we denote its partial derivative at (x_0, y_0) with respect to x by $D_x f(x_0, y_0)$. Likewise, $D_y f(x_0, y_0)$ is the partial derivative with respect to y. We reserve the notation $f' = \partial_x f$ for a function $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$.
- If $\mathcal{X}$ and $\mathcal{Y}$ are sets and $f\colon \mathcal{U} \subseteq \mathcal{X} \to \mathcal{Y}$ is a function, then for any $\mathcal{U}_0 \subseteq \mathcal{U}$ we denote by $f\big|_{\mathcal{U}_0}$ the restriction of f to $\mathcal{U}_0$.
- If $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{Y}}$, respectively, then the adjoint of a bounded linear operator $\mathcal{T}\colon \mathcal{X} \to \mathcal{Y}$ is the bounded linear operator $\mathcal{T}^*\colon \mathcal{Y} \to \mathcal{X}$ satisfying $\langle \mathcal{T}x, y\rangle_{\mathcal{Y}} = \langle x, \mathcal{T}^* y\rangle_{\mathcal{X}}$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. If the range $\mathcal{T}(\mathcal{X})$ of $\mathcal{T}$ is closed, then $\mathcal{T}(\mathcal{X}) = \ker(\mathcal{T}^*)^\perp$, where $\mathcal{U}^\perp$ is the orthogonal complement of $\mathcal{U} \subseteq \mathcal{Y}$.
- If $\mathcal{X}$ is a normed space, $x \in \mathcal{X}$, and $r > 0$, then $\mathcal{B}(x; r)$ is the open ball

$$\mathcal{B}(x; r) := \left\{ y \in \mathcal{X} \;\middle|\; \|x - y\|_{\mathcal{X}} < r \right\}.$$

- If $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces, then $\mathbf{B}(\mathcal{X}, \mathcal{Y})$ is the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ with operator norm $\|\mathcal{T}\|_{\mathcal{X}\to\mathcal{Y}}$.

2. Linear analysis

We assume familiarity here with the notation and conventions of Appendix A.2 on periodic Sobolev spaces and Fourier coefficients. Briefly, $\widehat{\boldsymbol{\phi}}(k)$ is the kth Fourier coefficient of $\boldsymbol{\phi} \in L^2_{\text{per}}(\mathbb{R}^2)$, and $\langle\cdot,\cdot\rangle$ is the L^2_{per}-inner product (we no longer retain the subscript here).

Our bifurcation analysis naturally hinges on a careful understanding of the linearization

$$\mathcal{L}_c[\omega] := D_{\boldsymbol{\phi}} \boldsymbol{\Phi}_c(0, \omega) \tag{2.1}$$

of the problem $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) = 0$ at $\boldsymbol{\phi} = 0$, where $\boldsymbol{\Phi}_c$ was defined in (1.9). Using that definition of $\boldsymbol{\Phi}_c$ and recalling from the hypotheses (1.1) that the spring potentials satisfy

$$\mathcal{V}_1'(r) = r + \mathcal{O}(r^2) \qquad \text{and} \qquad \mathcal{V}_2'(r) = \kappa r + \mathcal{O}(r^2),$$

we have

$$\mathcal{L}_c[\omega]\boldsymbol{\phi} = c^2\omega^2 M \boldsymbol{\phi}'' + \mathcal{D}[\omega]\boldsymbol{\phi},$$

where

$$\mathcal{D}[\omega] := \begin{bmatrix} (1+\kappa) & -(S^{\omega} + \kappa S^{-\omega}) \\ -(\kappa S^{\omega} + S^{-\omega}) & (1+\kappa). \end{bmatrix}. \tag{2.2}$$

We follow the strategy of the existing bifurcation arguments [20, App. C], [16, Sec. 3], [27, Sec. 5], [18, Sec. 3], [14, Sec. 3] and begin by considering the kernel of $\mathcal{L}_c[\omega]$. We have $\mathcal{L}_c[\omega]\boldsymbol{\phi} = 0$ if and only if

$$\widetilde{\mathcal{L}}_c(\omega k)\widehat{\boldsymbol{\phi}}(k) = 0 \tag{2.3}$$

for all $k \in \mathbb{Z}$, where

$$\widetilde{\mathcal{L}}_c(K) := -c^2 K^2 M + \widetilde{\mathcal{D}}(K) \tag{2.4}$$

and

$$\widetilde{\mathcal{D}}(K) := \begin{bmatrix} (1+\kappa) & -(e^{iK}+\kappa e^{-iK}) \\ -(\kappa e^{iK}+e^{-iK}) & (1+\kappa) \end{bmatrix}. \tag{2.5}$$

Then (2.3) is equivalent to

$$c^2(\omega k)^2\widehat{\boldsymbol{\phi}}(k) = M^{-1}\widetilde{\mathcal{D}}(\omega k)\widehat{\boldsymbol{\phi}}(k),$$

and so, if $\widehat{\boldsymbol{\phi}}(k) \neq 0$, then $c^2(\omega k)^2$ must be an eigenvalue of $M^{-1}\widetilde{\mathcal{D}}(\omega k)$. Any eigenvalue λ of $M^{-1}\widetilde{\mathcal{D}}(K)$ must satisfy the characteristic equation

$$\lambda^2 - (1+\kappa)(1+w)\lambda + 4\kappa w(1-\cos^2(K)) = 0,$$

and so the eigenvalues are

$$\widetilde{\lambda}_{\pm}(K) := \frac{(1+\kappa)(1+w)}{2} \pm \frac{\widetilde{\varrho}(K)}{2}, \tag{2.6}$$

where

$$\widetilde{\varrho}(K) := \sqrt{(1+w)^2(1-\kappa)^2 + 4\kappa((1-w)^2+4w\cos^2(K))} \qquad \text{and} \qquad w := \frac{1}{m}. \tag{2.7}$$

The following is proved in [15, Prop. 2.2.1] about these eigenvalues.

Theorem 2.1. *Suppose that at least one of the inequalities $\kappa > 1$ or $w > 1$ holds and define the 'speed of sound' to be*

$$c_\star := \sqrt{\frac{4\kappa w}{(1+\kappa)(1+w)}}. \tag{2.8}$$

(i) *If $|c| > c_\star$, then $c^2K^2 = \widetilde{\lambda}_-(K)$ if and only if $K = 0$.*
(ii) *If $|c| > c_\star$, then there exists $\omega_c > 0$ such that $c^2K^2 = \widetilde{\lambda}_+(K)$ if and only if $K = \pm\omega_c$.*
(iii) *This 'critical frequency' ω_c satisfies the estimates*

$$\frac{\sqrt{\widetilde{\lambda}_+(\pi/2)}}{c} \leq \omega_c \leq \frac{\sqrt{(1+\kappa)(1+w)}}{c} \tag{2.9}$$

and

$$\inf_{|c|>c_\star} 2c^2\omega_c - \widetilde{\lambda}_+'(\omega_c) > 0. \tag{2.10}$$

We sketch in Figure 3 graphs of the eigenvalues $\widetilde{\lambda}_\pm(K)$ against parabolas c^2K^2 for $|c| < c_\star$ and $|c| > c_\star$. In the case that $|c| > c_\star$, we see that the only intersections of c^2K^2 and $\widetilde{\lambda}_\pm(K)$ are those promised by parts (i) and (ii) of Theorem 2.1. Additionally, we can view the inequality (2.10) as a quantitative condition on the angle of intersection of c^2K^2 and $\widetilde{\lambda}_+(K)$. However, for $|c| < c_\star$, there can be more intersections, which breaks the utility of the one 'critical frequency' for the subsequent bifurcation arguments. These bifurcation arguments retain the style of the Classical Crandall–Rabinowitz–Zeidler

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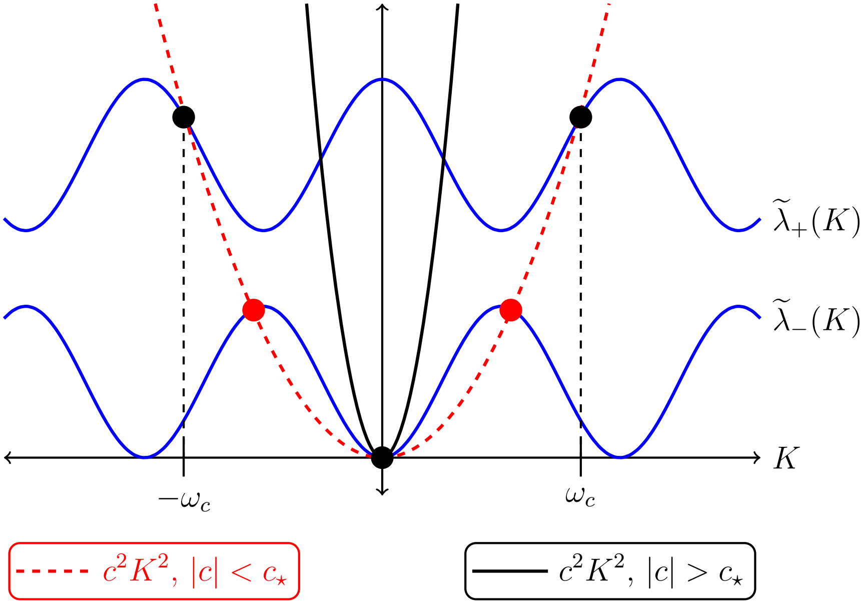

Figure 3. *Graphs of the two branches $\widetilde{\lambda}_\pm(K)$ of the dispersion relation against c^2K^2 for $|c| < c_\star$ and $|c| > c_\star$. Solid black circles indicate intersections of c^2K^2 and $\widetilde{\lambda}_-(K)$ at K = 0 for all c and of c^2K^2 and $\widetilde{\lambda}_+(K)$ only at $K = \pm\omega_c$ when $|c| > c_\star$. Solid red circles indicate potential intersections of c^2K^2 and $\widetilde{\lambda}_+(K)$ for $K \neq 0$ when $|c| < c_\star$. While not graphed, c^2K^2 and $\widetilde{\lambda}_+(K)$ could also have intersections in addition to $K = \pm\omega_c$ when $|c| < c_\star$.*

'bifurcation from a simple eigenvalue' argument but now take into account the presence of the two-dimensional cokernel of $\mathcal{L}_c[\omega_c]$.

We will need a good understanding of the kernel and cokernel of $\mathcal{L}_c[\omega_c]$, and so we carefully compute the following in Appendix A.1.

Corollary 2.2. *The kernels of both*

$$\mathcal{L}_c[\omega_c]\colon H^2_{\text{per}}(\mathbb{R}^2) \to L^2_{\text{per}}(\mathbb{R}^2) \qquad \textit{and} \qquad \mathcal{L}_c[\omega_c]^*\colon L^2_{\text{per}}(\mathbb{R}^2) \to H^2_{\text{per}}(\mathbb{R}^2)$$

are spanned by the orthonormal vectors $\boldsymbol{\nu}_0$, $\boldsymbol{\nu}_1^c$ and $\boldsymbol{\nu}_2^c$ defined by

$$\boldsymbol{\nu}_0 := \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \tag{2.11}$$

$$\boldsymbol{\nu}_1^c(x) := \frac{e^{-ix}}{\mathrm{N}_c}\begin{pmatrix}e^{-i\omega_c} + \kappa e^{i\omega_c}\\ 1 + \kappa - c^2\omega_c^2\end{pmatrix} + \frac{e^{ix}}{\mathrm{N}_c}\begin{pmatrix}e^{i\omega_c} + \kappa e^{-i\omega_c}\\ 1 + \kappa - c^2\omega_c^2\end{pmatrix} \tag{2.12}$$

and

$$\boldsymbol{\nu}_2^c(x) := \frac{e^{-ix}}{\mathrm{N}_c}\begin{pmatrix}i(e^{-i\omega_c} + \kappa e^{i\omega_c})\\ i(1 + \kappa - c^2\omega_c^2)\end{pmatrix} + \frac{e^{ix}}{\mathrm{N}_c}\begin{pmatrix}-i(e^{i\omega_c} + \kappa e^{-i\omega_c})\\ -i(1 + \kappa - c^2\omega_c^2)\end{pmatrix}, \tag{2.13}$$

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where

$$\mathrm{N}_c := \sqrt{2}\left(\left[(1-\kappa)^2 + 4\kappa\cos(\omega_c)\right]^2 + \left[\frac{(1+\kappa)(1-w)+\widetilde{\varrho}(\omega_c)}{2}\right]^2\right)^{1/2}. \tag{2.14}$$

The eigenfunctions $\boldsymbol{\nu}_1^c$ and $\boldsymbol{\nu}_2^c$ satisfy the derivative identities

$$\partial_x \boldsymbol{\nu}_1^c = -\boldsymbol{\nu}_2^c \qquad \textit{and} \qquad \partial_x \boldsymbol{\nu}_2^c = \boldsymbol{\nu}_1^c \tag{2.15}$$

and the shift identity

$$\boldsymbol{\nu}_2^c = S^{-\pi/2}\boldsymbol{\nu}_1^c. \tag{2.16}$$

A function $\boldsymbol{\phi} \in L^2_{\mathrm{per}}(\mathbb{R}^2)$ satisfies

$$\langle \boldsymbol{\phi}, \boldsymbol{\nu}_1^c\rangle = \langle \boldsymbol{\phi}, \boldsymbol{\nu}_2^c\rangle = 0 \qquad \textit{if and only if} \qquad \widehat{\boldsymbol{\phi}}(1)\cdot\widehat{\boldsymbol{\nu}_1^c}(1) = 0. \tag{2.17}$$

Thus the kernel and the cokernel of the linearization of the travelling wave problem (1.8) are ostensibly three-dimensional. Translation invariance (Corollary 3.3) will allow us to rule out $\boldsymbol{\nu}_0$ from both kernel and cokernel, and so we are down to two dimensions in each. We can and will simplify the kernel further: there are $\mathbf{v}_1^c$, $\mathbf{v}_2^c \in \mathbb{C}^2$ such that any function $\mathbf{f}$ in the span of $\boldsymbol{\nu}_1^c$ and $\boldsymbol{\nu}_2^c$ can be written in the form $\mathbf{f}(x) = \sin(x)\mathbf{v}_1^c + \cos(x)\mathbf{v}_2^c$. In the case of a mass or spring dimer, symmetry effectively causes the problem (1.8) to preserve function parity, and so we could consider the kernel as one-dimensional and spanned by either $\sin(\cdot)\mathbf{v}_1^c$ or $\cos(\cdot)\mathbf{v}_2^c$. We discuss this in much more precise detail in Section 5.

Without symmetry, we could still use trigonometric addition formulas to rewrite

$$\sin(x)\mathbf{v}_1^c + \cos(x)\mathbf{v}_2^c = a\sin(x+\theta)\mathbf{w}_1^c$$

for some a, $\theta \in \mathbb{R}$ and $\mathbf{w}_1^c \in \mathbb{C}^2$. Then shift invariance of (1.8) allows us to consider the kernel as one-dimensional and spanned by $\sin(\cdot)\mathbf{w}_1^c$. We discuss this in much more precise detail in Lemma 3.8, where we employ those trigonometric identities to do this rewriting. For this reason, it suffices to assume that any solution $\boldsymbol{\phi}$ to the travelling wave problem $\boldsymbol{\Phi}_c(\boldsymbol{\phi},\omega) = 0$ has the form $\boldsymbol{\phi} = a\sin(x)\mathbf{w}_1^c + \boldsymbol{\psi}$, where $\widehat{\boldsymbol{\psi}}(0) = 0$ (which encodes translation invariance) and $\langle \sin(\cdot)\mathbf{w}_1^c, \boldsymbol{\psi}\rangle_{L^2_{\mathrm{per}}} = 0$. This is effectively the structure that we select in Section 3.4. Unfortunately, none of these reductions help with the cokernel, as without symmetry (as discussed in Section 5) $\boldsymbol{\Phi}_c$ does not possess any other helpful mapping properties to winnow down the remaining two dimensions of the cokernel.

We also need to understand the interaction of the mixed partial derivative $\mathcal{L}_c'[\omega_c] := D_{\boldsymbol{\phi}\omega}\boldsymbol{\Phi}_c(0,\omega_c)$ with the eigenfunction $\boldsymbol{\nu}_1$. It follows from Appendix A.4, specifically the identity (A.6), that $\mathcal{L}_c'[\omega_c]$ is the Fourier multiplier given by

$$\widehat{\mathcal{L}_c'[\omega_c]\boldsymbol{\phi}}(k) = k\widetilde{\mathcal{L}}_c'(\omega_c k)\widehat{\boldsymbol{\phi}}(k) \tag{2.18}$$

with $\widetilde{\mathcal{L}}_c'$ as the componentwise derivative of the matrix $\widetilde{\mathcal{L}}_c$ from (2.4). We prove the following estimate in Appendix A.2. This is the direct analogue of the classical Crandall–Rabinowitz–Zeidler transversality condition [31, Eqn. (I.5.3)] for our approach.

Corollary 2.3. $\inf_{|c|>c_\star} |\langle \mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_1^c\rangle| > 0.$

Last, we will need the following estimate on $\mathcal{L}_c[\omega_c]$, proved in Appendix A.3.

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Corollary 2.4. *There is* $C > 0$ *such that the following holds for all* c *with* $|c| > c_\star$ *and all* $r \geq 0$*. If* $\mathcal{L}_c[\omega_c]\boldsymbol{\psi} = \boldsymbol{\eta}$ *for* $\boldsymbol{\psi} \in H^{r+2}_{\mathrm{per}}(\mathbb{R}^2)$ *and* $\boldsymbol{\eta} \in H^r_{\mathrm{per}}(\mathbb{R}^2)$ *with*

$$\langle \boldsymbol{\psi}, \boldsymbol{\nu}_0 \rangle = \langle \boldsymbol{\psi}, \boldsymbol{\nu}_1^c \rangle = \langle \boldsymbol{\psi}, \boldsymbol{\nu}_2^c \rangle = 0 \qquad \text{and} \qquad \langle \boldsymbol{\eta}, \boldsymbol{\nu}_0 \rangle = \langle \boldsymbol{\eta}, \boldsymbol{\nu}_1^c \rangle = \langle \boldsymbol{\eta}, \boldsymbol{\nu}_2^c \rangle = 0,$$

then

$$\|\boldsymbol{\psi}\|_{H^{r+2}_{\mathrm{per}}} \leq C \|\boldsymbol{\eta}\|_{H^r_{\mathrm{per}}}.$$

3. The gradient formulation

3.1. The gradient structure of the travelling wave problem

We rewrite the travelling wave operator $\boldsymbol{\Phi}_c$ from (1.9) as the L^2_{per}-gradient of a certain 'kinetic + potential energy' functional on $H^2_{\mathrm{per}}(\mathbb{R}^2)$. This formulation yields transparent proofs of certain properties of $\boldsymbol{\Phi}_c$ from shift invariance, and from these properties follow our first existence proof in Sections 3.3 and 3.4.

Firstly, we need some new notation; all of the consequences below of this notation are straightforward calculations, which we omit. For $\omega \in \mathbb{R}$, put

$$\Delta_+(\omega) := \begin{bmatrix} -1 & S^{\omega} \\ 1 & -S^{-\omega} \end{bmatrix} \qquad \text{and} \qquad \Delta_-(\omega) = \begin{bmatrix} 1 & -1 \\ -S^{-\omega} & S^{\omega} \end{bmatrix}. \tag{3.1}$$

We then have the adjoint relationship

$$\langle \Delta_+(\omega)\boldsymbol{\phi}, \boldsymbol{\eta} \rangle = -\langle \boldsymbol{\phi}, \Delta_-(\omega)\boldsymbol{\eta} \rangle \tag{3.2}$$

for any $\boldsymbol{\phi}, \boldsymbol{\eta} \in L^2_{\mathrm{per}}(\mathbb{R}^2)$.

Next, let

$$\boldsymbol{\mathcal{V}}(\mathbf{p}) := \begin{pmatrix} \mathcal{V}_1(p_1) \\ \mathcal{V}_2(p_2) \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\mathcal{V}}'(\mathbf{p}) := \begin{pmatrix} \mathcal{V}_1'(p_1) \\ \mathcal{V}_2'(p_2) \end{pmatrix},$$

where $\mathcal{V}_1$ and $\mathcal{V}_2$ are the spring potentials from (1.1), and $\mathbf{p} = (p_1, p_2) \in L^2_{\mathrm{per}}(\mathbb{R}^2)$. For $\mathbf{v} = (v_1, v_2)$, $\mathbf{w} = (v_1, v_2) \in \mathbb{R}^2$, define componentwise multiplication as

$$\mathbf{v}.\mathbf{w} := \begin{pmatrix} v_1 v_1 \\ v_2 v_2 \end{pmatrix}.$$

We then have the derivative formula

$$D_{\mathbf{p}}\boldsymbol{\mathcal{V}}(\mathbf{p})\dot{\mathbf{p}} = \boldsymbol{\mathcal{V}}'(\mathbf{p}).\dot{\mathbf{p}} \tag{3.3}$$

for any $\dot{\mathbf{p}} \in L^2_{\mathrm{per}}(\mathbb{R}^2)$. Define $\mathbf{1}(x) := (1, 1)$; then since $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ have real entries, the useful identity

$$\langle \mathbf{v}.\mathbf{w}, \mathbf{1} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle \tag{3.4}$$

is true.

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Last, we have

$$\Delta_-(\omega)\boldsymbol{\mathcal{V}}'(\Delta_+(\omega)\boldsymbol{\phi}) = \begin{pmatrix} \mathcal{V}_1'(S^{\omega}\phi_2 - \phi_1) - \mathcal{V}_2'(\phi_1 - S^{-\omega}\phi_2) \\ \mathcal{V}_2'(S^{\omega}\phi_1 - \phi_2) - \mathcal{V}_1'(\phi_2 - S^{-\omega}\phi_1) \end{pmatrix}.$$

Comparing this to the second term in $\mathbf{\Phi}_c$ from (1.9), we conclude

$$\mathbf{\Phi}_c(\boldsymbol{\phi}, \omega) = c^2\omega^2 M\boldsymbol{\phi}'' - \Delta_-(\omega)\boldsymbol{\mathcal{V}}'(\Delta_+(\omega)\boldsymbol{\phi}). \tag{3.5}$$

This version of $\mathbf{\Phi}_c$ allows us to recognize it as a gradient; similar calculations for the monatomic lattice appear in [36, Prop. 3.2] and for mass dimers with the FPUT β-model in [37, Lem. 3.1].

Theorem 3.1. *Let $c \in \mathbb{R}$. Define*

$$\mathcal{T}\colon H^2_{\text{per}}(\mathbb{R}^2) \times \mathbb{R} \to \mathbb{R}\colon (\boldsymbol{\phi}, \omega) \mapsto \frac{\omega^2}{2}\langle M\boldsymbol{\phi}'', \boldsymbol{\phi}\rangle \tag{3.6}$$

and, with $\mathbf{1}(x) := (1, 1)$,

$$\mathcal{P}\colon H^2_{\text{per}}(\mathbb{R}^2) \times \mathbb{R} \to \mathbb{R}\colon (\boldsymbol{\phi}, \omega) \mapsto \langle \boldsymbol{\mathcal{V}}(\Delta_+(\omega)\boldsymbol{\phi}), \mathbf{1}\rangle \tag{3.7}$$

Put

$$\mathcal{G}_c := c^2\mathcal{T} + \mathcal{P}. \tag{3.8}$$

Then

$$\mathbf{\Phi}_c(\boldsymbol{\phi}, \omega) = \nabla\mathcal{G}_c(\boldsymbol{\phi}, \omega)$$

in the sense that

$$D_{\boldsymbol{\phi}}\mathcal{G}_c(\boldsymbol{\phi}, \omega)\boldsymbol{\eta} = \langle \mathbf{\Phi}_c(\boldsymbol{\phi}, \omega), \boldsymbol{\eta}\rangle \tag{3.9}$$

for all $\boldsymbol{\phi}$, $\boldsymbol{\eta} \in H^2_{\text{per}}(\mathbb{R}^2)$ and $\omega \in \mathbb{R}$.

Proof. The proof is just a careful calculation using the definition of the derivative and the inner product $\langle\cdot,\cdot\rangle$ and the various identities stated above. More precisely, we compute the following.

Firstly, for $\boldsymbol{\phi}$, $\boldsymbol{\eta} \in H^2_{\text{per}}(\mathbb{R}^2)$ and ω, $h \in \mathbb{R}$, we use the definition of $\mathcal{T}$ in (3.6) to compute

$$\mathcal{T}(\boldsymbol{\phi} + h\boldsymbol{\eta}, \omega) - \mathcal{T}(\boldsymbol{\phi}, \omega) = h\langle \omega^2 M\boldsymbol{\phi}'', \boldsymbol{\eta}\rangle + h^2\frac{\omega^2\langle M\boldsymbol{\eta}'', \boldsymbol{\eta}\rangle}{2}.$$

This uses two applications of the integration by parts identity (A.3) to compute $\langle\boldsymbol{\phi}'', \boldsymbol{\eta}\rangle = \langle\boldsymbol{\phi}, \boldsymbol{\eta}''\rangle$, the symmetry of M, and the assumption that $\boldsymbol{\phi}$ and $\boldsymbol{\eta}$ are $\mathbb{R}^2$-valued. It follows that

$$D_{\boldsymbol{\phi}}\mathcal{T}(\boldsymbol{\phi}, \omega)\boldsymbol{\eta} = \langle \omega^2 M\boldsymbol{\phi}'', \boldsymbol{\eta}\rangle.$$

Next, we use the definition of $\mathcal{P}$ in (3.7) to compute

$$\begin{aligned} D_{\boldsymbol{\phi}}\mathcal{P}(\boldsymbol{\phi}, \omega)\boldsymbol{\eta} &= \langle D_{\boldsymbol{\phi}}\boldsymbol{\mathcal{V}}(\Delta_+(\omega)\boldsymbol{\phi})\Delta_+(\omega)\boldsymbol{\eta}, \mathbf{1}\rangle \text{ by the chain rule} \\ &= \langle \boldsymbol{\mathcal{V}}'(\Delta_+(\omega)\boldsymbol{\phi}).\Delta_+(\omega)\boldsymbol{\eta}, \mathbf{1}\rangle \text{ by (3.3)} \\ &= \langle \boldsymbol{\mathcal{V}}'(\Delta_+(\omega)\boldsymbol{\phi}, \Delta_+(\omega)\boldsymbol{\eta}\rangle \text{ by (3.4)} \\ &= -\langle \Delta_-(\omega)\boldsymbol{\mathcal{V}}'(\Delta_+(\omega)\boldsymbol{\phi}), \boldsymbol{\eta}\rangle \text{ by (3.2)}. \end{aligned}$$

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All together, we have

$$D_{\boldsymbol{\phi}}\mathcal{G}_c(\boldsymbol{\phi},\omega)\boldsymbol{\eta} = D_{\boldsymbol{\phi}}\mathcal{T}(\boldsymbol{\phi},\omega)\boldsymbol{\eta} + D_{\boldsymbol{\phi}}\mathcal{P}(\boldsymbol{\phi},\omega)\boldsymbol{\eta} = \langle c^2\omega^2 M\boldsymbol{\phi}'',\boldsymbol{\eta}\rangle - \langle \Delta_-(\omega)\boldsymbol{\mathcal{V}}'(\Delta_+(\omega)\boldsymbol{\phi}),\boldsymbol{\eta}\rangle.$$

By our rewritten formula for $\mathbf{\Phi}_c$ in (3.5), this proves (3.9). □

We collect two families of properties of $\mathcal{G}_c$ and $\mathbf{\Phi}_c$. The proofs for $\mathcal{G}_c$ are easy consequences of its definition in Theorem 3.1, while those for $\mathbf{\Phi}_c$ are also straightforward and could in fact be done (somewhat more laboriously) using just the definition of $\mathbf{\Phi}_c$ in (1.9). However, we present proofs for $\mathbf{\Phi}_c$ here as a consequence of the gradient formulation to emphasize the utility and efficiency of this formulation.

Corollary 3.2 (Shift invariance). *The following hold for all* $\boldsymbol{\phi} \in H^2_{\mathrm{per}}(\mathbb{R}^2)$ *and* $\omega \in \mathbb{R}$.

(i) *The functional* $\mathcal{G}_c$ *is shift-invariant:*

$$\mathcal{G}_c(S^\theta\boldsymbol{\phi},\omega) = \mathcal{G}_c(\boldsymbol{\phi},\omega) \tag{3.10}$$

for all $\theta \in \mathbb{R}$.

(ii) *The operator* $\mathbf{\Phi}_c$ *is also shift-invariant:*

$$\mathbf{\Phi}_c(S^\theta\boldsymbol{\phi},\omega) = S^\theta\mathbf{\Phi}_c(\boldsymbol{\phi},\omega) \tag{3.11}$$

for all $\theta \in \mathbb{R}$.

(iii) *The operator* $\mathbf{\Phi}_c$ *has the 'derivative orthogonality property'*

$$\langle \mathbf{\Phi}_c(\boldsymbol{\phi},\omega),\boldsymbol{\phi}'\rangle = 0. \tag{3.12}$$

Proof.

(i) We use the identities

$$\langle \boldsymbol{\phi}'',\boldsymbol{\nu}_0\rangle = 0 \qquad \text{and} \qquad \Delta_+(\omega)\boldsymbol{\nu}_0 = 0$$

to obtain $\mathcal{T}(\boldsymbol{\phi}+\alpha\boldsymbol{\nu}_0,\omega) = \mathcal{T}(\boldsymbol{\phi},\omega)$ and $\mathcal{P}(\boldsymbol{\phi}+\alpha\boldsymbol{\nu}_0,\omega) = \mathcal{P}(\boldsymbol{\phi},\omega)$, respectively. Since $\mathcal{G}_c = c^2\mathcal{T}+\mathcal{P}$, the identity (3.13) follows.

(ii) The chain rule and the identity (3.10) imply

$$D_{\boldsymbol{\phi}}\mathcal{G}_c(\boldsymbol{\phi},\omega)\boldsymbol{\eta} = D_{\boldsymbol{\phi}}\mathcal{G}_c(S^\theta\boldsymbol{\phi},\omega)S^\theta\boldsymbol{\eta}$$

for all $\boldsymbol{\eta} \in H^2_{\mathrm{per}}(\mathbb{R}^2)$. At the level of gradients, this reads

$$\langle \mathbf{\Phi}_c(\boldsymbol{\phi},\omega),\boldsymbol{\eta}\rangle = \langle \mathbf{\Phi}_c(S^\theta\boldsymbol{\phi},\omega),S^\theta\boldsymbol{\eta}\rangle.$$

On the right, we use the adjoint relation (A.2) for shifts to rewrite

$$\langle \mathbf{\Phi}_c(S^\theta\boldsymbol{\phi},\omega),S^\theta\boldsymbol{\eta}\rangle = \langle S^{-\theta}\mathbf{\Phi}_c(S^\theta\boldsymbol{\phi},\omega),\boldsymbol{\eta}\rangle.$$

It follows that

$$\langle \mathbf{\Phi}_c(\boldsymbol{\phi},\omega),\boldsymbol{\eta}\rangle = \langle S^{-\theta}\mathbf{\Phi}_c(S^\theta\boldsymbol{\phi},\omega),\boldsymbol{\eta}\rangle$$

for all $\boldsymbol{\eta} \in H^2_{\mathrm{per}}(\mathbb{R}^2)$, and so $\mathbf{\Phi}_c(\boldsymbol{\phi},\omega) = S^{-\theta}\mathbf{\Phi}_c(S^\theta\boldsymbol{\phi},\omega)$. Applying S^θ to both sides yields (3.11).

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(iii) Now we differentiate the identity $\mathcal{G}_c(S^\theta\boldsymbol{\phi},\omega) = \mathcal{G}_c(\boldsymbol{\phi},\omega)$ from (3.10) with respect to θ and evaluate the result at $\theta=0$. This yields

$$D_{\boldsymbol{\phi}}\mathcal{G}_c(S^0\boldsymbol{\phi},\omega)\left(\frac{\partial}{\partial\theta}[S^\theta\boldsymbol{\phi}]\bigg|_{\theta=0}\right) = 0.$$

Differentiating the shift operator yields the identity

$$\frac{\partial}{\partial\theta}[S^\theta\boldsymbol{\phi}]\bigg|_{\theta=0} = \boldsymbol{\phi}',$$

which is valid in the L^2_{per}-norm since $\boldsymbol{\phi}\in H^2_{\text{per}}$.

Thus

$$D_{\boldsymbol{\phi}}\mathcal{G}_c(\boldsymbol{\phi},\omega)\boldsymbol{\phi}' = 0,$$

and in the language of the gradient formulation, this says

$$\langle\boldsymbol{\Phi}_c(\boldsymbol{\phi},\omega),\boldsymbol{\phi}'\rangle = 0.$$

□

Corollary 3.3 (Translation invariance). *The following hold for all* $\boldsymbol{\phi}\in H^2_{\text{per}}(\mathbb{R}^2)$ *and* $\omega\in\mathbb{R}$.

(i) *The functional* $\mathcal{G}_c$ *is translation-invariant in the sense that*

$$\mathcal{G}_c(\boldsymbol{\phi}+\alpha\boldsymbol{\nu}_0,\omega) = \mathcal{G}_c(\boldsymbol{\phi},\omega) \tag{3.13}$$

for all $\alpha\in\mathbb{R}$, *where* $\boldsymbol{\nu}_0$ *is defined in (2.11).*

(ii) *The operator* $\boldsymbol{\Phi}_c$ *is also translation-invariant:*

$$\boldsymbol{\Phi}_c(\boldsymbol{\phi}+\alpha\boldsymbol{\nu}_0,\omega) = \boldsymbol{\Phi}_c(\boldsymbol{\phi},\omega) \tag{3.14}$$

for all $\alpha\in\mathbb{R}$.

(iii) *The range of* $\boldsymbol{\Phi}_c$ *is orthogonal to* $\boldsymbol{\nu}_0$*:*

$$\langle\boldsymbol{\Phi}_c(\boldsymbol{\phi},\omega),\boldsymbol{\nu}_0\rangle = 0. \tag{3.15}$$

Proof.

(i) This follows from the integral structure of $\mathcal{G}_c = c^2\mathcal{T}+\mathcal{P}$ from Theorem 3.1, the 2π-periodicity of $\boldsymbol{\phi}$ and the identity

$$\int_{-\pi}^{\pi} f(x+\theta)\,dx = \int_{-\pi}^{\pi} f(x)\,dx,$$

which is valid for all $\theta\in\mathbb{R}$ and all integrable, 2π-periodic $f\colon[-\pi,\pi]\to\mathbb{C}$.

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(ii) We differentiate the identity $\mathcal{G}_c(\boldsymbol{\phi} + \alpha \boldsymbol{\nu}_0, \omega) = \mathcal{G}_c(\boldsymbol{\phi}, \omega)$ from (3.13) with respect to $\boldsymbol{\phi}$ to find

$$D_{\boldsymbol{\phi}} \mathcal{G}_c(\boldsymbol{\phi} + \alpha \boldsymbol{\nu}_0, \omega)\boldsymbol{\eta} = D_{\boldsymbol{\phi}} \mathcal{G}_c(\boldsymbol{\phi}, \omega)\boldsymbol{\eta}$$

for all $\boldsymbol{\eta} \in H^2_{\text{per}}(\mathbb{R}^2)$ and thus

$$\langle \boldsymbol{\Phi}_c(\boldsymbol{\phi} + \alpha \boldsymbol{\nu}_0, \omega), \boldsymbol{\eta} \rangle = \langle \boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega), \boldsymbol{\eta} \rangle.$$

Since this holds for all $\boldsymbol{\eta}$, we obtain (3.14).

(iii) Now we differentiate the identity $\mathcal{G}_c(\boldsymbol{\phi} + \alpha \boldsymbol{\nu}_0, \omega) = \mathcal{G}_c(\boldsymbol{\phi}, \omega)$ with respect to α and evaluate the result at $\alpha = 0$. This yields

$$0 = D_{\boldsymbol{\phi}} \mathcal{G}_c(\boldsymbol{\phi} + (0 \cdot \boldsymbol{\nu}_0), \omega) \left(\frac{\partial}{\partial \alpha} [\boldsymbol{\phi} + \alpha \boldsymbol{\nu}_0] \bigg|_{\alpha=0} \right) = \langle \boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega), \boldsymbol{\nu}_0 \rangle.$$

□

An immediate consequence of the translation invariance of $\boldsymbol{\Phi}_c$ from (3.14) is that solutions $\boldsymbol{\phi}$ to $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) = 0$ are only unique up to translation by $\boldsymbol{\nu}_0$, as claimed in Theorem 1.1.

Remark 3.4. The derivative orthogonality property (3.12) of $\boldsymbol{\Phi}_c$ is the key to resolving the overdetermined periodic problem. While this property follows quickly from the shift invariance of $\mathcal{G}_c$, as proved above, it is not quite as easy to prove directly from the definition of $\boldsymbol{\Phi}_c$ as are all the other consequences of shift and translation invariance. We discuss that direction of proof further in Lemma 4.1. A similar derivative orthogonality property, deployed in somewhat different language, enabled Wright and Scheel [40, Sec. 4, p. 548] to complete a Lyapunov–Schmidt analysis in which the linearization also had a two-dimensional kernel that, in the absence of symmetry, could not be reduced in dimension.

3.2. Function spaces and projection operators

The translation invariance identities (3.14) and (3.15) mean that we can effectively ignore the contributions of $\boldsymbol{\nu}_0$ to the problem $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) = 0$. So, we put

$$\mathcal{Y} := \left\{ \boldsymbol{\eta} \in L^2_{\text{per}}(\mathbb{R}^2) \,\middle|\, \langle \boldsymbol{\eta}, \boldsymbol{\nu}_0 \rangle = 0 \right\}, \tag{3.16}$$

$$\mathcal{X} := H^2_{\text{per}}(\mathbb{R}^2) \cap \mathcal{Y}, \tag{3.17}$$

and

$$\mathcal{Z}_c := \text{span}(\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_2^c). \tag{3.18}$$

It follows that $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) \in \mathcal{Y}$ for all $\boldsymbol{\phi} \in \mathcal{X}$ and $\omega \in \mathbb{R}$, and also

$$\mathcal{Z}_c = \ker(\mathcal{L}_c[\omega_c]) \cap \mathcal{X} = \ker(\mathcal{L}_c[\omega_c]^*) \cap \mathcal{Y}.$$

Define

$$\Pi_c \colon \mathcal{Y} \to \mathcal{Z}_c \colon \boldsymbol{\phi} \mapsto \langle \boldsymbol{\phi}, \boldsymbol{\nu}_1^c \rangle \boldsymbol{\nu}_1^c + \langle \boldsymbol{\phi}, \boldsymbol{\nu}_2^c \rangle \boldsymbol{\nu}_2^c. \tag{3.19}$$

Since $\boldsymbol{\nu}_1^c$ and $\boldsymbol{\nu}_2^c$ are orthogonal, per Corollary 2.2, the operator Π_c is the orthogonal projection of $\mathcal{Y}$ (and $\mathcal{X}$) onto $\mathcal{Z}_c$. In particular,

$$\langle \Pi_c \boldsymbol{\phi}, \boldsymbol{\eta} \rangle = \langle \boldsymbol{\phi}, \Pi_c \boldsymbol{\eta} \rangle \tag{3.20}$$

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for all $\boldsymbol{\phi}, \boldsymbol{\psi} \in \mathcal{Y}$.

It turns out to be quite useful for us that the projection Π_c and the first derivative ∂_x commute.

Lemma 3.5. *Let $\boldsymbol{\phi} \in H^1_{\text{per}}(\mathbb{R}^2)$. Then*

$$\Pi_c \partial_x \boldsymbol{\phi} = \partial_x \Pi_c \boldsymbol{\phi}. \tag{3.21}$$

Proof. We use the integration by parts identity $\langle \boldsymbol{\phi}', \boldsymbol{\eta} \rangle = -\langle \boldsymbol{\phi}, \boldsymbol{\eta}' \rangle$ from (A.3) and the derivative identities (2.15) to compute

$$\begin{aligned}
\Pi_c \partial_x \boldsymbol{\phi} &= \langle \boldsymbol{\phi}', \boldsymbol{\nu}_1^c \rangle \boldsymbol{\nu}_1^c + \langle \boldsymbol{\phi}', \boldsymbol{\nu}_2^c \rangle \boldsymbol{\nu}_2^c \\
&= -\langle \boldsymbol{\phi}, \partial_x \boldsymbol{\nu}_1^c \rangle \boldsymbol{\nu}_1^c - \langle \boldsymbol{\phi}, \partial_x \boldsymbol{\nu}_2^c \rangle \boldsymbol{\nu}_2^c \\
&= \langle \boldsymbol{\phi}, \boldsymbol{\nu}_2^c \rangle \boldsymbol{\nu}_1^c - \langle \boldsymbol{\phi}, \boldsymbol{\nu}_1^c \rangle \boldsymbol{\nu}_2^c \\
&= \langle \boldsymbol{\phi}, \boldsymbol{\nu}_2^c \rangle \partial_x \boldsymbol{\nu}_2^c + \langle \boldsymbol{\phi}, \boldsymbol{\nu}_1^c \rangle \partial_x \boldsymbol{\nu}_1^c \\
&= \partial_x \Pi_c \boldsymbol{\phi}.
\end{aligned}$$

□

Last, we state precisely the regularity of $\boldsymbol{\Phi}_c$ and some of its derivatives on periodic Sobolev spaces. The technical challenge here is that $\boldsymbol{\Phi}_c$ is infinitely differentiable from $H^2_{\text{per}}(\mathbb{R}^2)$ to $L^2_{\text{per}}(\mathbb{R}^2)$ with respect to $\boldsymbol{\phi}$, but any order derivative with respect to $\boldsymbol{\phi}$ is only once continuously differentiable with respect to ω. This is ultimately a consequence of the limited differentiability of shift operators between periodic Sobolev spaces, as we discuss in Appendix A.4. We prove the next lemma in Appendix A.4.

Lemma 3.6. $\boldsymbol{\Phi}_c \in \mathcal{C}^1(H^2_{\text{per}}(\mathbb{R}^2) \times \mathbb{R}, L^2_{\text{per}}(\mathbb{R}^2))$ *and* $D_{\boldsymbol{\phi}} \boldsymbol{\Phi}_c \in \mathcal{C}^1(H^2_{\text{per}}(\mathbb{R}^2) \times \mathbb{R}, L^2_{\text{per}}(\mathbb{R}^2))$.

3.3. The Lyapunov–Schmidt decomposition: infinite-dimensional analysis

The approach here is classical and follows, for example, the proof of the Crandall–Rabinowitz–Zeidler theorem in [31, Thm. 1.5.1]. The difference appears in the following section, when we manage the two-dimensional kernel.

We use the projection operator Π_c from (3.19) to make a Lyapunov–Schmidt decomposition for our problem $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) = 0$. Firstly, with the spaces $\mathcal{X}$ and $\mathcal{Y}$ defined in (3.17) and (3.16), let

$$\mathcal{X}_c^\infty := (\mathcal{I}_\mathcal{X} - \Pi_c)(\mathcal{X}) \qquad \text{and} \qquad \mathcal{Y}_c^\infty := (\mathcal{I}_\mathcal{Y} - \Pi_c)(\mathcal{Y}), \tag{3.22}$$

where $\mathcal{I}_\mathcal{X}$ and $\mathcal{I}_\mathcal{Y}$ are the identity operators on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Consequently,

$$\mathcal{Z}_c \cap \mathcal{X}_c^\infty = \mathcal{Z}_c \cap \mathcal{Y}_c^\infty = \{0\}. \tag{3.23}$$

Next, write $\boldsymbol{\phi} = \boldsymbol{\nu} + \boldsymbol{\psi}$, where $\boldsymbol{\nu} \in \mathcal{Z}_c$ and $\boldsymbol{\psi} \in \mathcal{X}_c^\infty$. Then $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) = 0$ if and only if

$$\begin{cases} (\mathcal{I}_\mathcal{Y} - \Pi_c) \boldsymbol{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\psi}, \omega) = 0 & (3.24a) \\ \Pi_c \boldsymbol{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\psi}, \omega) = 0. & (3.24b) \end{cases} \tag{3.24}$$

We solve (3.24a) quickly with a direct application of the implicit function theorem [31, Thm. I.1.1].

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Define

$$\mathcal{F}_c^\infty \colon \mathcal{X}_c^\infty \times \mathcal{Z}_c \times \mathbb{R} \to \mathcal{Y}_c^\infty \colon (\boldsymbol{\psi}, \boldsymbol{\nu}, \omega) \mapsto (\mathcal{I}_\mathcal{Y} - \Pi_c)\boldsymbol{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\psi}, \omega). \tag{3.25}$$

Certainly $\mathcal{F}_c^\infty(0, 0, \omega) = 0$ for all ω, and we have

$$D_{\boldsymbol{\psi}}\mathcal{F}_c^\infty(0, 0, \omega_c) = (\mathcal{I}_\mathcal{Y} - \Pi_c)\,\mathcal{L}_c[\omega_c]\big|_{\mathcal{X}_c^\infty}.$$

This operator has trivial kernel in $\mathcal{X}_c^\infty$ and trivial cokernel in $\mathcal{Y}_c^\infty$ by (3.23), and so it is invertible. (More precisely, the closure of its range is the orthogonal complement of its cokernel, which is all of $\mathcal{Y}_c^\infty$. But the range is closed by the estimate in Corollary 2.4.) Since $\boldsymbol{\Phi}_c \in \mathcal{C}^1(H^2_{\text{per}}(\mathbb{R}^2), L^2_{\text{per}}(\mathbb{R}^2))$ by Lemma 3.6, with $\boldsymbol{\Phi}_c(0, \omega) = 0$ for all ω, the implicit function theorem yields δ_c, $\epsilon_c > 0$ and a map $\boldsymbol{\Psi}_c \in \mathcal{C}^1\big(\mathcal{B}_{\mathcal{Z}_c\times\mathbb{R}}((0, \omega_c); \delta_c), \mathcal{B}_{\mathcal{X}_c^\infty}(0; \epsilon_c)\big)$ such that

$$\mathcal{F}_c^\infty(\boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega), \boldsymbol{\nu}, \omega) = 0. \tag{3.26}$$

Moreover, if $\mathcal{F}_c^\infty(\boldsymbol{\psi}, \boldsymbol{\nu}, \omega) = 0$ for some $\boldsymbol{\psi} \in \mathcal{B}_{\mathcal{X}_c^\infty}(0; \epsilon_c)$ and $(\boldsymbol{\nu}, \omega) \in \mathcal{B}_{\mathcal{Z}_c\times\mathbb{R}}((0, \omega_c); \delta_c)$, then $\boldsymbol{\psi} = \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega)$. (Recall that $\mathcal{B}_\mathcal{X}(x_0; r) = \big\{x \in \mathcal{X} \;\big|\; \|x - x_0\|_\mathcal{X} < r\big\}$ for $x_0 \in \mathcal{X}$ and $r > 0$.) We pause to collect some useful properties of this map $\boldsymbol{\Psi}_c$.

Lemma 3.7. *Let $\boldsymbol{\nu} \in \mathcal{Z}_c$ and $\omega \in \mathbb{R}$ with $\|\boldsymbol{\nu}\|_{H^2_{\text{per}}} + |\omega - \omega_c| < \delta_c$. Then the following identities hold.*

(i) $\langle \boldsymbol{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega), \omega), \partial_x\boldsymbol{\nu}\rangle = 0$ *for* $(\boldsymbol{\nu}, \omega) \in \mathcal{B}_{\mathcal{Z}_c\times\mathbb{R}}((0, \omega_c); \delta_c)$.
(ii) $\boldsymbol{\Psi}_c(0, \omega) = 0$ *for* $(0, \omega) \in \mathcal{B}_{\mathcal{Z}_c\times\mathbb{R}}((0, \omega_c); \delta_c)$.
(iii) $D_{\boldsymbol{\nu}}\boldsymbol{\Psi}_c(0, \omega_c) = 0$.
(iv) $D_{\boldsymbol{\nu}}\boldsymbol{\Psi}_c \in \mathcal{C}^1(\mathcal{B}_{\mathcal{Z}_c\times\mathbb{R}}((0, \omega_c); \delta_c), \mathbf{B}(\mathcal{Z}_c, \mathcal{X}_c^\infty))$.

Proof.

(i) The derivative orthogonality property of $\boldsymbol{\Phi}_c$ from (3.12) implies

$$\langle \boldsymbol{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega), \omega), \partial_x[\boldsymbol{\nu} + \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega)]\rangle = 0, \tag{3.27}$$

and by (3.26), we have

$$(\mathcal{I}_\mathcal{Y} - \Pi_c)\boldsymbol{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega), \omega) = 0. \tag{3.28}$$

Then we compute

$$\begin{aligned} 0 &= \langle \Pi_c\boldsymbol{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega), \omega), \partial_x[\boldsymbol{\nu} + \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega)]\rangle \text{ using (3.28) in (3.27)} \\ &= \langle \boldsymbol{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega), \omega), \Pi_c\partial_x[\boldsymbol{\nu} + \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega)]\rangle \text{ by (3.20)} \\ &= \langle \boldsymbol{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega), \omega), \partial_x\Pi_c[\boldsymbol{\nu} + \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega)]\rangle \text{ since } \partial_x \text{ and } \Pi_c \text{ commute} \\ &= \langle \boldsymbol{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega), \omega), \partial_x\boldsymbol{\nu}\rangle \text{ since } \Pi_c\boldsymbol{\nu} = \boldsymbol{\nu} \text{ and } \Pi_c\boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega) = 0. \end{aligned} \tag{3.29}$$

(ii) By definition of $\mathcal{F}_c^\infty$ in (3.25), we have

$$\mathcal{F}_c(0, 0, \omega) = (\mathcal{I}_\mathcal{Y} - \Pi_c)\boldsymbol{\Phi}_c(0, \omega) = 0$$

for all ω. By the uniqueness property of $\boldsymbol{\Psi}_c$, we have $\boldsymbol{\Psi}(0, \omega) = 0$ for all ω.

(iii) We differentiate (3.28) with respect to $\boldsymbol{\nu} \in \mathcal{Z}_c$ to find the operator-valued identity

$$(\mathcal{I}_{\mathcal{Y}} - \Pi_c) D_{\boldsymbol{\phi}} \boldsymbol{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega), \omega) \big(\mathcal{I}_{\mathcal{Z}_c} + D_{\boldsymbol{\nu}} \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega) \big) = 0. \tag{3.30}$$

Here $\mathcal{I}_{\mathcal{Z}_c}$ is the identity operator on $\mathcal{Z}_c = \ker(\mathcal{L}_c[\omega_c])$. Taking $\boldsymbol{\nu} = 0$ and $\omega = \omega_c$ collapses (3.30) to

$$(\mathcal{I}_{\mathcal{Y}} - \Pi_c) \mathcal{L}_c[\omega_c] \big(\mathcal{I}_{\mathcal{Z}_c} + D_{\boldsymbol{\nu}} \boldsymbol{\Psi}_c(0, \omega_c) \big) = 0,$$

recalling $\mathcal{L}_c[\omega_c] = D_{\boldsymbol{\phi}} \boldsymbol{\Phi}_c(0, \omega_c)$ from (2.1). Since $\mathcal{L}_c[\omega_c]\big|_{\mathcal{Z}_c} = 0$, this further reduces to

$$\mathcal{L}_c[\omega_c] D_{\boldsymbol{\nu}} \boldsymbol{\Psi}_c(0, \omega_c) - \Pi_c \mathcal{L}_c[\omega_c] D_{\boldsymbol{\nu}} \boldsymbol{\Psi}_c(0, \omega_c) = 0.$$

Because $\ker(\mathcal{L}_c[\omega_c]) = \ker(\mathcal{L}_c[\omega_c]^*)$, it follows from the definition of Π_c in (3.19) that $\Pi_c \mathcal{L}_c[\omega_c] = 0$. Thus

$$\mathcal{L}_c[\omega_c] D_{\boldsymbol{\nu}} \boldsymbol{\Psi}_c(0, \omega_c) = 0,$$

and so the range of $D_{\boldsymbol{\nu}} \boldsymbol{\Psi}_c(0, \omega_c)$ is contained in $\ker(\mathcal{L}_c[\omega_c]) = \mathcal{Z}_c$. But the range of $D_{\boldsymbol{\nu}} \boldsymbol{\Psi}_c(0, \omega_c)$ is also contained in $\mathcal{Y}_c^\infty$, and $\mathcal{Y}_c^\infty \cap \mathcal{Z}_c = \{0\}$ by (3.23). Thus the range of $D_{\boldsymbol{\nu}} \boldsymbol{\Psi}_c(0, \omega_c)$ is trivial.

(iv) This follows from the implicit function theorem, which guarantees that $\boldsymbol{\Psi}_c$ is as regular as $\boldsymbol{\Phi}_c$. Since $D_{\boldsymbol{\phi}} \boldsymbol{\Phi}_c \in \mathcal{C}^1(H^2_{\text{per}}(\mathbb{R}^2), L^2_{\text{per}}(\mathbb{R}^2))$ by Lemma 3.6, $D_{\boldsymbol{\nu}} \boldsymbol{\Psi}_c$ inherits this regularity on $\mathcal{B}_{\mathcal{Z}_c \times \mathbb{R}}((0, \omega_c); \delta_c)$.

□

3.4. The Lyapunov–Schmidt decomposition: finite-dimensional analysis

Now we solve the second equation (3.24) in the Lyapunov–Schmidt decomposition with $\boldsymbol{\psi} = \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega)$. This amounts to solving the pair of equations

$$\begin{cases} \langle \boldsymbol{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega), \omega), \boldsymbol{\nu}_1^c \rangle = 0 \\ \langle \boldsymbol{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega), \omega), \boldsymbol{\nu}_2^c \rangle = 0, \end{cases} \tag{3.31}$$

where $\boldsymbol{\nu}$ is a linear combination of the two linearly independent eigenfunctions $\boldsymbol{\nu}_1^c$ and $\boldsymbol{\nu}_2^c$. The apparent quandary is that we want solutions parametrized in amplitude, so formally this suggests $\boldsymbol{\nu} + \boldsymbol{\Psi}_c(\boldsymbol{\nu}, \omega) = \mathcal{O}(a)$. This leads to our taking $\boldsymbol{\nu} = a\boldsymbol{\nu}_1^c$ below, which may appear to remove a degree of freedom from the ansatz. In turn, this could appear to be problematic, given that we have two equations to solve above in (3.31). Ostensibly, we could have stayed with $\boldsymbol{\nu}$ as a combination of $\boldsymbol{\nu}_1^c$ and $\boldsymbol{\nu}_2^c$. None of this, however, is a problem, and we discuss at length in Section 3.5 why.

With the choice of

$$\boldsymbol{\nu} = a\boldsymbol{\nu}_1^c \tag{3.32}$$

for $a \in \mathbb{R}$ sufficiently small, we convert (3.31) to

$$\begin{cases} \langle \boldsymbol{\Phi}_c(a\boldsymbol{\nu}_1^c + \boldsymbol{\Psi}_c(a\boldsymbol{\nu}_1^c, \omega), \omega), \boldsymbol{\nu}_1^c \rangle = 0 & (3.33a) \\ \langle \boldsymbol{\Phi}_c(a\boldsymbol{\nu}_1^c + \boldsymbol{\Psi}_c(a\boldsymbol{\nu}_1^c, \omega), \omega), \boldsymbol{\nu}_2^c \rangle = 0. & (3.33b) \end{cases} \tag{3.33}$$

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We claim that (3.33b) always holds. Indeed, for $a = 0$, it is trivially true, since $\mathbf{\Phi}_c(0, \omega) = 0$, while for $a \neq 0$ we have

$$\begin{aligned}\langle \mathbf{\Phi}_c(a\boldsymbol{\nu}_1^c + \mathbf{\Psi}_c(a\boldsymbol{\nu}_1^c, \omega), \omega), \boldsymbol{\nu}_2^c\rangle &= a^{-1}\langle \mathbf{\Phi}_c(a\boldsymbol{\nu}_1^c + \mathbf{\Psi}_c(a\boldsymbol{\nu}_1^c, \omega), \omega), a\boldsymbol{\nu}_2^c\rangle \\ &= -a^{-1}\langle \mathbf{\Phi}_c(a\boldsymbol{\nu}_1^c + \mathbf{\Psi}_c(a\boldsymbol{\nu}_1^c, \omega), \omega), \partial_x[a\boldsymbol{\nu}_1^c]\rangle \text{ by (2.15)} \\ &= 0 \text{ by part (i) of Lemma 3.7.}\end{aligned}$$

We emphasize that our success here traces back to the derivative orthogonality property (3.12).

We conclude by solving (3.33a) with another application of the implicit function theorem, and this, again, is effectively the remainder of the proof of the Crandall–Rabinowitz–Zeidler theorem [31, Thm. 1.5.1]. Define

$$\mathcal{F}_c^0 \colon \mathcal{B}_{\mathbb{R}^2}((\omega_c, 0); \delta_c/2) \to \mathbb{R} \colon (\omega, a) \mapsto \langle \mathbf{\Phi}_c(a\boldsymbol{\nu}_1^c + \mathbf{\Psi}_c(a\boldsymbol{\nu}_1^c, \omega), \omega), \boldsymbol{\nu}_1^c\rangle.$$

The threshold δ_c arose from the infinite-dimensional implicit function theorem argument in Section 3.3.

Since $\mathcal{F}_c^0(\omega, 0) = 0$ for all ω, we have

$$\mathcal{F}_c^0(\omega, a) = a\mathcal{H}_c(\omega, a), \qquad \mathcal{H}_c(\omega, a) := \int_0^1 D_a\mathcal{F}_c^0(\omega, a\alpha)\, d\alpha.$$

It therefore suffices to solve $\mathcal{H}_c(\omega, a) = 0$ by selecting ω as a function of a, and we do this by checking $\mathcal{H}_c(\omega_c, 0) = 0$ and $D_\omega\mathcal{H}_c(\omega_c, 0) \neq 0$.

Toward this end, we first differentiate

$$D_a\mathcal{F}_c^0(\omega, a) = \langle D_{\boldsymbol{\phi}}\mathbf{\Phi}_c(a\boldsymbol{\nu}_1^c + \mathbf{\Psi}_c(a\boldsymbol{\nu}_1^c, \omega))\big(\boldsymbol{\nu}_1^c + D_{\boldsymbol{\nu}}\mathbf{\Psi}_c(a\boldsymbol{\nu}_1^c, \omega)\boldsymbol{\nu}_1^c\big), \boldsymbol{\nu}_1^c\rangle.$$

We put $a = 0$ and use $\mathbf{\Psi}_c(0, \omega) = 0$ to find for any ω that

$$\mathcal{H}_c(\omega, 0) = \int_0^1 D_a\mathcal{F}_c^0(\omega, 0)\, d\alpha = D_a\mathcal{F}_c(\omega, 0) = \langle \mathcal{L}_c[\omega]\big(\boldsymbol{\nu}_1^c + D_{\boldsymbol{\nu}}\mathbf{\Psi}_c(0, \omega)\boldsymbol{\nu}_1^c\big), \boldsymbol{\nu}_1^c\rangle. \tag{3.34}$$

In the special case of $\omega = \omega_c$, we can use either $D_{\boldsymbol{\nu}}\mathbf{\Psi}_c(0, \omega) = 0$ from part (iii) of Lemma 3.7 or the condition $\mathcal{L}_c[\omega_c]^*\boldsymbol{\nu}_1^c = 0$ to reduce (3.34) to

$$\mathcal{H}_c(\omega_c, 0) = \langle \mathcal{L}_c[\omega_c]\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_1^c\rangle = 0.$$

Next, with ω arbitrary, we differentiate (3.34) with respect to ω and use the product rule to find

$$D_\omega\mathcal{H}_c(\omega, 0) = \langle \mathcal{L}_c'[\omega]\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_1^c\rangle + \langle \mathcal{L}_c'[\omega]D_{\boldsymbol{\nu}}\mathbf{\Psi}_c(0, \omega)\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_1^c\rangle + \langle \mathcal{L}_c[\omega]D_{\boldsymbol{\nu}\omega}\mathbf{\Psi}_c(0, \omega)\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_1^c\rangle.$$

Here we are using the shorter notation from (2.18) of $\mathcal{L}_c'[\omega] = D_{\boldsymbol{\phi}\omega}\mathbf{\Phi}_c(0, \omega)$. Taking $\omega = \omega_c$, we use $D_{\boldsymbol{\nu}}\mathbf{\Psi}_c(0, \omega_c) = 0$ to find that the second term is 0. At ω_c, the third term is 0 since $\mathcal{L}_c[\omega_c]^*\boldsymbol{\nu}_1^c = 0$. And so

$$D_\omega\mathcal{H}_c(\omega_c, 0) = \langle \mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_1^c\rangle \neq 0,$$

by Corollary 2.3.

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We are now in position to invoke the implicit function theorem once more, and we find $a_c, b_c > 0$ and a map $\Omega_c \colon (-a_c, a_c) \to \mathbb{R}$ such that

$$\mathcal{H}_c(\Omega_c(a), a) = 0$$

for $|a| < a_c$, while if $|\omega - \omega_c| < b_c$ and $|a| < a_c$ and $\mathcal{H}_c(\omega, a) = 0$, then $\omega = \Omega_c(a)$. In particular, $\Omega_c(0) = \omega_c$.

In short, taking

$$\boldsymbol{\phi}_c^a := a\boldsymbol{\nu}_1 + \boldsymbol{\Psi}_c(a\boldsymbol{\nu}_1, \Omega_c(a)) \qquad \text{and} \qquad \omega_c^a := \Omega_c(a)$$

solves our original problem $\boldsymbol{\Phi}_c(\boldsymbol{\phi}_c^a, \omega_c^a) = 0$. We can expose uniformly the 'amplitude' parameter of a in $\boldsymbol{\phi}_c^a$ by setting

$$\boldsymbol{\psi}_c(a) := \boldsymbol{\Psi}_c(a\boldsymbol{\nu}_1, \Omega_c(a)),$$

and computing

$$\boldsymbol{\psi}_c(0) = \boldsymbol{\Psi}_c(0, \omega_c) = 0, \quad \boldsymbol{\psi}_c(a) = a\int_0^1 D_a\boldsymbol{\psi}_c(a\alpha)\, d\alpha, \quad \text{and} \quad \boldsymbol{\phi} = a\left(\boldsymbol{\nu}_1 + \int_0^1 D_a\boldsymbol{\psi}_c(a\alpha)\, d\alpha\right).$$

Likewise, we can write

$$\omega_c^a = \omega_c + a\xi_c^a \qquad \text{and} \qquad \xi_c^a := \int_0^1 D_a\Omega_c(a\alpha)\, d\alpha.$$

This concludes our first proof of Theorem 1.1.

3.5. *Proof of local uniqueness up to shifts and translations*

We discuss our decision at the start of Section 3.4 to specialize the finite-dimensional component $\boldsymbol{\nu}$ to $\boldsymbol{\nu} = a\boldsymbol{\nu}_1^c$. We consider two aspects of this choice to allay any concerns about its peculiarity or restrictiveness.

Firstly, up to a shift, any solution $\boldsymbol{\phi}$ to $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) = 0$ has this form $\boldsymbol{\phi} = a\boldsymbol{\nu}_1^c + \boldsymbol{\psi}$, with $a \in \mathbb{R}$ and $\boldsymbol{\psi}$ orthogonal to $\boldsymbol{\nu}_0$, $\boldsymbol{\nu}_1^c$, and $\boldsymbol{\nu}_2^c$.

Lemma 3.8. *Let $\omega \in \mathbb{R}$. If $\boldsymbol{\phi} \in \mathcal{X}$ solves $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) = 0$, then there exist* a, $\theta \in \mathbb{R}$ *and $\boldsymbol{\psi} \in \mathcal{X}_c^\infty$ such that*

$$\boldsymbol{\phi} = S^\theta(a\boldsymbol{\nu}_1^c + \boldsymbol{\psi}). \tag{3.35}$$

In particular, $\boldsymbol{\Phi}_c(a\boldsymbol{\nu}_1^c + \boldsymbol{\psi}, \omega) = 0$, as well.

Proof. We prove the last sentence first. If a solution $\boldsymbol{\phi}$ to $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) = 0$ has the form (3.35), then the shift invariance of $\boldsymbol{\Phi}_c$ from (3.11) implies

$$\boldsymbol{\Phi}_c(a\boldsymbol{\nu}_1^c + \boldsymbol{\psi}, \omega) = S^{-\theta}\boldsymbol{\Phi}_c(S^\theta(a\boldsymbol{\nu}_1^c + \boldsymbol{\psi}), \omega) = S^{-\theta}\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) = 0.$$

It remains to prove the decomposition (3.35). We can always write

$$\boldsymbol{\phi} = a_1\boldsymbol{\nu}_1^c + a_2\boldsymbol{\nu}_2^c + \widetilde{\boldsymbol{\psi}} \tag{3.36}$$

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for some $a_1, a_2 \in \mathbb{R}$ and $\widetilde{\boldsymbol{\psi}} \in \mathcal{X}_c^\infty$. Write

$$a_1 - ia_2 = ae^{i\theta}$$

in polar coordinates, where $a, \theta \in \mathbb{R}$. (If $a_1 = a_2 = 0$, just take $a=0$ and $\boldsymbol{\psi} = \widetilde{\boldsymbol{\psi}}$.) The identities $\boldsymbol{v}_1^c(x) = 2\,\mathrm{Re}[e^{ix}\widehat{\boldsymbol{v}_1^c}(1)]$ from (2.12) and $\boldsymbol{v}_2^c = S^{-\pi/2}\boldsymbol{v}_1^c$ from (2.16) then imply

$$a_1\boldsymbol{v}_1^c(x) + a_2\boldsymbol{v}_2^c(x) = a_1\boldsymbol{v}_1^c(x) + a_2S^{-\pi/2}\boldsymbol{v}_1^c(x) = 2\,\mathrm{Re}[(a_1 - ia_2)e^{ix}\widehat{\boldsymbol{v}_1^c}(1)] = a(S^\theta\boldsymbol{v}_1^c)(x).$$

Returning to (3.36), we have

$$\boldsymbol{\phi} = S^\theta(a\boldsymbol{v}_1^c + \boldsymbol{\psi}), \qquad \boldsymbol{\psi} := S^{-\theta}\widetilde{\boldsymbol{\psi}}.$$

We conclude by checking that if $\widetilde{\boldsymbol{\psi}} \in \mathcal{X}_c^\infty$, then $\boldsymbol{\psi} = S^{-\theta}\widetilde{\boldsymbol{\psi}} \in \mathcal{X}_c^\infty$. That is, we assume

$$\langle\widetilde{\boldsymbol{\psi}}, \boldsymbol{v}_1^c\rangle = \langle\widetilde{\boldsymbol{\psi}}, \boldsymbol{v}_2^c\rangle = 0, \tag{3.37}$$

and we want to show

$$\langle S^{-\theta}\widetilde{\boldsymbol{\psi}}, \boldsymbol{v}_1^c\rangle = \langle S^{-\theta}\widetilde{\boldsymbol{\psi}}, \boldsymbol{v}_2^c\rangle = 0. \tag{3.38}$$

By the orthogonality condition (2.17), our assumption (3.37) is equivalent to

$$\widehat{\widetilde{\boldsymbol{\psi}}}(1) \cdot \widehat{\boldsymbol{v}_1^c}(1) = 0, \tag{3.39}$$

and our desired conclusion (3.38) is equivalent to

$$(e^{-i\theta}\widehat{\widetilde{\boldsymbol{\psi}}}(1)) \cdot \widehat{\boldsymbol{v}_1^c}(1) = 0. \tag{3.40}$$

Certainly (3.39) implies (3.40). □

This lemma provides the local uniqueness of our solutions up to shifts, which combines with the local uniqueness up to translations (as discussed after the proof of Corollary 3.3) to give the statement at the end of Theorem 1.1. Specifically, let $\boldsymbol{\phi}$ and ω solve $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) = 0$ with $\|\boldsymbol{\phi}\|_{H^2_{\mathrm{per}}}$ and $|\omega - \omega_c|$ sufficiently small. Then $\boldsymbol{\phi} = S^\theta(a\boldsymbol{v}_1^c + \boldsymbol{\psi})$ for some $a \in \mathbb{R}$ and $\boldsymbol{\psi} \in \mathcal{X}_c^\infty$, and $\|a\boldsymbol{v}_1^c + \boldsymbol{\psi}\|_{H^2_{\mathrm{per}}} = \|S^\theta(a\boldsymbol{v}_1^c + \boldsymbol{\psi})\|$. By orthogonality, this ensures that $|a|$ and $\|\boldsymbol{\psi}\|_{H^2_{\mathrm{per}}}$ are sufficiently small. The uniqueness result from Section 3.3 implies $\boldsymbol{\psi} = \boldsymbol{\Psi}_c(a\boldsymbol{v}_1^c, \omega)$, and then the uniqueness result from Section 3.4 implies $\omega = \Omega_c(a)$.

Another consequence of this lemma is that it shows why trying an ansatz of the form $\boldsymbol{\phi} = a\boldsymbol{v}_1^c + b\boldsymbol{v}_2^c + \boldsymbol{\psi}$ in the hope that a and b would be enough to manage the two equations in (3.31) will not be effective. Informally, the problem simply does not 'see' the two unknowns a and b simultaneously. And such an ansatz would not expose the single uniform amplitude parameter that we desire, anyway.

Additionally, there is nothing special about $\boldsymbol{v}_1^c$ here, and we could just as easily show that any solution to $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega)$ is a shifted version of a solution of the form $a\boldsymbol{v}_2^c + \boldsymbol{\psi}$. In fact, we could have run the bifurcation argument above using $\boldsymbol{v}_2^c$ throughout in place of $\boldsymbol{v}_1^c$. This hinges on expressing the transversality inequality of Corollary 2.3 in terms of $\boldsymbol{v}_2^c$, which is possible because of the calculation

$$\langle \mathcal{L}_c'[\omega_c]\boldsymbol{v}_2^c, \boldsymbol{v}_2^c\rangle = \langle \mathcal{L}_c'[\omega_c]\partial_x \boldsymbol{v}_1^c, \partial_x \boldsymbol{v}_1^c\rangle = \langle \partial_x \mathcal{L}_c'[\omega_c]\boldsymbol{v}_1^c, \partial_x \boldsymbol{v}_1^c\rangle = -\langle \mathcal{L}_c'[\omega_c]\boldsymbol{v}_1^c, \partial_x^2 \boldsymbol{v}_1^c\rangle = \langle \mathcal{L}_c'[\omega_c]\boldsymbol{v}_1^c, \boldsymbol{v}_1^c\rangle. \tag{3.41}$$

The second equality relies on the commutativity of the Fourier multipliers ∂_x and $\mathcal{L}_c'[\omega_c]$ on $H^3_{\text{per}}(\mathbb{R}^2)$. However, since we can write any solution in the form

$$S^\theta(a\boldsymbol{v}_1^c + \boldsymbol{\psi}) = S^\theta(aS^{\pi/2}\boldsymbol{v}_2^c + \boldsymbol{\psi}) = S^{\theta+\pi/2}(a\boldsymbol{v}_2^c + S^{-\pi/2}\boldsymbol{\psi})$$

with $\boldsymbol{\psi}$, and thus (by the end of the proof of Lemma 3.8) $S^{-\pi/2}\boldsymbol{\psi}$, orthogonal to $\boldsymbol{v}_1^c$ and $\boldsymbol{v}_2^c$, there is not much point to this line of inquiry.

Next, we consider further the special form of solutions to $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) = 0$ as given in Theorem 1.1: they are $\boldsymbol{\phi} = a(\boldsymbol{v}_1^c + \boldsymbol{\psi})$ with $\boldsymbol{\psi}$ again orthogonal to $\boldsymbol{v}_1^c$ and $\boldsymbol{v}_2^c$. This may appear to be less general than the result of Lemma 3.8, which says that, up to a shift, any solution has the form $a\boldsymbol{v}_1^c + \widetilde{\boldsymbol{\psi}}$, with $\widetilde{\boldsymbol{\psi}}$ satisfying the perennial orthogonality conditions. Of course, if $a \neq 0$, then this solution factors as $a(\boldsymbol{v}_1^c + a^{-1}\widetilde{\boldsymbol{\psi}})$, and that has the form given by Theorem 1.1. It turns out that all *small* nontrivial solutions to $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) = 0$ have this special factored form (again, up to a shift from Lemma 3.8).

We prove a negative version of this result, which says that if the shifted solution from Lemma 3.8 has the form $\boldsymbol{\phi} = \widetilde{\boldsymbol{\psi}}$ alone, i.e., if $a = 0$, and if this solution is sufficiently small, then it is trivial.

Lemma 3.9. *There exists $\delta_c^\infty > 0$ such that if $\boldsymbol{\psi} \in \mathcal{X}_c^\infty$ and $\omega \in \mathbb{R}$ with $\|\boldsymbol{\psi}\|_{H^2_{\text{per}}} + |\omega - \omega_c| < \delta_c^\infty$, and if $\boldsymbol{\Phi}_c(\boldsymbol{\psi}, \omega) = 0$, then $\boldsymbol{\psi} = 0$.*

Proof. Define

$$\widetilde{\mathcal{F}}_c^\infty : \mathcal{X}_c^\infty \times \mathbb{R} \to \mathcal{Y}_c^\infty : (\boldsymbol{\psi}, \omega) \mapsto (\mathcal{I}_{\mathcal{Y}} - \Pi_c)\boldsymbol{\Phi}_c(\boldsymbol{\psi}, \omega).$$

The notation and structure of this map are intentionally similar to those of $\mathcal{F}_c^\infty$ in (3.25), and the spaces $\mathcal{X}_c^\infty$ and $\mathcal{Y}_c^\infty$ are defined in (3.22). Then $\widetilde{\mathcal{F}}_c^\infty(0, \omega) = 0$ for all ω and

$$D_{\boldsymbol{\psi}} \widetilde{\mathcal{F}}_c^\infty(0, \omega_c) = (\mathcal{I}_{\mathcal{Y}} - \Pi_c)\, \mathcal{L}_c[\omega_c]\big|_{\mathcal{X}_c^\infty}$$

As with the analogous linearization in Section 3.3, this operator has trivial kernel and cokernel and therefore is invertible. The implicit function theorem gives $\delta_c^\infty, \epsilon_c^\infty > 0$ and a map

$$\boldsymbol{\Psi}_c^\infty : (\omega_c - \delta_c^\infty, \omega_c + \delta_c^\infty) \to \mathcal{X}_c^\infty$$

such that $\widetilde{\mathcal{F}}_c^\infty(\boldsymbol{\Psi}_c^\infty(\omega), \omega) = 0$ for $|\omega - \omega_c| < \delta_c^\infty$. Moreover, if $|\omega - \omega_c| < \delta_c^\infty$ and $\|\boldsymbol{\psi}\|_{H^2_{\text{per}}} < \epsilon_c^\infty$, and if $\widetilde{\mathcal{F}}_c^\infty(\boldsymbol{\psi}, \omega) = 0$, then $\boldsymbol{\psi} = \boldsymbol{\Psi}_c^\infty(\omega)$. But $\widetilde{\mathcal{F}}_c^\infty(0, \omega) = 0$ for all ω, and so we must have $\boldsymbol{\Psi}_c^\infty(\omega) = 0$ for all ω. Conversely, if $\boldsymbol{\Phi}_c(\boldsymbol{\psi}, \omega) = 0$ then $\widetilde{\mathcal{F}}_c^\infty(\boldsymbol{\psi}, \omega) = 0$, too, and so if $|\omega - \omega_c| < \delta_c^\infty$ and $\|\boldsymbol{\psi}\|_{H^2_{\text{per}}} < \epsilon_c^\infty$, then $\boldsymbol{\psi} = \boldsymbol{\Psi}_c^\infty(\omega) = 0$. □

Lemmas 3.8 and 3.9 together effectively tell us that the only worthwhile form of solutions to $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) = 0$ is $\boldsymbol{\phi} = a(\boldsymbol{v}_1^c + \boldsymbol{\psi})$. When we study this problem quantitatively in Section 6.1, we will start directly with an ansatz of this form.

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4. The Lyapunov centre formulation

We solve the problem

$$\mathbf{\Phi}_c(\boldsymbol{\phi}, \omega) + \gamma\boldsymbol{\phi}' = 0 \tag{4.1}$$

for $\boldsymbol{\phi} \in H^2_{\text{per}}(\mathbb{R}^2)$ and ω, $\gamma \in \mathbb{R}$. The extra unknown γ closes the overdetermined system that results from the two solvability conditions induced by the two-dimensional cokernel; we show momentarily that any solution to (4.1) necessarily has $\gamma = 0$, and so solving (4.1) really returns solutions to our original problem $\mathbf{\Phi}_c(\boldsymbol{\phi}, \omega) = 0$. This strategy is based on the work of Wright and Scheel in [40, Sec. 8]; in their words, '[t]he idea is to augment the Hamiltonian equation with a dissipation term, for instance $\gamma\nabla H$, so that for $\gamma \neq 0$, the system is gradient-like and does not possess any small non-equilibrium solutions'. In turn, the proof there was motivated by a proof of the Lyapunov centre theorem [2, Thm. 3.2].

Some (though not all) of the implicit function theorem arguments are quite similar to those in Sections 3.3 and 3.4, so we move rather more briskly here. We emphasize that while the existence proof developed in this section is not strictly necessary for logical completeness of our argument, we see it as a potentially useful alternative to the first proof in that it is completely independent of the gradient structure.

First we show that any nonconstant solution to (4.1) has $\gamma = 0$; the following calculation is similar to [2, Lem. 3.1], which was done in preparation for their proof of the Lyapunov centre theorem. If (4.1) holds, then

$$0 = \langle \mathbf{\Phi}_c(\boldsymbol{\phi}, \omega) + \gamma\boldsymbol{\phi}', \boldsymbol{\phi}' \rangle = \langle \mathbf{\Phi}_c(\boldsymbol{\phi}, \omega), \boldsymbol{\phi}' \rangle + \gamma\|\boldsymbol{\phi}'\|^2 = \gamma\|\boldsymbol{\phi}'\|^2. \tag{4.2}$$

Since $\boldsymbol{\phi}$ is nonconstant, we must have $\gamma = 0$.

In (4.2) we used the derivative orthogonality property

$$\langle \mathbf{\Phi}_c(\boldsymbol{\phi}, \omega), \boldsymbol{\phi}' \rangle = 0,$$

as established in Corollary 3.2 using the gradient formulation. However, with some more work, this can be checked directly from the definition of $\mathbf{\Phi}_c$.

Lemma 4.1. *Let $\boldsymbol{\phi} \in H^1_{\text{per}}(\mathbb{R}^2)$ and define*

$$\mathcal{J}_c(\boldsymbol{\phi}, \omega) := c^2\omega^2 \frac{(\phi_1')^2}{2} + c^2\omega^2 \frac{(\phi_2')^2}{2w} + \mathcal{V}_1(\phi_2 - S^{-\omega}\phi_1) + \mathcal{V}_2(\phi_1 - S^{-\omega}\phi_2) + \mathcal{I}(\boldsymbol{\phi}, \omega), \tag{4.3}$$

where

$$\mathcal{I}(\boldsymbol{\phi}, \omega)(x) := \int_x^{x-\omega} \mathcal{V}_1'(S^{\omega}\phi_2 - \phi_1)\phi_1' + \int_x^{x-\omega} \mathcal{V}_2'(S^{\omega}\phi_1 - \phi_2)\phi_2'.$$

Then

$$\langle \mathbf{\Phi}_c(\boldsymbol{\phi}, \omega), \boldsymbol{\phi}' \rangle = \int_{-\pi}^{\pi} \partial_x \mathcal{J}_c(\boldsymbol{\phi}, \omega). \tag{4.4}$$

In particular, since $\mathcal{J}_c(\boldsymbol{\phi}, \omega)$ is 2π-periodic,

$$\langle \mathbf{\Phi}_c(\boldsymbol{\phi}, \omega), \boldsymbol{\phi}' \rangle = 0.$$

Proof. The proof of (4.4) is a direct calculation using the definition of $\mathcal{J}_c$ above and the definition of $\mathbf{\Phi}_c$ in (1.9), but, for clarity, we provide some details as to how $\mathcal{J}_c$ naturally arises. Firstly, computing the dot product yields

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$$\mathbf{\Phi}_c(\boldsymbol{\phi}, \omega) \cdot \boldsymbol{\phi}' = c^2\omega^2\phi_1''\phi_1' + \frac{c^2}{w}\omega^2\phi_2''\phi_2' + \mathcal{V}_1'(\phi_2 - S^{-\omega}\phi_1)\phi_2' - \mathcal{V}_1'(S^{\omega}\phi_2 - \phi_1)\phi_1' + \mathcal{V}_2'(\phi_1 - S^{-\omega}\phi_2)\phi_1' - \mathcal{V}_2'(S^{\omega}\phi_1 - \phi_2)\phi_2'.$$

The first two terms are perfect derivatives, but the others involving $\mathcal{V}_1'$ and $\mathcal{V}_2'$ need some modification. We work with just the $\mathcal{V}_1'$ terms to show the origin of the first of the two integrals in $\mathcal{I}$. Adding zero, we have

$$\begin{aligned}\mathcal{V}_1'(\phi_2 - S^{-\omega}\phi_1)\phi_2' - \mathcal{V}_1'(S^{\omega}\phi_2 - \phi_1)\phi_1' &= \mathcal{V}_1'(\phi_2 - S^{-\omega}\phi_1)\phi_2' - \mathcal{V}_1'(\phi_2 - S^{-\omega}\phi_1)S^{-\omega}\phi_1' \\ &\quad + \mathcal{V}_1'(\phi_2 - S^{-\omega}\phi_1)S^{-\omega}\phi_1' - \mathcal{V}_1'(S^{\omega}\phi_2 - \phi_1)\phi_1' \\ &= \mathcal{V}_1'(\phi_2 - S^{-\omega}\phi_1)(\phi_2' - S^{-\omega}\phi_1') \\ &\quad + (S^{-\omega} - 1)[\mathcal{V}_1'(S^{\omega}\phi_2 - \phi_1)\phi_1'].\end{aligned}$$

Here we have factored

$$\mathcal{V}_1'(\phi_2 - S^{-\omega}\phi_1)S^{-\omega}\phi_1' = S^{-\omega}[\mathcal{V}_1'(S^{\omega}\phi_2 - \phi_1)\phi_1'].$$

to get the second term in the second equality above. In the first term of that second equality, we immediately recognize the perfect derivative

$$\mathcal{V}_1'(\phi_2 - S^{-\omega}\phi_1)(\phi_2' - S^{-\omega}\phi_1') = \partial_x[\mathcal{V}_1(\phi_2 - S^{-\omega}\phi_1)].$$

Finally, we use the identity

$$\partial_x\left[\int_x^{x-\omega} f\right] = f(x-\omega) - f(x) = [(S^{-\omega} - 1)f](x)$$

to rewrite

$$(S^{-\omega} - 1)[\mathcal{V}_1'(S^{\omega}\phi_2 - \phi_1)\phi_1'] = \partial_x\left[\int_x^{x-\omega} \mathcal{V}_1'(S^{\omega}\phi_2 - \phi_1)\phi_1'\right].$$

Repeating these calculations on the $\mathcal{V}_2'$ terms shows $\mathbf{\Phi}_c(\boldsymbol{\phi}, \omega) \cdot \boldsymbol{\phi}' = \partial_x\mathcal{J}_c(\boldsymbol{\phi}, \omega)$, and that is (4.4). □

Remark 4.2. *The structure of the operator $\mathcal{J}_c$ in (4.3) bears some resemblance to the first integral in [19, Prop. 3.10] for the spatial dynamics formulation of the travelling wave problem. Indeed, the existence of that conserved quantity from the spatial dynamics viewpoint inspired us to search for a related conserved quantity in this travelling wave framework, and $\mathcal{J}_c$ naturally emerged. Moreover, $\mathcal{J}_c$ is constant on solutions to $\mathbf{\Phi}_c(\boldsymbol{\phi}, \omega) = 0$ in the sense that if $\boldsymbol{\phi}$ and ω satisfy this equation, it can be checked that $\partial_x\mathcal{J}_c(\boldsymbol{\phi}, \omega) = 0$. This leads to another (related) proof that the existence of a nonconstant solution to (4.1) forces $\gamma = 0$: if $\boldsymbol{\phi}$ and ω meet (4.1), it follows that*

$$\partial_x\mathcal{J}_c(\boldsymbol{\phi}, \omega) = -\gamma\left((\phi_1')^2 + \frac{(\phi_2')^2}{w}\right).$$

If $\gamma > 0$, then $\partial_x\mathcal{J}_c(\boldsymbol{\phi}, \omega)$ is nonpositive and not identically zero; since $\mathcal{J}_c(\boldsymbol{\phi}, \omega)$ is periodic, this is impossible. A similar contradiction results if $\gamma < 0$.

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Now we study the problem (4.1) with a Lyapunov–Schmidt decomposition as in Sections 3.3 and 3.4. Using the projection operator Π_c and the function spaces $\mathcal{X}_c^\infty$, $\mathcal{Y}_c^\infty$, and $\mathcal{Z}_c$ from Section 3.2, we split (4.1) into the pair of equations

$$\begin{cases}(\mathcal{I}_{\mathcal{Y}} - \Pi_c)\mathbf{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\psi}, \omega) + \gamma(\mathcal{I}_{\mathcal{X}} - \Pi_c)(\boldsymbol{\nu}' + \boldsymbol{\psi}') = 0 \\ \Pi_c\mathbf{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\psi}, \omega) + \gamma\Pi_c(\boldsymbol{\nu}' + \boldsymbol{\psi}') = 0,\end{cases}$$

where $\boldsymbol{\phi} = \boldsymbol{\nu} + \boldsymbol{\psi}$ and $\boldsymbol{\nu} \in \mathcal{Z}_c$, $\boldsymbol{\psi} \in \mathcal{X}_c^\infty$. We can simplify the terms involving γ:

$$\Pi_c\boldsymbol{\nu}' = \partial_x\Pi_c\boldsymbol{\nu} = \boldsymbol{\nu}' \qquad \text{and} \qquad \Pi_c\boldsymbol{\psi}' = \partial_x\Pi_c\boldsymbol{\psi} = 0,$$

since Π_c and ∂_x commute by Lemma 3.5 and since $\Pi_c\boldsymbol{\nu} = \boldsymbol{\nu}$ while $\Pi_c\boldsymbol{\psi} = 0$. Then the decomposition reads

$$\begin{cases}(\mathcal{I}_{\mathcal{X}} - \Pi_c)\mathbf{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\psi}, \omega) + \gamma\boldsymbol{\psi}' = 0 & (4.5a) \\ \Pi_c\mathbf{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\psi}, \omega) + \gamma\boldsymbol{\nu}' = 0, & (4.5b)\end{cases}$$

and this is the problem that we will solve here.

First we address the infinite-dimensional equation (4.5a). Using the same notation as in Section 3.3, define

$$\mathcal{F}_c^\infty\colon \mathcal{X}_c^\infty \times \mathcal{Z}_c \times \mathbb{R}^2 \to \mathcal{Y}_c^\infty\colon (\boldsymbol{\psi}, \boldsymbol{\nu}, \omega, \gamma) \mapsto (\mathcal{I}_{\mathcal{X}} - \Pi_c)\mathbf{\Phi}_c(\boldsymbol{\nu} + \boldsymbol{\psi}, \omega) + \gamma\boldsymbol{\psi}'.$$

Since $\Pi_c\boldsymbol{\psi}' = 0$ as computed above, and since

$$\langle \boldsymbol{\psi}', \boldsymbol{\nu}_0\rangle = -\langle \boldsymbol{\psi}, \boldsymbol{\nu}_0'\rangle = 0$$

by integration by parts and the identity $\boldsymbol{\nu}_0' = 0$ from (2.11), we do indeed have $\boldsymbol{\psi}' \in \mathcal{Y}_c^\infty$ for $\boldsymbol{\psi} \in \mathcal{X}_c^\infty$. That is, $\mathcal{F}_c^\infty$ does indeed map into $\mathcal{Y}_c^\infty$. Next, $\mathcal{F}_c^\infty(0, 0, \omega, \gamma) = 0$ for all ω and γ, and $D_{\boldsymbol{\psi}}\mathcal{F}_c^\infty(0, \omega_c, 0, 0) = (\mathcal{I}_{\mathcal{X}} - \Pi_c)\,\mathcal{L}_c[\omega_c]\big|_{\mathcal{X}_c^\infty}$ is invertible. Consequently, by the implicit function theorem, all suitably small solutions to $\mathcal{F}_c^\infty(\boldsymbol{\psi}, \boldsymbol{\nu}, \omega, \gamma) = 0$ have the form $\boldsymbol{\psi} = \mathbf{\Psi}_c(\boldsymbol{\nu}, \omega, \gamma)$ with $\mathbf{\Psi}_c(0, \omega, \gamma) = 0$ for all ω and γ. Before proceeding, we note that the same proof as for part (iii) of Lemma 3.7 (which did not rely on the gradient structure at all) yields

$$D_{\boldsymbol{\nu}}\mathbf{\Psi}(0, \omega_c, 0) = 0. \tag{4.6}$$

Now we specialize to $\boldsymbol{\nu} = a\boldsymbol{\nu}_1^c$ and solve the finite-dimensional equation (4.5b) by studying

$$\mathcal{F}_c^0(\omega, \gamma, a) := \Pi_c\mathbf{\Phi}_c(a\boldsymbol{\nu}_1 + \mathbf{\Psi}_c(a\boldsymbol{\nu}_1^c, \omega, \gamma), \omega) - \gamma a\boldsymbol{\nu}_2^c = 0.$$

Here we have used $\partial_x\boldsymbol{\nu}_1^c = -\boldsymbol{\nu}_2^c$. Since $\mathcal{F}_c^0(\omega, \gamma, 0) = 0$, as in Section 3.4 we have the factorization

$$\mathcal{F}_c^0(\omega, \gamma, a) = a\mathcal{H}_c(\omega, \gamma, a), \qquad \mathcal{H}_c(\omega, \gamma, a) := \int_0^1 D_a\mathcal{F}_c^0(\omega, \gamma, a\alpha)\, d\alpha.$$

We solve $\mathcal{H}_c(\omega, \gamma, a) = 0$.

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Firstly, we compute

$$D_a\mathcal{F}_c^0(\omega,\gamma,a) = \Pi_c D_{\boldsymbol{\phi}}\boldsymbol{\Phi}_c(a\boldsymbol{\nu}_1 + \boldsymbol{\Psi}_c(a\boldsymbol{\nu}_1^c,\omega,\gamma),\omega)\big(\boldsymbol{\nu}_1^c + D_{\boldsymbol{\nu}}\boldsymbol{\Psi}_c(a\boldsymbol{\nu}_1^c,\omega,\gamma)\boldsymbol{\nu}_1^c\big) - \gamma\boldsymbol{\nu}_2^c.$$

This, together with (4.6) and $\Pi_c\mathcal{L}_c[\omega_c] = 0$, implies $\mathcal{H}_c(\omega_c,0,0) = 0$. Next, differentiate with respect to $(\omega,\gamma) \in \mathbb{R}^2$ and write this derivative as a linear combination of partial derivatives:

$$D_{(\omega,\gamma)}\mathcal{H}_c(\omega_c,0,0)(\omega,\gamma) = \omega D_\omega\mathcal{H}_c(\omega_c,0,0) + \gamma D_\gamma\mathcal{H}_c(\omega_c,0,0). \tag{4.7}$$

We compute each of these partial derivatives separately.

For $D_\omega\mathcal{H}_c$, we calculate

$$\mathcal{H}_c(\omega,0,0) = \Pi_c D_{\boldsymbol{\phi}}\boldsymbol{\Phi}_c(0,\omega)\big(\boldsymbol{\nu}_1^c + D_{\boldsymbol{\nu}}\boldsymbol{\Psi}_c(0,\omega,0)\boldsymbol{\nu}_1^c\big),$$

and use the product rule and the identities $\Pi_c\mathcal{L}_c[\omega_c] = 0$ and $D_{\boldsymbol{\nu}}\boldsymbol{\Psi}_c(0,\omega_c,0) = 0$ to obtain

$$D_\omega\mathcal{H}_c(\omega_c,0,0) = \Pi_c\mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c = \langle\mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c,\boldsymbol{\nu}_1^c\rangle\boldsymbol{\nu}_1^c. \tag{4.8}$$

Above we used the following lemma to simplify the projection calculation.

Lemma 4.3. $\langle\mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c,\boldsymbol{\nu}_2^c\rangle = 0$.

We give two proofs of this lemma in Appendix A.5, one using the gradient formulation, and one using directly the definitions of $\mathcal{L}_c'[\omega_c]$, $\boldsymbol{\nu}_1^c$, and $\boldsymbol{\nu}_2^c$.

Next, we work on $D_\gamma\mathcal{H}_c$. Since $\boldsymbol{\Psi}_c(0,\omega,\gamma) = 0$ for all ω and γ, we have

$$\mathcal{H}_c(\omega_c,\gamma,0) = \Pi_c\mathcal{L}_c[\omega_c]\big(\boldsymbol{\nu}_1^c + D_{\boldsymbol{\nu}}\boldsymbol{\Psi}_c(0,\omega_c,\gamma)\boldsymbol{\nu}_1^c\big) - \gamma\boldsymbol{\nu}_2 = -\gamma\boldsymbol{\nu}_2,$$

thanks to $\Pi_c\mathcal{L}_c[\omega_c] = 0$ once again. Thus

$$D_\gamma\mathcal{H}_c(\omega_c,0,0) = -\boldsymbol{\nu}_2^c. \tag{4.9}$$

We combine (4.7), (4.8) and (4.9) to find

$$D_{(\omega,\gamma)}\mathcal{H}_c(\omega_c,0,0)(\omega,\gamma) = \omega\langle\mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c,\boldsymbol{\nu}_1^c\rangle\boldsymbol{\nu}_1^c - \gamma\boldsymbol{\nu}_2^c.$$

Since $\langle\mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c,\boldsymbol{\nu}_1^c\rangle \neq 0$ by Corollary 2.3, and since $\boldsymbol{\nu}_1^c$ and $\boldsymbol{\nu}_2^c$ form a basis for $\mathcal{Z}_c$, we conclude that $D_{(\omega,\gamma)}\mathcal{H}_c(\omega_c,0,0)$ is an invertible linear operator from $\mathbb{R}^2$ to $\mathcal{Z}_c$. By the implicit function theorem, for suitably small ω, γ and a, we can solve $\mathcal{H}_c(\omega,\gamma,a) = 0$ with $\omega = \Omega_c(a)$ and $\gamma = \Gamma_c(a)$ for some maps Ω_c and Γ_c with $\Omega_c(0) = \omega_c$ and $\Gamma_c(0) = 0$.

It follows that taking

$$\boldsymbol{\phi}_c^a := a\boldsymbol{\nu}_1^c + \boldsymbol{\Psi}_c(a\boldsymbol{\nu}_1^c,\Omega_c(a),\Gamma_c(a)) \qquad \text{and} \qquad \omega_c^a := \Omega_c(a)$$

solves $\boldsymbol{\Phi}_c(\boldsymbol{\phi}_c^a,\omega_c^a)+\Gamma_c(a)\boldsymbol{\phi}' = 0$. Since $\widehat{\boldsymbol{\phi}_c^a}(\pm 1) \neq 0$, $\boldsymbol{\phi}_c^a$ is nonconstant, and so by the calculation in (4.2) we really have $\Gamma_c(a) = 0$ for all a. Additionally, if we put $\boldsymbol{\psi}_c(a) = \boldsymbol{\Psi}_c(a\boldsymbol{\nu}_1^c,\Omega_c(a),0)$, then $\boldsymbol{\psi}_c(0) = 0$, and so

$$\boldsymbol{\phi}_c^a = a\left(\boldsymbol{\nu}_1^c + \int_0^1 D_a\boldsymbol{\psi}_c(a\alpha)\,d\alpha\right),$$

which is the representation that we want. This concludes our second proof of Theorem 1.1.

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5. Periodic solutions with symmetry

We first work out an abstract notion of symmetry in Section 5.1 and quickly show in Section 5.2 how bifurcation unfolds in its presence. Then we prove in Section 5.3 that mass and spring dimers actually possess such symmetries. The point of this analysis is that when the lattice has a symmetry, the periodic travelling wave solutions can be chosen to respect that symmetry. This proves Theorem 1.1 for mass and spring dimers, which recovers the results of [16, 20].

5.1. Symmetry operators and their properties

Definition 5.1. *A bounded linear operator $\mathcal{S}\colon L^2_{\mathrm{per}}(\mathbb{R}^2) \to L^2_{\mathrm{per}}(\mathbb{R}^2)$ is a* ***symmetry*** *if the following hold.*

(i) *$\mathcal{G}_c(\mathcal{S}\boldsymbol{\phi}, \omega) = \mathcal{G}_c(\boldsymbol{\phi}, \omega)$ for all $\boldsymbol{\phi} \in H^2_{\mathrm{per}}(\mathbb{R}^2)$ and any $c \in \mathbb{R}$, where $\mathcal{G}_c$ is defined in (3.8).*
(ii) *$\mathcal{S}^2\boldsymbol{\phi} = \boldsymbol{\phi}$ for all $\boldsymbol{\phi} \in L^2_{\mathrm{per}}(\mathbb{R}^2)$.*
(iii) *$\langle \mathcal{S}\boldsymbol{\phi}, \boldsymbol{\eta}\rangle = \langle \boldsymbol{\phi}, \mathcal{S}\boldsymbol{\eta}\rangle$ for all $\boldsymbol{\phi}$, $\boldsymbol{\eta} \in L^2_{\mathrm{per}}(\mathbb{R}^2)$.*
(iv) *$\partial_x \mathcal{S}\boldsymbol{\phi} = -\mathcal{S}\boldsymbol{\phi}'$ for all $\boldsymbol{\phi} \in H^2_{\mathrm{per}}(\mathbb{R}^2)$.*

We point out that while shift operators S^d do satisfy the invariance property (i) above, and while $S^{\pm\pi}$ also satisfies (ii), shifts in general do not meet (iii) and (iv). The symmetries that we construct will not rely on shift operators.

Here are some useful properties of symmetries for our problem.

Lemma 5.2. *Let $\mathcal{S}$ be a symmetry.*

(i) *$\boldsymbol{\Phi}_c(\mathcal{S}\boldsymbol{\phi}, \omega) = \mathcal{S}\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega)$ for all $\boldsymbol{\phi} \in H^2_{\mathrm{per}}(\mathbb{R}^2)$ and $\omega \in \mathbb{R}$.*
(ii) *$\mathcal{L}_c[\omega]\mathcal{S} = \mathcal{S}\mathcal{L}_c[\omega]$ and $\mathcal{L}_c'[\omega]\mathcal{S} = \mathcal{S}\mathcal{L}_c'[\omega]$ for all ω.*
(iii) *$\mathcal{S}\boldsymbol{\nu}_1^c = \pm\boldsymbol{\nu}_1^c$ if and only if $\mathcal{S}\boldsymbol{\nu}_2^c = \mp\boldsymbol{\nu}_2^c$.*

Proof.

(i) Since $\mathcal{G}_c(\mathcal{S}\boldsymbol{\phi}, \omega) = \mathcal{G}_c(\boldsymbol{\phi}, \omega)$ for all $\boldsymbol{\phi} \in H^2_{\mathrm{per}}(\mathbb{R}^2)$ and $\omega \in \mathbb{R}$, we differentiate with respect to $\boldsymbol{\phi}$ and use the chain rule (much as we did in the proof of part (ii) of Corollary 3.2) to find

$$D_{\boldsymbol{\phi}}\mathcal{G}_c(\boldsymbol{\phi}, \omega)\boldsymbol{\eta} = D_{\boldsymbol{\phi}}\mathcal{G}_c(\mathcal{S}\boldsymbol{\phi}, \omega)\mathcal{S}\boldsymbol{\eta}$$

for all $\boldsymbol{\eta} \in H^2_{\mathrm{per}}(\mathbb{R}^2)$. Using the gradient formulation, this reads

$$\langle \boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega), \boldsymbol{\eta}\rangle = \langle \boldsymbol{\Phi}_c(\mathcal{S}\boldsymbol{\phi}, \omega), \mathcal{S}\boldsymbol{\eta}\rangle = \langle \mathcal{S}\boldsymbol{\Phi}_c(\mathcal{S}\boldsymbol{\phi}, \omega), \boldsymbol{\eta}\rangle,$$

where the second equality is the adjoint property of $\mathcal{S}$. Since this is true for all $\boldsymbol{\eta} \in H^2_{\mathrm{per}}(\mathbb{R}^2)$, we have $\mathcal{S}\boldsymbol{\Phi}_c(\mathcal{S}\boldsymbol{\phi}, \omega) = \boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega)$.
(ii) This follows from part (i) and the chain rule.
(iii) We use the relations $\partial_x \boldsymbol{\nu}_1^c = -\boldsymbol{\nu}_2^c$ and $\partial_x \boldsymbol{\nu}_2^c = \boldsymbol{\nu}_1^c$ from Corollary 2.2. If $\mathcal{S}\boldsymbol{\nu}_1^c = \pm\boldsymbol{\nu}_1^c$, then

$$\mathcal{S}\boldsymbol{\nu}_2^c = -\mathcal{S}\partial_x\boldsymbol{\nu}_1^c = \partial_x\mathcal{S}\boldsymbol{\nu}_1^c = \pm\partial_x\boldsymbol{\nu}_1^c = \mp\partial_x\boldsymbol{\nu}_2^c.$$

Conversely, if $\mathcal{S}\boldsymbol{\nu}_2^c = \mp\boldsymbol{\nu}_2^c$, then

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$$\mathcal{S}\boldsymbol{v}_1^c = \mathcal{S}\partial_x \boldsymbol{v}_2^c = -\partial_x \mathcal{S}\boldsymbol{v}_2^c = -(\mp\partial_x \boldsymbol{v}_2^c) = \pm\partial_x \boldsymbol{v}_2^c = \pm\boldsymbol{v}_1^c.$$

□

Now we adapt the nonconstant eigenfunctions $\boldsymbol{v}_1^c$ and $\boldsymbol{v}_2^c$ from Corollary 2.2 so that they respect symmetry.

Lemma 5.3. *Let $\mathcal{S}$ be a symmetry and define*

$$\boldsymbol{v}_+^c := \begin{cases} \boldsymbol{v}_1^c, \ \mathcal{S}\boldsymbol{v}_1^c = \boldsymbol{v}_1^c \\ \boldsymbol{v}_2^c, \ \mathcal{S}\boldsymbol{v}_1^c = -\boldsymbol{v}_1^c \\ (\boldsymbol{v}_1^c + \mathcal{S}\boldsymbol{v}_1^c)/\|\boldsymbol{v}_1^c + \mathcal{S}\boldsymbol{v}_1^c\|_{L^2_{\text{per}}}, \ \mathcal{S}\boldsymbol{v}_1^c \neq \pm\boldsymbol{v}_1^c \end{cases}$$

and

$$\boldsymbol{v}_-^c := \begin{cases} \boldsymbol{v}_2^c, \ \mathcal{S}\boldsymbol{v}_1^c = \boldsymbol{v}_1^c \\ \boldsymbol{v}_1^c, \ \mathcal{S}\boldsymbol{v}_1^c = -\boldsymbol{v}_1^c \\ (\boldsymbol{v}_2^c - \mathcal{S}\boldsymbol{v}_2^c)/\|\boldsymbol{v}_2^c - \mathcal{S}\boldsymbol{v}_2^c\|_{L^2_{\text{per}}}, \ \mathcal{S}\boldsymbol{v}_1^c \neq \pm\boldsymbol{v}_1^c. \end{cases}$$

(i) $\mathcal{S}\boldsymbol{v}_+^c = \boldsymbol{v}_+^c$ *and* $\mathcal{S}\boldsymbol{v}_-^c = \boldsymbol{v}_-^c$.
(ii) *The vectors $\boldsymbol{v}_+^c$ and $\boldsymbol{v}_-^c$ form an orthonormal basis for $\mathcal{Z}_c$ as defined in (3.18).*
(iii) $\inf_{|c|>c_\star} \langle \mathcal{L}_c'[\omega_c]\boldsymbol{v}_+^c, \boldsymbol{v}_+^c\rangle > 0$.

Proof. We first remark that part (iii) of Lemma 5.2 ensures that $\boldsymbol{v}_\pm^c$ is defined in the third case of $\mathcal{S}\boldsymbol{v}_1^c \neq \pm\boldsymbol{v}_1^c$: if $\mathcal{S}\boldsymbol{v}_1^c \neq \pm\boldsymbol{v}_1^c$, then also $\mathcal{S}\boldsymbol{v}_2^c \neq \pm\boldsymbol{v}_2^c$, and so both $\boldsymbol{v}_1^c + \mathcal{S}\boldsymbol{v}_1^c$ and $\boldsymbol{v}_2^c - \mathcal{S}\boldsymbol{v}_2^c$ are nonzero.

(i) This is a direct calculation.
(ii) This is obvious in the cases $\mathcal{S}\boldsymbol{v}_1^c = \pm\boldsymbol{v}_1^c$. In the third case, we use part (ii) of Lemma 5.2 to compute

$$\mathcal{L}_c[\omega_c]\mathcal{S}\boldsymbol{v}_1^c = \mathcal{S}\mathcal{L}_c[\omega_c]\boldsymbol{v}_1^c = 0$$

and likewise $\mathcal{L}_c[\omega_c]\mathcal{S}\boldsymbol{v}_2^c = 0$. This shows $\boldsymbol{v}_\pm^c \in \ker(\mathcal{L}_c[\omega_c])$. Next,

$$\langle \mathcal{S}\boldsymbol{v}_1^c, \boldsymbol{v}_0\rangle = \langle \mathcal{S}\partial_x \boldsymbol{v}_2^c, \boldsymbol{v}_0\rangle = -\langle \partial_x \mathcal{S}\boldsymbol{v}_2^c, \boldsymbol{v}_0\rangle = \langle \mathcal{S}\boldsymbol{v}_2^c, \partial_x \boldsymbol{v}_0\rangle = 0$$

and likewise $\langle \mathcal{S}\boldsymbol{v}_2^c, \boldsymbol{v}_0\rangle = 0$. This shows $\boldsymbol{v}_\pm^c \in \mathcal{Z}_c$.

For orthogonality, we compute

$$\langle \boldsymbol{v}_1^c + \mathcal{S}\boldsymbol{v}_1^c, \boldsymbol{v}_2^c - \mathcal{S}\boldsymbol{v}_2^c\rangle = \langle \boldsymbol{v}_1^c, \boldsymbol{v}_2^c\rangle - \langle \boldsymbol{v}_1^c, \mathcal{S}\boldsymbol{v}_2^c\rangle + \langle \mathcal{S}\boldsymbol{v}_1^c, \boldsymbol{v}_2^c\rangle - \langle \mathcal{S}\boldsymbol{v}_1^c, \mathcal{S}\boldsymbol{v}_2^c\rangle. \tag{5.1}$$

Now we use properties of $\mathcal{S}$ to rewrite

$$\langle \boldsymbol{v}_1^c, \mathcal{S}\boldsymbol{v}_2^c\rangle = \langle \mathcal{S}\boldsymbol{v}_1^c, \boldsymbol{v}_2^c\rangle \qquad \text{and} \qquad \langle \mathcal{S}\boldsymbol{v}_1^c, \mathcal{S}\boldsymbol{v}_2^c\rangle = \langle \mathcal{S}^2\boldsymbol{v}_1^c, \boldsymbol{v}_2^c\rangle = \langle \boldsymbol{v}_1^c, \boldsymbol{v}_2^c\rangle.$$

From this and (5.1), we obtain $\langle \boldsymbol{v}_1^c + \mathcal{S}\boldsymbol{v}_1^c, \boldsymbol{v}_2^c - \mathcal{S}\boldsymbol{v}_2^c\rangle = 0$. Since $\mathcal{Z}_c$ is already two-dimensional, it follows from orthogonality and linear independence that $\boldsymbol{v}_+^c$ and $\boldsymbol{v}_-^c$ are a basis.
(iii) The first case that $\mathcal{S}\boldsymbol{v}_1^c = \boldsymbol{v}_1^c$ is Corollary 2.3. The second case that $\mathcal{S}\boldsymbol{v}_1^c = -\boldsymbol{v}_1^c$ is equivalent to $\mathcal{S}\boldsymbol{v}_2^c = \boldsymbol{v}_2^c$ by part (iii) of Lemma 5.2, and then we can use the calculation in (3.41). For the third

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case that $\mathcal{S}\boldsymbol{\nu}_1^c \neq \pm\boldsymbol{\nu}_1^c$, we start by taking $\boldsymbol{\nu} \in \mathcal{Z}_c = \operatorname{span}(\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_2^c)$ and then computing via the orthonormality of $\boldsymbol{\nu}_1^c$ and $\boldsymbol{\nu}_2^c$, (3.41), and Lemma 4.3 that

$$\langle \mathcal{L}_c'[\omega_c]\boldsymbol{\nu}, \boldsymbol{\nu}\rangle = \|\boldsymbol{\nu}\|_{L^2_{\text{per}}}^2 \langle \mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_1^c\rangle.$$

With $\boldsymbol{\nu} = \boldsymbol{\nu}_+^c$, we have $\boldsymbol{\nu}_+^c \in \operatorname{span}(\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_2^c)$ and $\|\boldsymbol{\nu}_+^c\|_{L^2_{\text{per}}} = 1$, so

$$\langle \mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_+^c, \boldsymbol{\nu}_+^c\rangle = \langle \mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_1^c\rangle,$$

from which the positive infimum follows.

□

5.2. *Bifurcation in the presence of symmetry*

Let $\mathcal{S}$ be a symmetry and define

$$\mathcal{Y}_{\mathcal{S}} := \left\{\boldsymbol{\phi} \in L^2_{\text{per}}(\mathbb{R}^2) \;\middle|\; \langle \boldsymbol{\phi}, \boldsymbol{\nu}_0\rangle = 0 \text{ and } \mathcal{S}\boldsymbol{\phi} = \boldsymbol{\phi}\right\} \qquad \text{and} \qquad \mathcal{X}_{\mathcal{S}} := H^2_{\text{per}}(\mathbb{R}^2) \cap \mathcal{Y}_{\mathcal{S}}.$$

Part (i) of Lemma 5.2 shows that $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) \in \mathcal{Y}_{\mathcal{S}}$ for each $\boldsymbol{\phi} \in \mathcal{X}_{\mathcal{S}}$ and $\omega \in \mathbb{R}$. The effect of restricting $\boldsymbol{\Phi}_c$ to map from $\mathcal{X}_{\mathcal{S}} \times \mathbb{R}$ to $\mathcal{Y}_{\mathcal{S}}$ is that the restriction $\mathcal{L}_c[\omega_c]\big|_{\mathcal{X}_{\mathcal{S}}}$ now has a one-dimensional kernel and cokernel. This, along with the transversality condition from part (iii) of Lemma 5.3, puts us in a position to use the classical Crandall–Rabinowitz–Zeidler theorem directly, without the work in Sections 3.4 or 4 to manage the extra finite-dimensional equation.

Remark 5.4. *While the Crandall–Rabinowitz–Zeidler theorem is often used to solve a problem of the form $F(x, \lambda) = 0$ with F twice continuously differentiable, this regularity is not strictly necessary; the proof in [31, Thm. 1.5.1] really hinges on having F and F_x once continuously differentiable. This allows us to avoid the annoying insufficient regularity in the frequency parameter ω in our problem; recall Lemma 3.6.*

More precisely, we know that the three vectors $\boldsymbol{\nu}_0$, $\boldsymbol{\nu}_+^c$, and $\boldsymbol{\nu}_-^c$ form an orthonormal basis for $\ker(\mathcal{L}_c[\omega_c])$ and $\ker(\mathcal{L}_c[\omega_c]^*)$; now suppose that $\boldsymbol{\phi} \in \mathcal{Y}_{\mathcal{S}} \cap \operatorname{span}(\boldsymbol{\nu}_0, \boldsymbol{\nu}_+^c, \boldsymbol{\nu}_-^c)$. Then by orthonormality

$$\boldsymbol{\phi} = \langle \boldsymbol{\phi}, \boldsymbol{\nu}_0\rangle\boldsymbol{\nu}_0 + \langle \boldsymbol{\phi}, \boldsymbol{\nu}_+^c\rangle\boldsymbol{\nu}_+^c + \langle \boldsymbol{\phi}, \boldsymbol{\nu}_-^c\rangle\boldsymbol{\nu}_-^c.$$

By definition of $\mathcal{Y}_{\mathcal{S}}$, we already have $\langle \boldsymbol{\phi}, \boldsymbol{\nu}_0\rangle = 0$, and now we compute

$$\langle \boldsymbol{\phi}, \boldsymbol{\nu}_-^c\rangle = \langle \mathcal{S}\boldsymbol{\phi}, \boldsymbol{\nu}_-^c\rangle = \langle \boldsymbol{\phi}, \mathcal{S}\boldsymbol{\nu}_-^c\rangle = -\langle \boldsymbol{\phi}, \boldsymbol{\nu}_-^c\rangle. \tag{5.2}$$

Thus $\langle \boldsymbol{\phi}, \boldsymbol{\nu}_-^c\rangle = 0$, and so $\boldsymbol{\phi} \in \operatorname{span}(\boldsymbol{\nu}_+^c)$. This proves our claim above that $\boldsymbol{\nu}_+^c$ spans both the kernel and cokernel of $\mathcal{L}_c[\omega_c]$.

Alternatively, we could follow the bifurcation argument in Sections 3.3 and 3.4 and replace $\boldsymbol{\nu}_1^c$ with $\boldsymbol{\nu}_+^c$ and $\boldsymbol{\nu}_2^c$ with $\boldsymbol{\nu}_-^c$. The only change would be the new version of the finite-dimensional problem (3.31)

$$\begin{cases} \langle \boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega), \boldsymbol{\nu}_+^c\rangle = 0 & (5.3a) \\ \langle \boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega), \boldsymbol{\nu}_-^c\rangle = 0. & (5.3b) \end{cases}$$

By a calculation similar to (5.2), we always have (5.3b). Specifically, for $\boldsymbol{\phi} \in \mathcal{X}_{\mathcal{S}}$, we have

$$\langle \boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega), \boldsymbol{\nu}_-^c\rangle = \langle \boldsymbol{\Phi}_c(\mathcal{S}\boldsymbol{\phi}, \omega), \boldsymbol{\nu}_-^c\rangle = \langle \mathcal{S}\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega), \boldsymbol{\nu}_-^c\rangle = \langle \boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega), \mathcal{S}\boldsymbol{\nu}_-^c\rangle = -\langle \boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega), \boldsymbol{\nu}_-^c\rangle,$$

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thus $\langle \mathbf{\Phi}_c(\boldsymbol{\phi}, \omega), \boldsymbol{\nu}_-^c \rangle = 0$ regardless of the form of $\boldsymbol{\phi} \in \mathcal{X}_{\mathcal{S}}$. (This is actually a stronger result than our managing of the second finite-dimensional equation (3.33a) in Section 3.4, as here there are no restrictions on the form of $\boldsymbol{\phi}$.) Last, we can solve (5.3a) using the transversality condition from part (iii) of Lemma 5.3, exactly as we did (3.33a) in Section 3.4. The major difference in the results here is that the solutions $\boldsymbol{\phi}$ now respect the symmetry.

5.3. Existence of symmetries for the mass and spring dimers

We will build the symmetries primarily on a 'reflection' operator and a 'flip' operator.

Lemma 5.5. *The operator*

$$(R\boldsymbol{\phi})(x) := \boldsymbol{\phi}(-x) \tag{5.4}$$

has the following properties.

(i) $R\partial_x = -\partial_x R$.
(ii) $RS^\theta = S^{-\theta}R$ *for all* $\theta \in \mathbb{R}$.
(iii) $\langle R\boldsymbol{\eta}, R\boldsymbol{\phi}\rangle = \langle \boldsymbol{\phi}, \boldsymbol{\eta}\rangle$ *for all* $\boldsymbol{\phi}$, $\boldsymbol{\eta} \in L^2_{\text{per}}(\mathbb{R}^2)$.

Proof.

(i) This follows from the chain rule.
(ii) We compute

$$(RS^\theta\boldsymbol{\phi})(x) = (S^\theta\boldsymbol{\phi})(-x) = \boldsymbol{\phi}(-x+\theta) = \boldsymbol{\phi}(-(x-\theta)) = (R\boldsymbol{\phi})(x-\theta) = (S^{-\theta}R\boldsymbol{\phi})(x).$$

(iii) This follows from substitution.

□

We will also use the 'flip' operator

$$J := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{5.5}$$

which commutes with R. These reflection and flip operators also appeared in the manifestation of symmetries for spatial dynamics coordinates [19 Sec. 3.2].

5.3.1. Symmetry in the mass dimer

The mass dimer symmetry is

$$\mathcal{S}_{\mathbf{M}}\boldsymbol{\phi} := -R\boldsymbol{\phi}.$$

The subscript here is meant to emphasize the role of the mass ratio $m = w^{-1}$ in the mass dimer analysis.

We show that $\mathcal{S}_{\mathbf{M}}$ satisfies property (i) of Definition 5.1; all of the other properties of symmetries are quick and direct calculations. That is, we show

$$\mathcal{G}_c(\mathcal{S}_{\mathbf{M}}\boldsymbol{\phi}, \omega) = \mathcal{G}_c(\boldsymbol{\phi}, \omega)$$

with $\mathcal{G}_c = c^2\mathcal{T} + \mathcal{P}$ as defined in (3.8). The operator $\mathcal{T}$ is defined in (3.6) and $\mathcal{P}$ in (3.7), and it is important here that in (3.7) we are assuming $\mathcal{V}_1 = \mathcal{V}_2 =: \mathcal{V}$. In particular, we take $\kappa = 1$.

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First we compute

$$\frac{2}{\omega^2}\mathcal{T}(\mathcal{S}_{\mathbf{M}}\boldsymbol{\phi},\omega) = \langle -M(R\boldsymbol{\phi})'', -R\boldsymbol{\phi}\rangle = \langle RM\boldsymbol{\phi}'', R\boldsymbol{\phi}\rangle = \langle M\boldsymbol{\phi}'', \boldsymbol{\phi}\rangle = \frac{2}{\omega^2}\mathcal{T}(\boldsymbol{\phi},\omega).$$

Here we have used $\partial_x^2 R = R\partial_x^2$, which follows from part (i) of Lemma 5.5, and also part (iii) of that lemma to get the penultimate equality.

Next,

$$\mathcal{P}(\mathcal{S}_{\mathbf{M}}\boldsymbol{\phi},\omega) = \langle \boldsymbol{\mathcal{V}}(\Delta_+(\omega)\mathcal{S}_{\mathbf{M}}\boldsymbol{\phi}), \mathbf{1}\rangle,$$

where $\boldsymbol{\mathcal{V}}(\mathbf{p}) = (\mathcal{V}(p_1), \mathcal{V}(p_2))$ for $\mathbf{p} = (p_1, p_2)$, $\mathbf{1} = (1,1)$ and $\Delta_+(\omega)$ is defined in (3.1). Since $S^{\pm\omega}R = RS^{\mp\omega}$ by part (ii) of Lemma 5.5, we have

$$\Delta_+(\omega)\mathcal{S}_{\mathbf{M}}\boldsymbol{\phi} = -\Delta_+(\omega)R\boldsymbol{\phi} = -R\Delta_+(-\omega)\boldsymbol{\phi} = RJ\Delta_+(\omega)\boldsymbol{\phi}. \tag{5.6}$$

Here we used the property that

$$-\Delta_+(-\omega) = J\Delta_+(\omega) \tag{5.7}$$

with J from (5.5).

Thus

$$\mathcal{P}(\mathcal{S}_{\mathbf{M}}\boldsymbol{\phi},\omega) = \langle \boldsymbol{\mathcal{V}}(RJ\Delta_+(\omega)\boldsymbol{\phi}), \mathbf{1}\rangle = \langle R\boldsymbol{\mathcal{V}}(J\Delta_+(\omega)\boldsymbol{\phi}), \mathbf{1}\rangle.$$

Since $\mathbf{1}$ is constant, $\mathbf{1} = R\mathbf{1}$, and so part (iii) of Lemma 5.5 implies

$$\mathcal{P}(\mathcal{S}_{\mathbf{M}}\boldsymbol{\phi},\omega) = \langle R\boldsymbol{\mathcal{V}}(J\Delta_+(\omega)\boldsymbol{\phi}), R\mathbf{1}\rangle = \langle \boldsymbol{\mathcal{V}}(J\Delta_+(\omega)\boldsymbol{\phi}), \mathbf{1}\rangle.$$

Last, since $\mathcal{V}(\mathbf{p}) = (\mathcal{V}(p_1), \mathcal{V}(p_2))$ for $\mathbf{p} = (p_1, p_2)$, we have

$$\langle \boldsymbol{\mathcal{V}}(J\mathbf{p}), \mathbf{1}\rangle = \int_{-\pi}^{\pi} \big(\mathcal{V}(p_2) + \mathcal{V}(p_1)\big) = \langle \mathcal{V}(\mathbf{p}), \mathbf{1}\rangle.$$

We conclude

$$\mathcal{P}(\mathcal{S}_{\mathbf{M}}\boldsymbol{\phi},\omega) = \langle \boldsymbol{\mathcal{V}}(\Delta_+(\omega)\boldsymbol{\phi}), \mathbf{1}\rangle = \mathcal{P}(\boldsymbol{\phi},\omega).$$

Last, we use the definitions of $\boldsymbol{\nu}_1^c$ and $\boldsymbol{\nu}_2^c$ from (2.12) and (2.13) to compute, assuming $\kappa = 1$,

$$\boldsymbol{\nu}_1^c(x) = \frac{2\cos(x)}{\mathrm{N}_c}\begin{pmatrix} 2\cos(\omega_c) \\ 2 - c^2\omega_c^2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\nu}_2^c(x) = \frac{2\sin(x)}{\mathrm{N}_c}\begin{pmatrix} 2\cos(\omega_c) \\ 2 - c^2\omega_c^2 \end{pmatrix}.$$

This shows $\mathcal{S}_{\mathbf{M}}\boldsymbol{\nu}_1^c = \boldsymbol{\nu}_1^c$ and $\mathcal{S}_{\mathbf{M}}\boldsymbol{\nu}_2^c = \boldsymbol{\nu}_2^c$ directly. Consequently, when we run the symmetric bifurcation argument for the mass dimer, we can just use the first case for $\boldsymbol{\nu}_\pm^c$ in Lemma 5.3.

5.3.2. Symmetry in the spring dimer

The spring dimer symmetry is

$$\mathcal{S}_{\mathbf{K}} := -RJ = -JR \tag{5.8}$$

with R defined in (5.4) and J defined in (5.5). The subscript is meant to emphasize the role of the linear spring coefficient ratio κ in the spring dimer analysis.

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Again, we just check that $\mathcal{G}_c(\mathcal{S}_{\mathbf{K}}\boldsymbol{\phi},\omega) = \mathcal{G}_c(\boldsymbol{\phi},\omega)$ in the case $w = m^{-1} = 1$, as the other symmetry properties from Definition 5.1 are evident. With $\mathcal{G}_c = c^2\mathcal{T} + \mathcal{P}$ and $\mathcal{T}$ defined in (3.6) and $\mathcal{P}$ in (3.7), we have

$$\frac{2}{\omega^2}\mathcal{T}(\mathcal{S}_{\mathbf{K}}\boldsymbol{\phi},\omega) = \langle -(RJ\boldsymbol{\phi})'', -RJ\boldsymbol{\phi}\rangle = \langle RJ\boldsymbol{\phi}'', RJ\boldsymbol{\phi}\rangle = \langle J\boldsymbol{\phi}'', J\boldsymbol{\phi}\rangle = \langle J^2\boldsymbol{\phi}'', \boldsymbol{\phi}\rangle = \langle \boldsymbol{\phi}'', \boldsymbol{\phi}\rangle$$
$$= \frac{2}{\omega^2}\mathcal{T}(\boldsymbol{\phi},\omega).$$

Here we have again used properties of R from Lemma 5.5 and also $J^* = J^{-1} = J$.

Next,

$$\mathcal{P}(\mathcal{S}_{\mathbf{K}}\boldsymbol{\phi},\omega) = \langle \boldsymbol{\mathcal{V}}(\Delta_+(\omega)\mathcal{S}_{\mathbf{K}}\boldsymbol{\phi}), \mathbf{1}\rangle,$$

with $\boldsymbol{\mathcal{V}}(\mathbf{p}) = (\mathcal{V}_1(p_1), \mathcal{V}_2(p_2))$ for $\mathbf{p} = (p_1,p_2)$, $\mathbf{1} = (1,1)$, and $\Delta_+(\omega)$ defined in (3.1). We have

$$\Delta_+(\omega)\mathcal{S}_{\mathbf{K}}\boldsymbol{\phi} = -\Delta_+(\omega)RJ = RJ\Delta_+(\omega)J$$

by (5.6) and (5.7), and so

$$\mathcal{P}(\mathcal{S}_{\mathbf{K}}\boldsymbol{\phi},\omega) = \langle \boldsymbol{\mathcal{V}}(RJ\Delta_+(\omega)J\boldsymbol{\phi}), \mathbf{1}\rangle = \langle \boldsymbol{\mathcal{V}}(J\Delta_+(\omega)J\boldsymbol{\phi}), \mathbf{1}\rangle.$$

We compute

$$J\Delta_+(\omega)J = \Lambda(\omega)\Delta_+(\omega), \qquad \Lambda(\omega) := \begin{bmatrix} S^{\omega} & 0 \\ 0 & S^{-\omega}\end{bmatrix},$$

and, for $\mathbf{p} = (p_1,p_2) \in L^2_{\mathrm{per}}(\mathbb{R}^2)$,

$$\langle \boldsymbol{\mathcal{V}}(\Lambda(\omega)\mathbf{p}), \mathbf{1}\rangle = \int_{-\pi}^{\pi} \mathcal{V}_1(S^{\omega}p_1) + \int_{-\pi}^{\pi} \mathcal{V}_2(S^{-\omega}p_2) = \int_{-\pi}^{\pi} \big(\mathcal{V}_1(p_1) + \mathcal{V}_2(p_2)\big) = \langle \boldsymbol{\mathcal{V}}(\mathbf{p}), \mathbf{1}\rangle.$$

We conclude

$$\mathcal{P}(\mathcal{S}_{\mathbf{K}}\boldsymbol{\phi},\omega) = \langle \boldsymbol{\mathcal{V}}(\Lambda(\omega)\Delta_+(\omega)\boldsymbol{\phi}), \mathbf{1}\rangle = \langle \boldsymbol{\mathcal{V}}(\Delta_+(\omega)\boldsymbol{\phi}), \mathbf{1}\rangle = \mathcal{P}(\boldsymbol{\phi},\omega).$$

Unlike in the mass dimer, it is not always the case that $\mathcal{S}_{\mathbf{K}}\boldsymbol{\nu}_1^c = \boldsymbol{\nu}_1^c$ for the spring dimer. Indeed, the situation is rather more complicated here, as we outline below. It is for this reason that we developed Lemma 5.3, which is unnecessarily elaborate for the mass dimer.

Lemma 5.6. *Assume* w = *1.*

(i) $\mathcal{S}_{\mathbf{K}}\boldsymbol{\nu}_1^c = \boldsymbol{\nu}_1^c$ *if and only if* $\omega_c = j\pi$ *for some even* $j \in \mathbb{Z}$.
(ii) $\mathcal{S}_{\mathbf{K}}\boldsymbol{\nu}_1^c = -\boldsymbol{\nu}_1^c$ *if and only if* $\omega_c = j\pi$ *for some odd* $j \in \mathbb{Z}$.

We prove this lemma in Appendix B.6. A consequence is that outside the isolated situations $\omega_c = j\pi$ for some $j \in \mathbb{Z}$, we must use the third, more complicated case of Lemma 5.3 to obtain symmetric eigenfunctions for the spring dimer.

We conclude this discussion of symmetry by noting that not all solutions to the travelling wave problem are symmetric. Indeed, since the travelling wave problem is shift invariant (part (ii) of

Corollary 3.2), any solution $\boldsymbol{\phi}$ to $\boldsymbol{\Phi}_c(\boldsymbol{\phi}, \omega) = 0$ generates other solutions $S^{\theta}\boldsymbol{\phi}$ for $\theta \in \mathbb{R}$. Still working in the spring dimer, suppose that $\boldsymbol{\phi}$ is symmetric with respect to $\mathcal{S}_{\mathbf{K}}$, so $\mathcal{S}_{\mathbf{K}}\boldsymbol{\phi} = \boldsymbol{\phi}$. We compute

$$\mathcal{S}_{\mathbf{K}} S^{\theta}\boldsymbol{\phi} = -JRS^{\theta}\boldsymbol{\phi} = -JS^{-\theta}R\boldsymbol{\phi} = S^{-\theta}(-JR)\boldsymbol{\phi} = S^{-\theta}\mathcal{S}_{\mathbf{K}}\boldsymbol{\phi} = S^{-\theta}\boldsymbol{\phi}.$$

Typically $S^{-\theta}\boldsymbol{\phi} \neq \boldsymbol{\phi}$ unless θ is an even integer multiple of π. Thus the shifted solution need not be symmetric.

6. Quantitative results

Our previous proofs have fixed the wave speed c to be greater than the speed of sound and yielded families of periodic solutions parametrized in 'amplitude', where the range of the amplitude has been allowed to depend on c. In Section 6.1, we develop tools to track dependence on c and its variation from the speed of sound. Such quantitative results have been essential to all of the existing nanopteron proofs that incorporate periodic solutions, and we expect the same to be necessary in any future constructions. We illustrate such an application in Section 6.2 for the long wave limit in dimer FPUT.

6.1. An abstract quantitative bifurcation theorem

We first prove a very abstract bifurcation result from which our quantitative result for lattice periodics follows easily. This result subsumes all of the existing quantitative periodic constructions for lattices – all of which, we emphasize, relied on symmetry to control (co)kernel dimensionality – and does not strictly depend on the long wave structure of the problem considered more broadly here. That is, we claim that any of the prior quantitative periodic proofs follows from Theorem 6.2 below.

We rely on the following fixed-point theorem, which was proved as [20, Lem. C.1].

Lemma 6.1. *For $0 < \epsilon < \epsilon_0$, let $\mathcal{X}^{\rho}$ be a Banach space and let $\mathcal{F}_{\epsilon} : \mathcal{X}^{\rho} \times \mathbb{R} \to \mathcal{X}^{\rho}$ be a family of maps. Suppose that for some C_0, a_0, $b_0 > 0$, if x, $\grave{x} \in \mathcal{X}^{\rho}$ and $a \in \mathbb{R}$ with $\|x\|_{\mathcal{X}^{\rho}}$, $\|\grave{x}\|_{\mathcal{X}^{\rho}} \leq b_0$ and $|a| \leq a_0$, then*

$$\|\mathcal{F}_{\epsilon}(x, a)\|_{\mathcal{X}^{\rho}} \leq C_0\big(|a| + \|x\|^2_{\mathcal{X}^{\rho}}\big) \tag{6.1}$$

and

$$\|\mathcal{F}_{\epsilon}(x, a) - \mathcal{F}_{\epsilon}(\grave{x}, a)\|_{\mathcal{X}^{\rho}} \leq C_0\big(\|x\|_{\mathcal{X}^{\rho}} + \|\grave{x}\|_{\mathcal{X}^{\rho}} + |a|\big)\|x - \grave{x}\|_{\mathcal{X}^{\rho}} \tag{6.2}$$

for all $0 < \epsilon < \epsilon_0$. Then there exist a_1, $r_1 > 0$ such that for each $|a| \leq a_1$ and $0 < \epsilon < \epsilon_0$, there is a unique $x^a_{\epsilon} \in \mathcal{X}^{\rho}$ with $x^a_{\epsilon} = \mathcal{F}_{\epsilon}(x^a_{\epsilon}, a)$ and $\|x^a_{\epsilon}\|_{\mathcal{X}^{\rho}} \leq r_1$.

Moreover, suppose that there is $L_0 > 0$ such that

$$\|\mathcal{F}_{\epsilon}(x, a) - \mathcal{F}_{\epsilon}(x, \grave{a})\|_{\mathcal{X}^{\rho}} \leq L_0|a - \grave{a}| \tag{6.3}$$

for all $x \in \mathcal{X}^{\rho}$ with $\|x\|_{\mathcal{X}^{\rho}} \leq b_0$, a, $\grave{a} \in \mathbb{R}$ with $|a|$, $|\grave{a}| \leq a_0$, and $0 < \epsilon < \epsilon_0$. Then there is $L_1 > 0$ such that

$$\|x^a_{\epsilon} - x^{\grave{a}}_{\epsilon}\|_{\mathcal{X}^{\rho}} \leq L_1|a - \grave{a}| \tag{6.4}$$

for all $0 < \epsilon < \epsilon_0$ and a, $\grave{a} \in \mathbb{R}$ with $|a|$, $|\grave{a}| \leq a_1$.

Here is our primary abstract result. It is very technical. We discuss the application of this result to our long wave problem in Section 6.2 below, but for now we encourage the reader to think of the

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map $\mathbf{\Phi}_\epsilon$ in the theorem as $\mathbf{\Phi}_{c_\epsilon}$ from (1.9) with $c_\epsilon^2 = c_\star^2 + \epsilon^2$ and to think of the spaces $\mathcal{X}^r$ below as $\left\{\boldsymbol{\phi} \in H^r_{\mathrm{per}}(\mathbb{R}^2) \,\middle|\, \langle \boldsymbol{\phi}, \boldsymbol{\nu}_0 \rangle = 0\right\}$ with the special case of $\rho = 2$. The conclusions of this theorem are a quantitative version of the results of Theorem 1.1 in a much more abstract context.

Theorem 6.2. *Let $\{\mathcal{X}^r\}_{r\geq 0}$ be a family of Hilbert spaces such that $\mathcal{X}^{r+s}$ is continuously embedded in $\mathcal{X}^r$ for each $s \geq 0$. Denote the inner product on $\mathcal{X}^r$ by $\langle\cdot,\cdot\rangle_r$ and, for simplicity, let $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_0$; denote the norm on $\mathcal{X}^r$ by $\|\cdot\|_r$. Suppose that for $0 < \epsilon < \epsilon_0$ and some $\rho > 0$, there is a map*

$$\mathbf{\Phi}_\epsilon : \mathcal{X}^\rho \times \mathbb{R} \to \mathcal{X}^0 : (\boldsymbol{\phi}, \omega) \mapsto \mathbf{\Phi}_\epsilon(\boldsymbol{\phi}, \omega)$$

with the following properties.

(i) *[Branch of trivial solutions] $\mathbf{\Phi}_\epsilon(0,\omega) = 0$ for all $\omega \in \mathbb{R}$.*

(ii) *[Regularity] The partial derivatives $D_{\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon$ and $D^2_{\boldsymbol{\phi}\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon$ exist and are continuous from $\mathcal{X}^\rho \times \mathbb{R}$ to $\mathcal{X}^0$, and the partial derivative $D^2_{\boldsymbol{\phi}\omega}\mathbf{\Phi}_\epsilon(0,\cdot)$ exists and is continuous on $\mathbb{R}$.*

(iii) *[(Co)kernel dimensionality] There is $\omega_\epsilon \in \mathbb{R}$ such that*

$$\ker(D_{\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon(0,\omega_\epsilon)) = \operatorname{span}(\boldsymbol{\nu}^\epsilon) \qquad \text{and} \qquad \ker(D_{\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon(0,\omega_\epsilon)^*) = \operatorname{span}(\boldsymbol{\mu}_1^\epsilon, \boldsymbol{\mu}_2^\epsilon)$$

for some vectors $\boldsymbol{\nu}^\epsilon$, $\boldsymbol{\mu}_1^\epsilon$, $\boldsymbol{\mu}_2^\epsilon \in \mathcal{X}^0$ with $\|\boldsymbol{\nu}^\epsilon\|_0 = \|\boldsymbol{\mu}_1^\epsilon\|_0 = 1$ (the case $\boldsymbol{\mu}_2^\epsilon = 0$ is allowed).

(iv) *[Uniform transversality] $\inf_{0<\epsilon<\epsilon_0} |\langle D^2_{\boldsymbol{\phi}\omega}\mathbf{\Phi}_\epsilon(0,\omega_\epsilon)\boldsymbol{\nu}^\epsilon, \boldsymbol{\mu}_1^\epsilon\rangle| > 0$.*

(v) *[Uniform coercivity] For each $r \geq 0$, there is $C_r > 0$ such that if $\boldsymbol{\psi} \in \mathcal{X}^{r+\rho}$ and $\boldsymbol{\eta} \in \mathcal{X}^r$ with*

$$D_{\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon(0,\omega_\epsilon)\boldsymbol{\psi} = \boldsymbol{\eta}, \quad \langle \boldsymbol{\psi}, \boldsymbol{\nu}^\epsilon\rangle = \langle \boldsymbol{\psi}, \boldsymbol{\mu}_1^\epsilon\rangle = \langle \boldsymbol{\psi}, \boldsymbol{\mu}_2^\epsilon\rangle = 0, \quad \text{and}$$
$$\langle \boldsymbol{\eta}, \boldsymbol{\nu}^\epsilon\rangle = \langle \boldsymbol{\eta}, \boldsymbol{\mu}_1^\epsilon\rangle = \langle \boldsymbol{\eta}, \boldsymbol{\mu}_2^\epsilon\rangle = 0,$$

then $\|\boldsymbol{\psi}\|_{r+\rho} \leq C_r\|\boldsymbol{\eta}\|_r$.

(vi) *[Bootstrapping] If $\boldsymbol{\phi} \in \mathcal{X}^\rho$ such that $D_{\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon(0,\omega_\epsilon)\boldsymbol{\phi} \in \mathcal{X}^r$ for some $r \geq 0$, then $\boldsymbol{\phi} \in \mathcal{X}^{r+\rho}$.*

(vii) *[Uniform mapping and Lipschitz estimates] There is $b_0 > 0$ such that the following estimates hold for each $r \geq 0$ (not necessarily uniformly in* r*):*

$$\sup_{0<\epsilon<\epsilon_0} \|D^2_{\boldsymbol{\phi}\omega}\mathbf{\Phi}_\epsilon(0,\omega_\epsilon)\|_{\mathcal{X}^{r+\rho}\to\mathcal{X}^r} < \infty,$$

$$\sup_{\substack{0<\epsilon<\epsilon_0 \\ |\omega-\omega_\epsilon|<b_0,\ |\grave{\omega}-\omega_\epsilon|<b_0 \\ \omega\neq\grave{\omega}}} \frac{\|D^2_{\boldsymbol{\phi}\omega}\mathbf{\Phi}_\epsilon(0,\omega) - D^2_{\boldsymbol{\phi}\omega}\mathbf{\Phi}_\epsilon(0,\grave{\omega})\|_{\mathcal{X}^{r+\rho}\to\mathcal{X}^r}}{|\omega-\grave{\omega}|} < \infty,$$

$$\sup_{\substack{0<\epsilon<\epsilon_0 \\ \|\boldsymbol{\phi}\|_{r+\rho}+|\omega-\omega_\epsilon|<b_0}} \|D^2_{\boldsymbol{\phi}\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon(\boldsymbol{\phi},\omega)\|_{\mathcal{X}^{r+\rho}\times\mathcal{X}^{r+\rho}\to\mathcal{X}^r} < \infty$$

and

$$\sup_{\substack{0<\epsilon<\epsilon_0 \\ \|\boldsymbol{\phi}\|_{r+\rho}+|\omega-\omega_\epsilon|<b_0,\ \|\grave{\boldsymbol{\phi}}\|_{r+\rho}+|\grave{\omega}-\omega_\epsilon|<b_0 \\ (\boldsymbol{\phi},\omega)\neq(\grave{\boldsymbol{\phi}},\grave{\omega})}} \frac{\|D^2_{\boldsymbol{\phi}\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon(\boldsymbol{\phi},\omega) - D^2_{\boldsymbol{\phi}\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon(\grave{\boldsymbol{\phi}},\grave{\omega})\|_{\mathcal{X}^{r+\rho}\times\mathcal{X}^{r+\rho}\to\mathcal{X}^r}}{\|\boldsymbol{\phi}-\grave{\boldsymbol{\phi}}\|_{r+\rho} + |\omega-\grave{\omega}|} < \infty.$$

(viii) *If $\boldsymbol{\mu}_2^\epsilon \neq 0$, then there are a Banach space $\mathcal{W}_\epsilon$ with $\mathcal{X}^\rho \subseteq \mathcal{W}_\epsilon \subseteq \mathcal{X}^0$ and a nonzero linear operator $\mathcal{T}_\epsilon : \mathcal{W}_\epsilon \to \mathcal{X}^0$ with the following properties.*

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- $\langle \mathbf{\Phi}_\epsilon(\boldsymbol{\phi},\omega), \mathcal{T}_\epsilon \boldsymbol{\phi}\rangle = 0$ *for all* $\boldsymbol{\phi} \in \mathcal{X}^\rho$ *and* $\omega \in \mathbb{R}$.
- $\mathcal{T}_\epsilon \boldsymbol{\nu}^\epsilon = \boldsymbol{\mu}_1^\epsilon$.
- *There is* $\tau_\epsilon \in \mathbb{R} \setminus \{0\}$ *such that* $\mathcal{T}_\epsilon \boldsymbol{\mu}_1^\epsilon = \pm\tau_\epsilon \boldsymbol{\mu}_2^\epsilon$ *and* $\mathcal{T}_\epsilon \boldsymbol{\mu}_2^\epsilon = \mp\tau_\epsilon \boldsymbol{\mu}_1^\epsilon$.

Then there is $a_\star > 0$ *such that for* $|a| < a_\star$ *and* $0 < \epsilon < \epsilon_0$, *there exist* $\boldsymbol{\phi}_\epsilon^a \in \cap_{r=0}^\infty \mathcal{X}^r$ *and* $\omega_\epsilon^a \in \mathbb{R}$ *such that* $\mathbf{\Phi}_\epsilon(\boldsymbol{\phi}_\epsilon^a, \omega_\epsilon^a) = 0$ *with*

$$\boldsymbol{\phi}_\epsilon^a = a(\boldsymbol{\nu}^\epsilon + \boldsymbol{\psi}_\epsilon^a), \qquad \langle \boldsymbol{\psi}_\epsilon^a, \boldsymbol{\nu}^\epsilon\rangle = 0, \qquad \boldsymbol{\psi}_\epsilon^0 = 0 \qquad \text{and} \qquad \omega_\epsilon^a = \omega_\epsilon + \xi_\epsilon^a, \qquad \xi_\epsilon^0 = 0.$$

The following mapping and Lipschitz estimates also hold for each r *(not necessarily uniformly in* r*):*

$$\sup_{\substack{0<\epsilon<\epsilon_0 \\ |a|<a_\star}} \|\boldsymbol{\psi}_\epsilon^a\|_r + |\xi_\epsilon^a| < \infty \qquad \text{and} \qquad \sup_{\substack{0<\epsilon<\epsilon_0 \\ |a|<a_\star,\ |\grave{a}|<a_\star \\ a\neq\grave{a}}} \frac{\|\boldsymbol{\psi}_\epsilon^a - \boldsymbol{\psi}_\epsilon^{\grave{a}}\|_r}{|a-\grave{a}|} + \frac{|\xi_\epsilon^a - \xi_\epsilon^{\grave{a}}|}{|a-\grave{a}|} < \infty. \tag{6.5}$$

Proof. We break the proof into several steps.

(1) *The Lyapunov–Schmidt reduction.* Since $\mathbf{\Phi}_\epsilon(0,\omega) = 0$ for all ω by Hypothesis (i) and $D^2_{\boldsymbol{\phi}\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon$ exists and is continuous on $\mathcal{X}^\rho \times \mathbb{R}$ by Hypothesis (ii), the fundamental theorem of calculus implies

$$\mathbf{\Phi}_\epsilon(\boldsymbol{\phi},\omega) = D_{\boldsymbol{\phi}}(0,\omega)\boldsymbol{\phi} + \int_0^1\int_0^1 tD^2_{\boldsymbol{\phi}\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon(st\boldsymbol{\phi},\omega)[\boldsymbol{\phi},\boldsymbol{\phi}]\,ds\,dt$$

for all $\boldsymbol{\phi} \in \mathcal{X}^\rho$ and $\omega \in \mathbb{R}$. Next, since $D^2_{\boldsymbol{\phi}\omega}\mathbf{\Phi}_\epsilon(0,\cdot)$ exists and is continuous on $\mathbb{R}$ by Hypothesis (ii) again, another application of the fundamental theorem of calculus yieldsm

$$D_{\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon(0,\omega+\xi) = D_{\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon(0,\omega) + \xi D^2_{\boldsymbol{\phi}\omega}\mathbf{\Phi}_\epsilon(0,\omega) + \int_0^1 \xi\big(D^2_{\boldsymbol{\phi}\omega}\mathbf{\Phi}_\epsilon(0,\omega+t\xi) - D^2_{\boldsymbol{\phi}\omega}\mathbf{\Phi}_\epsilon(0,\omega)\big)\,dt$$

for all ω, $\xi \in \mathbb{R}$. Together with $D_{\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon(0,\omega_\epsilon)\boldsymbol{\nu}^\epsilon = 0$ from Hypothesis (iii), these two expansions give

$$\mathbf{\Phi}_\epsilon(a(\boldsymbol{\nu}^\epsilon+\boldsymbol{\psi}),\omega_\epsilon+\xi) = aD_{\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon(0,\omega_\epsilon)\boldsymbol{\psi} + a\xi D^2_{\boldsymbol{\phi}\omega}\mathbf{\Phi}_\epsilon(0,\omega_\epsilon)\boldsymbol{\nu}^\epsilon - a\mathcal{R}_\epsilon(\boldsymbol{\psi},\xi,a) \tag{6.6}$$

for all $\boldsymbol{\psi} \in \mathcal{X}^\rho$ with $\langle \boldsymbol{\psi}, \boldsymbol{\nu}^\epsilon\rangle = 0$ and ξ, $a \in \mathbb{R}$, where

$$\begin{aligned}\mathcal{R}_\epsilon(\boldsymbol{\psi},\xi,a) := &-\xi D^2_{\boldsymbol{\phi}\omega}\mathbf{\Phi}_\epsilon(0,\omega_\epsilon)\boldsymbol{\psi} \\ &- a\int_0^1\int_0^1 tD^2_{\boldsymbol{\phi}\boldsymbol{\phi}}\mathbf{\Phi}_\epsilon(sta(\boldsymbol{\nu}^\epsilon+\boldsymbol{\psi}),\omega_\epsilon+\xi)[\boldsymbol{\nu}^\epsilon+\boldsymbol{\psi},\boldsymbol{\nu}^\epsilon+\boldsymbol{\psi}]\,ds\,dt \\ &- \xi\int_0^1 \big(D^2_{\boldsymbol{\phi}\omega}\mathbf{\Phi}_\epsilon(0,\omega_\epsilon+t\xi) - D^2_{\boldsymbol{\phi}\omega}\mathbf{\Phi}_\epsilon(0,\omega_\epsilon)\big)(\boldsymbol{\nu}^\epsilon+\boldsymbol{\psi})\,dt.\end{aligned} \tag{6.7}$$

We will use the expansion (6.6) to obtain a pair of fixed-point equations for $\boldsymbol{\psi}$ and ξ. Put

$$\Pi_\epsilon\boldsymbol{\phi} := \langle \boldsymbol{\phi}, \boldsymbol{\mu}_1^\epsilon\rangle\boldsymbol{\mu}_1^\epsilon + \langle \boldsymbol{\phi}, \boldsymbol{\mu}_2^\epsilon\rangle\boldsymbol{\mu}_2^\epsilon. \tag{6.8}$$

Then $\mathbf{\Phi}_\epsilon(a(\boldsymbol{\nu}^\epsilon+\boldsymbol{\psi}),\omega_\epsilon+\xi) = 0$ if and only if

$$\begin{cases}(\mathcal{I}_{\mathcal{X}^0} - \Pi_\epsilon)\mathbf{\Phi}_\epsilon(a(\boldsymbol{\nu}^\epsilon+\boldsymbol{\psi}),\omega_\epsilon+\xi) = 0 & (6.9a) \\ \Pi_\epsilon\mathbf{\Phi}_\epsilon(a(\boldsymbol{\nu}^\epsilon+\boldsymbol{\psi}),\omega_\epsilon+\xi) = 0. & (6.9b)\end{cases} \tag{6.9}$$

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(2) *The preliminary equation for $\boldsymbol{\psi}$.* It follows from the expansion (6.6) that (6.9a) is equivalent to

$$(\mathcal{I}_{\mathcal{X}^0} - \Pi_\epsilon) D_{\boldsymbol{\phi}} \boldsymbol{\Phi}_\epsilon(0, \omega_\epsilon)\boldsymbol{\psi} = (\mathcal{I}_{\mathcal{X}^0} - \Pi_\epsilon)\big(\mathcal{R}_\epsilon(\boldsymbol{\psi}, \xi, a) - \xi D^2_{\boldsymbol{\phi}\omega} \boldsymbol{\Phi}_\epsilon(0, \omega_\epsilon)\boldsymbol{\psi}\big). \tag{6.10}$$

Put

$$\mathcal{X}^\infty_\epsilon := \big\{\boldsymbol{\psi} \in \mathcal{X}^\rho \,\big|\, \langle \boldsymbol{\psi}, \boldsymbol{\nu}^\epsilon \rangle = 0\big\} \qquad \text{and} \qquad \mathcal{Y}^\infty_\epsilon := (\mathcal{I}_{\mathcal{X}^0} - \Pi_\epsilon)(\mathcal{X}^0). \tag{6.11}$$

Since, by Hypothesis (iii), $D_{\boldsymbol{\phi}} \boldsymbol{\Phi}_\epsilon(0, \omega_\epsilon)$ has trivial kernel on $\mathcal{X}^\infty_\epsilon$ and trivial cokernel in $\mathcal{Y}^\infty_\epsilon$, for each $\boldsymbol{\eta} \in \mathcal{Y}^\infty_\epsilon$, there is a unique $\boldsymbol{\psi} \in \mathcal{X}^\infty_\epsilon$ such that $D_{\boldsymbol{\phi}}(0, \omega_\epsilon)\boldsymbol{\psi} = \boldsymbol{\eta}$. We write $\boldsymbol{\psi} := D_{\boldsymbol{\phi}}(0, \omega_\epsilon)^{-1}\boldsymbol{\eta}$. With this notation, (6.10) is equivalent to

$$\boldsymbol{\psi} = D_{\boldsymbol{\phi}} \boldsymbol{\Phi}_\epsilon(0, \omega_\epsilon)^{-1} (\mathcal{I}_{\mathcal{X}^0} - \Pi_\epsilon)\big(\mathcal{R}_\epsilon(\boldsymbol{\psi}, \xi, a) - \xi D^2_{\boldsymbol{\phi}\omega} \boldsymbol{\Phi}_\epsilon(0, \omega_\epsilon)\boldsymbol{\nu}^\epsilon\big). \tag{6.12}$$

This is our preliminary fixed-point equation for $\boldsymbol{\psi}$, but it will need some subsequent modification, as the term $\xi D^2_{\boldsymbol{\phi}\omega} \boldsymbol{\Phi}_\epsilon(0, \omega_\epsilon)\boldsymbol{\nu}^\epsilon$ is formally $\mathcal{O}(1)$ in ξ and thus not suitably small for contractive purposes.

(3) *The preliminary equation for ξ.* We now turn our attention to the second, finite-dimensional equation (6.9). From Hypothesis (iii), we have

$$\langle D_{\boldsymbol{\phi}} \boldsymbol{\Phi}_\epsilon(0, \omega_\epsilon)\boldsymbol{\psi}, \boldsymbol{\mu}^\epsilon_j \rangle = \langle D_{\boldsymbol{\phi}} \boldsymbol{\Phi}_\epsilon(0, \omega_\epsilon)\boldsymbol{\psi}, \boldsymbol{\mu}^\epsilon_j \rangle_0 = \langle \boldsymbol{\psi}, D_{\boldsymbol{\phi}} \boldsymbol{\Phi}_\epsilon(0, \omega_\epsilon)^* \boldsymbol{\mu}^\epsilon_j \rangle_\rho = 0, \ j = 1, \ 2, \tag{6.13}$$

If $\boldsymbol{\mu}^\epsilon_2 \neq 0$, the argument in Appendix B.5.1 that proved Lemma 4.3 can be adapted (take $\mathcal{T}_\epsilon = \partial_x$) using the properties of $\mathcal{T}_\epsilon$ in Hypothesis (viii) to show

$$\langle D^2_{\boldsymbol{\phi}\omega} \boldsymbol{\Phi}_\epsilon(0, \omega_\epsilon)\boldsymbol{\nu}^\epsilon, \boldsymbol{\mu}^\epsilon_2 \rangle = 0. \tag{6.14}$$

The calculations (6.13) and (6.14) then imply that (6.9) is equivalent to the two equations

$$\begin{cases} \xi \langle D^2_{\boldsymbol{\phi}\omega} \boldsymbol{\Phi}_\epsilon(0, \omega_\epsilon)\boldsymbol{\nu}^\epsilon, \boldsymbol{\mu}^\epsilon_1 \rangle = \langle \mathcal{R}_\epsilon(\boldsymbol{\psi}, \xi, a), \boldsymbol{\mu}^\epsilon_1 \rangle & (6.15a) \\ \langle \mathcal{R}_\epsilon(\boldsymbol{\psi}, \xi, a), \boldsymbol{\mu}^\epsilon_2 \rangle = 0. & (6.15b) \end{cases}$$

( )

Hypothesis (iv) implies that (6.15) is equivalent to

$$\xi = \mathcal{P}_\epsilon \mathcal{R}_\epsilon(\boldsymbol{\psi}, \xi, a), \qquad \mathcal{P}_\epsilon \boldsymbol{\eta} := \frac{\langle \boldsymbol{\eta}, \boldsymbol{\mu}^\epsilon_1 \rangle}{\langle D^2_{\boldsymbol{\phi}\omega} \boldsymbol{\Phi}_\epsilon(0, \omega_\epsilon)\boldsymbol{\nu}^\epsilon, \boldsymbol{\mu}^\epsilon_1 \rangle} \tag{6.16}$$

This is our preliminary fixed-point equation for ξ, but, like the preliminary equation for $\boldsymbol{\psi}$, it too needs some adjustment. The problem here is that estimates on $\mathcal{R}_\epsilon(\boldsymbol{\psi}, \xi, a)$ in $\| \cdot \|_r$ will depend on estimates in $\boldsymbol{\psi}$ in $\| \cdot \|_{r+\rho}$, and so we will not get estimates within the same norm for contractive purposes.

(4) *The final fixed-point system.* Put

$$\boldsymbol{\Psi}_\epsilon(\boldsymbol{\psi}, \xi, a) := D_{\boldsymbol{\phi}} \boldsymbol{\Phi}_\epsilon(0, \omega_\epsilon)^{-1} (\mathcal{I}_{\mathcal{X}^0} - \Pi_\epsilon)\big[\mathcal{R}_\epsilon(\boldsymbol{\psi}, \xi, a) - \big(\mathcal{P}_\epsilon \mathcal{R}_\epsilon(\boldsymbol{\psi}, \xi, a)\big) D^2_{\boldsymbol{\phi}\omega} \boldsymbol{\Phi}_\epsilon(0, \omega_\epsilon)\boldsymbol{\nu}^\epsilon\big] \tag{6.17}$$

and

$$\Xi_\epsilon(\boldsymbol{\psi}, \xi, a) := \mathcal{P}_\epsilon \mathcal{R}_\epsilon(\boldsymbol{\Psi}_\epsilon(\boldsymbol{\psi}, \xi, a), \xi, a). \tag{6.18}$$

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Then the preliminary fixed-fixed point equations (6.12) and (6.16) for $\boldsymbol{\psi} \in \mathcal{X}_\epsilon^\infty$ and $\xi \in \mathbb{R}$ are equivalent to

$$\begin{cases} \boldsymbol{\psi} = \boldsymbol{\Psi}_\epsilon(\boldsymbol{\psi}, \xi, a) \\ \xi = \Xi_\epsilon(\boldsymbol{\psi}, \xi, a), \end{cases} \tag{6.19}$$

and this system will turn out to have the right contraction estimates.

(5) *Solving the third equation (6.15).* Before we solve (6.19) with a quantitative contraction mapping argument that is uniform in ϵ and a, we need to be sure that solutions $\boldsymbol{\psi}$ and ξ to (6.19) really do yield solutions to our original problem $\boldsymbol{\Phi}_\epsilon(a(\boldsymbol{\nu}^\epsilon + \boldsymbol{\psi}), \omega_\epsilon + \xi) = 0$. That is, we need to show that solutions to (6.19) also meet the third equation (6.15). Certainly this third equation is met if $\boldsymbol{\mu}_2^\epsilon = 0$, so assume $\boldsymbol{\mu}_2^\epsilon \neq 0$ and invoke Hypothesis (viii).

We first redo the proof of Lemma 3.5 with ∂_x replaced by $\mathcal{T}_\epsilon$ to show that $\mathcal{T}_\epsilon$ and Π_ϵ commute. Next, we use the equivalence of (6.17) and (6.10) to replicate the calculation in (3.29) and conclude that if $\boldsymbol{\psi} = \boldsymbol{\Psi}_\epsilon(\boldsymbol{\psi}, \xi, a)$, then

$$\pm a\tau_\epsilon \langle \boldsymbol{\Phi}_\epsilon(a(\boldsymbol{\nu}^\epsilon + \boldsymbol{\psi}), \omega_\epsilon + \xi), \boldsymbol{\mu}_2^\epsilon \rangle = 0.$$

Since $\tau_\epsilon \neq 0$, we have

$$\langle \boldsymbol{\Phi}_\epsilon(a(\boldsymbol{\nu}^\epsilon + \boldsymbol{\psi}), \omega_\epsilon + \xi), \boldsymbol{\mu}_2^\epsilon \rangle = 0 \tag{6.20}$$

for all $\boldsymbol{\psi}$, ξ, and $a \neq 0$ with $\boldsymbol{\psi} = \boldsymbol{\Psi}_\epsilon(\boldsymbol{\psi}, \xi, a)$. Finally, for $a \neq 0$, by (6.6) we have

$$\mathcal{R}_\epsilon(\boldsymbol{\psi}, \xi, a) = D_{\boldsymbol{\phi}}\boldsymbol{\Phi}_\epsilon(0, \omega_\epsilon)\boldsymbol{\psi} + \xi D^2_{\boldsymbol{\phi}\omega}\boldsymbol{\Phi}_\epsilon(0, \omega_\epsilon)\boldsymbol{\nu}^\epsilon - a^{-1}\boldsymbol{\Phi}_\epsilon(a(\boldsymbol{\nu}^\epsilon + \boldsymbol{\psi}), \omega_\epsilon + \xi).$$

Combining (6.13), (6.14) and (6.20) yields $\langle \mathcal{R}_2^\epsilon(\boldsymbol{\psi}, \xi, a), \boldsymbol{\mu}_2^\epsilon \rangle = 0$. This is (6.15).

(6) *Applying Lemma 6.1.* To solve the fixed-point problem (6.19) and consequently our original problem, we will apply this lemma to the family of maps

$$\mathcal{F}_\epsilon \colon (\mathcal{X}_\epsilon^\infty \times \mathbb{R}) \times \mathbb{R} \to \mathcal{X}_\epsilon^\infty \times \mathbb{R} \colon (\boldsymbol{\psi}, \xi, a) \mapsto \big(\boldsymbol{\Psi}_\epsilon(\boldsymbol{\psi}, \xi, a), \Xi_\epsilon(\boldsymbol{\psi}, \xi, a)\big)$$

with $\mathcal{X}_\epsilon^\infty$ defined in (6.11). We put $\|(\boldsymbol{\psi}, \xi)\|_r := \|\boldsymbol{\psi}\|_r + |\xi|$.

All of our estimates for $\mathcal{F}_\epsilon$ are ultimately based on estimates for $\mathcal{R}_\epsilon$ from (6.7). We provide these estimates in the arbitrary norm $\|\cdot\|_r$ for the sake of 'bootstrapping' later. Let b_0 be as in Hypothesis (vii) and $r \geq 0$. The estimates from that hypothesis provide $C_r > 0$ such that, if $0 < \epsilon < \epsilon_0$, $\|\boldsymbol{\psi}\|_{r+\rho}$, $\|\grave{\boldsymbol{\psi}}\|_{r+\rho}$, $|a|$, $|\grave{a}| \leq b_0/2$ and $|\xi|$, $|\grave{\xi}| \leq b_0$, the following mapping and Lipschitz estimates hold:

$$\|\mathcal{R}_\epsilon(\boldsymbol{\psi}, \xi, a)\|_r \leq C_r\big(\|\boldsymbol{\psi}\|_{r+\rho}^2 + |\xi|^2 + |a|\big),$$

$$\|\mathcal{R}_\epsilon(\boldsymbol{\psi}, \xi, a) - \mathcal{R}_\epsilon(\grave{\boldsymbol{\psi}}, \grave{\xi}, a)\|_r \leq C_r\big(\|\boldsymbol{\psi}\|_{r+\rho} + \|\grave{\boldsymbol{\psi}}\|_{r+\rho} + |\xi| + |\grave{\xi}| + |a|\big)\big(\|\boldsymbol{\psi} - \grave{\boldsymbol{\psi}}\|_{r+\rho} + |\xi - \grave{\xi}|\big),$$

and

$$\|\mathcal{R}_\epsilon(\boldsymbol{\psi}, \xi, a) - \mathcal{R}_\epsilon(\boldsymbol{\psi}, \xi, \grave{a})\|_r \leq C_r|a - \grave{a}|.$$

With $r = 0$, the transversality estimate from Hypothesis (iv) and the 'smoothing' estimate from Hypothesis (v) then imply

$$\|\boldsymbol{\Psi}_\epsilon(\boldsymbol{\psi}, \xi, a)\|_\rho \leq C\big(\|\boldsymbol{\psi}\|_\rho^2 + |\xi|^2 + |a|\big), \tag{6.21}$$

$$\|\boldsymbol{\Psi}_\epsilon(\boldsymbol{\psi},\xi,a)-\boldsymbol{\Psi}_\epsilon(\grave{\boldsymbol{\psi}},\grave{\xi},a)\|_\rho \le C\big(\|\boldsymbol{\psi}\|_\rho+\|\grave{\boldsymbol{\psi}}\|_\rho+|\xi|+|\grave{\xi}|+|a|\big)\big(\|\boldsymbol{\psi}-\grave{\boldsymbol{\psi}}\|_\rho+|\xi-\grave{\xi}|\big) \quad (6.22)$$

and

$$\|\boldsymbol{\Psi}_\epsilon(\boldsymbol{\psi},\xi,a)-\boldsymbol{\Psi}_\epsilon(\boldsymbol{\psi},\xi,\grave{a})\|_\rho \le C|a-\grave{a}|. \quad (6.23)$$

Since

$$|\Xi_\epsilon(\boldsymbol{\psi},\xi,a)| \le C\|\mathcal{R}_\epsilon(\boldsymbol{\Psi}_\epsilon(\boldsymbol{\psi},\xi,a))\|_0$$

and

$$|\Xi_\epsilon(\boldsymbol{\psi},\xi,a)-\Xi_\epsilon(\grave{\boldsymbol{\psi}},\grave{\xi},\grave{a})| \le C\|\mathcal{R}_\epsilon(\boldsymbol{\Psi}_\epsilon(\boldsymbol{\psi},\xi,a))-\mathcal{R}_\epsilon(\boldsymbol{\Psi}_\epsilon(\grave{\boldsymbol{\psi}},\grave{\xi},\grave{a})\|_0,$$

the estimates (6.21), (6.22) and (6.23) hold with $\boldsymbol{\Psi}_\epsilon$ replaced by Ξ_ϵ (and the norm $\|\cdot\|_\rho$ on the left replaced by absolute value). It follows that on the space $\mathcal{X}_\epsilon^\infty \times \mathbb{R}$, the map $\mathcal{F}_\epsilon$ meets the estimates (6.1), (6.2) and (6.3) from Lemma 6.1. By that lemma, there are solutions $(\boldsymbol{\psi}_\epsilon^a, \xi_\epsilon^a)$ meeting $(\boldsymbol{\psi}_\epsilon^a, \xi_\epsilon^a) = \mathcal{F}_\epsilon(\boldsymbol{\psi}_\epsilon^a, \xi_\epsilon^a, a)$ and the mapping and Lipschitz estimates in (6.5) for $r=\rho$.

We then 'bootstrap' on $\boldsymbol{\psi}$ using the equation $\boldsymbol{\psi} = \boldsymbol{\Psi}_\epsilon(\boldsymbol{\psi},\xi,a)$, the definition of $\boldsymbol{\Psi}_\epsilon$ in (6.17), and Hypothesis (vi) to conclude that $\boldsymbol{\psi} \in \mathcal{X}^{n\rho}$ for any integer $n \ge 1$. Using the estimates on $\mathcal{R}_\epsilon$ above, which are valid for any r, and inducting, we obtain the estimates in (6.5) for $r = n\rho$. Interpolating, we conclude that $\boldsymbol{\psi} \in \mathcal{X}^r$ for all r and obtain the estimates in (6.5) for r arbitrary.

□

6.2. *Application to periodic travelling waves in dimer FPUT*

For long wave solutions, we are interested in rescaling the profiles as $\boldsymbol{\phi}(x) = \epsilon^2\boldsymbol{\varphi}(\epsilon x)$, where $\epsilon > 0$ measures the distance between the speed of sound $c_\star$ from (2.8) and the chosen wave speed c via $c^2 = c_\star^2 + \epsilon^2$. In [16, 20], the travelling wave problem was solved under this rescaling with the help of symmetry; here we obtain those long wave solutions as a consequence of Theorem 6.2 by introducing a rescaling of the amplitude parameter.

Specifically, let $\epsilon_0 = 1$ and let c_ϵ satisfy $c_\epsilon^2 = c_\star^2 + \epsilon^2$. Let

$$\mathcal{X}^r := \big\{\boldsymbol{\phi} \in H_{\mathrm{per}}^r(\mathbb{R}^2) \;\big|\; \langle \boldsymbol{\phi}, \boldsymbol{\nu}_0\rangle = 0\big\}$$

and set

$$\boldsymbol{\nu}_\epsilon := \boldsymbol{\nu}_1^{c_\epsilon}, \qquad \boldsymbol{\mu}_1^\epsilon := \boldsymbol{\nu}_1^{c_\epsilon}, \qquad \text{and} \qquad \boldsymbol{\mu}_2^\epsilon := \boldsymbol{\nu}_2^{c_\epsilon}.$$

Assume now that the spring potentials satisfy $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{C}^\infty(\mathbb{R})$. Then the map $\boldsymbol{\Phi}_{c_\epsilon}$ from (1.9) meets all of the hypotheses of Theorem 6.2. More precisely, Hypothesis (ii) follows from Lemma 3.6, Hypothesis (iii) from Corollary 2.2, Hypothesis (iv) from Corollary 2.3 and Hypothesis (v) from Corollary 2.4. The mapping and Lipschitz estimates in Hypothesis (vii) follow from the regularity properties of shift operators in Appendix A.4 and composition operators in Appendix A.5 and the uniform bounds on ω_c in c from (2.9).

We thus obtain solutions $\boldsymbol{\phi} = \boldsymbol{\phi}_{c_\epsilon}^a$ and $\omega = \omega_{c_\epsilon}^a$ to $\boldsymbol{\Phi}_{c_\epsilon}(\boldsymbol{\phi}_{c_\epsilon}^a, \omega_{c_\epsilon}^a) = 0$ for $0 < \epsilon < \epsilon_0$ and $|a| \le a_{\mathrm{per}}$ for some $a_{\mathrm{per}} > 0$. Returning to our original position coordinates, we see that

$$\mathbf{p}_\epsilon^a(x) := a\boldsymbol{\phi}_{c_\epsilon}^a(\omega_{c_\epsilon}^a x)$$

solves the original travelling wave problem (1.5).

Now we expose the long wave scaling. Write a in the form $a = \alpha\epsilon^2$ for $|\alpha| \leq a_{\text{per}}$ and $0 < \epsilon < 1$; this ensures $|a| < a_{\text{per}}$ and $0 < \epsilon < \epsilon_0$. Put

$$\boldsymbol{\varphi}_\epsilon^\alpha := \boldsymbol{\phi}_\epsilon^{\alpha\epsilon^2} \qquad \text{and} \qquad \Omega_\epsilon^\alpha := \frac{\omega_{c_\epsilon}^{\alpha\epsilon^2}}{\epsilon}.$$

Then the solutions to (1.5) have the form

$$\mathbf{p}_\epsilon^a(x) = \alpha\epsilon^2\boldsymbol{\varphi}_\epsilon^\alpha(\epsilon\Omega_\epsilon^\alpha x),$$

which reveals the long wave scaling. Moreover, these solutions have the same mapping and Lipschitz estimates previously established in [20, Thm. 4.1] and [16, Thm. 3.1]. For the frequency, the mapping and Lipschitz estimates from (6.5) and the bounds on ω_{c_ϵ} from (2.9) give

$$\sup_{\substack{0<\epsilon<\epsilon_0 \\ |\alpha|<a_{\text{per}}}} |\epsilon\Omega_\epsilon^\alpha| = \sup_{\substack{0<\epsilon<\epsilon_0 \\ |\alpha|<a_{\text{per}}}} |\omega_{c_\epsilon}^{\alpha\epsilon^2}| < \infty$$

and, for $0 < \epsilon < \epsilon_0$ and $|\alpha|, |\grave{\alpha}| < a_{\text{per}}$

$$|\Omega_\epsilon^\alpha - \Omega_\epsilon^{\grave{\alpha}}| = \frac{|\omega_{c_\epsilon}^{\alpha\epsilon^2} - \omega_{c_\epsilon}^{\grave{\alpha}\epsilon^2}|}{\epsilon} \leq \frac{C|\alpha - \grave{\alpha}|\epsilon^2}{\epsilon} = C\epsilon|\alpha - \grave{\alpha}|,$$

where $C > 0$ is independent of ϵ, α, and $\grave{\alpha}$. This Lipschitz estimate is an improvement on the original $\mathcal{O}(1)$ Lipschitz estimates from [20, Thm. 4.1] and [16, Thm. 3.1].

For the profile, we first introduce the norm

$$\|\boldsymbol{\phi}\|_{\mathcal{C}_{\text{per}}^r} := \|\boldsymbol{\phi}\|_{L^\infty} + \|\partial_x^r[\boldsymbol{\phi}]\|_{L^\infty}$$

for r-times continuously differentiable, 2π-periodic functions $\boldsymbol{\phi}$. The mapping and Lipschitz estimates

$$\sup_{\substack{0<\epsilon<\epsilon_0 \\ |\alpha|<a_{\text{per}}}} \|\boldsymbol{\varphi}_\epsilon^\alpha\|_{\mathcal{C}_{\text{per}}^r} < \infty \qquad \text{and} \qquad \sup_{0<\epsilon<\epsilon_0} \|\boldsymbol{\varphi}_\epsilon^\alpha - \boldsymbol{\varphi}_\epsilon^{\grave{\alpha}}\|_{\mathcal{C}_{\text{per}}^r} < C_r|\alpha - \grave{\alpha}|$$

follow again from (6.5) and the Sobolev embedding.

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Appendix A. Fourier analysis

A.1. Vectors and matrices

The following is wholly standard, but we include it in the hopes of completeness and clarity. For $\mathbf{v}$, $\mathbf{w} \in \mathbb{C}^n$, we set

$$\mathbf{v} \cdot \mathbf{w} := \sum_{k=1}^{n} v_k \overline{w_k} \qquad \text{and} \qquad |\mathbf{v}|_2 := \big(\mathbf{v} \cdot \mathbf{v}\big)^{1/2}.$$

Also, we define $\overline{\mathbf{v}} \in \mathbb{C}^n$ to be the vector whose entries are the conjugates of those in $\mathbf{v} \in \mathbb{C}^n$, and likewise if $A \in \mathbb{C}^{m \times n}$ (where $\mathbb{C}^{m \times n}$ is the space of all $m \times n$ matrices with entries in $\mathbb{C}$), then $\overline{A} \in \mathbb{C}^{m \times n}$ is the matrix whose entries are the conjugates of those in A. We denote by $A^* \in \mathbb{C}^{n \times m}$ the conjugate transpose of A.

For a matrix $A \in \mathbb{C}^{m \times n}$, we put

$$|A|_2 = \max_{\substack{\mathbf{v} \in \mathbb{C}^n \\ |\mathbf{v}|_2 = 1}} |A\mathbf{v}|_2 \qquad \text{and} \qquad |A|_\infty = \max_{\substack{1 \le i \le m \\ 1 \le j \le n}} |A_{ij}|$$

with A_{ij} as the entries of A. Then we have the inequalities

$$|A\mathbf{v}|_2 \le |A|_2 |\mathbf{v}|_2, \ \mathbf{v} \in \mathbb{C}^n, \qquad \text{and} \qquad |A|_2 \le \sqrt{mn}|A|_\infty. \tag{A.1}$$

Let $I_n \in \mathbb{C}^{n \times n}$ be the identity matrix. If $|A|_2 < 1$, then $I_n - A$ is invertible by the Neumann series, and

$$|(I_2 - A)^{-1}|_2 \le \frac{1}{1 - |A|_2}.$$

A.2. Periodic Sobolev spaces

This material is developed in [33, Sec. 8.1], [28] and [15, App. C.2]. Let $L^2_{\text{per}}(\mathbb{C}^n)$ be the completion of

$$\mathcal{C}^\infty_{\text{per}}(\mathbb{C}^n) := \big\{\phi \in \mathcal{C}^\infty([-\pi, \pi], \mathbb{C}^n) \,\big|\, \phi(-\pi) = \phi(\pi)\big\}$$

under the norm

$$\|\boldsymbol{\phi}\|_{L^2_{\text{per}}(\mathbb{C}^n)} := \big(\langle \boldsymbol{\phi}, \boldsymbol{\phi} \rangle_{L^2_{\text{per}}}\big)^{1/2}, \qquad \langle \boldsymbol{\phi}, \boldsymbol{\eta} \rangle_{L^2_{\text{per}}(\mathbb{C}^n)} := \frac{1}{2\pi} \int_{-\pi}^{\pi} \boldsymbol{\phi}(x) \cdot \boldsymbol{\eta}(x)\, dx.$$

For $k \in \mathbb{Z}$, the kth Fourier coefficient of $\boldsymbol{\phi} \in L^2_{\text{per}}(\mathbb{C}^n)$ is

$$\widehat{\boldsymbol{\phi}}(k) := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-ikx} \boldsymbol{\phi}(x)\, dx.$$

For $r \in \mathbb{R}$ and $\boldsymbol{\phi}, \boldsymbol{\eta} \in L^2_{\text{per}}(\mathbb{C}^n)$, let

$$\langle \boldsymbol{\phi}, \boldsymbol{\eta} \rangle_{H^r_{\text{per}}(\mathbb{C}^n)} := \sum_{k=-\infty}^{\infty} (1 + k^2)^r \big(\widehat{\boldsymbol{\phi}}(k) \cdot \widehat{\boldsymbol{\eta}}(k)\big), \qquad \text{and} \qquad \|\boldsymbol{\phi}\|_{H^r_{\text{per}}(\mathbb{C}^n)} := \big(\langle \boldsymbol{\phi}, \boldsymbol{\phi} \rangle_{H^r_{\text{per}}}\big)^{1/2}$$

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Finally, we put

$$H^r_{\mathrm{per}}(\mathbb{C}^n) := \left\{ \boldsymbol{\phi} \in L^2_{\mathrm{per}}(\mathbb{C}^n) \;\middle|\; \|\boldsymbol{\phi}\|_{H^r_{\mathrm{per}}(\mathbb{C}^n)} < \infty \right\}.$$

Since we will primarily use the L^2_{per}-inner product, we abbreviate it as

$$\langle \boldsymbol{\phi}, \boldsymbol{\eta} \rangle := \langle \boldsymbol{\phi}, \boldsymbol{\eta} \rangle_{L^2_{\mathrm{per}}(\mathbb{C}^n)} = \int_{-\pi}^{\pi} \boldsymbol{\phi}(x) \cdot \boldsymbol{\eta}(x)\, dx.$$

We will employ two elementary identities involving this inner product.

First, with $\boldsymbol{\phi}, \boldsymbol{\eta} \in L^2_{\mathrm{per}}(\mathbb{C}^n)$, we substitute to obtain

$$\langle S^{\theta} \boldsymbol{\phi}, \boldsymbol{\eta} \rangle = \langle \boldsymbol{\phi}, S^{-\theta} \boldsymbol{\eta} \rangle. \tag{A.2}$$

Secondly, with $\boldsymbol{\phi}, \boldsymbol{\eta} \in H^1_{\mathrm{per}}(\mathbb{C}^n)$, we integrate by parts to find

$$\langle \boldsymbol{\phi}', \boldsymbol{\eta} \rangle = -\langle \boldsymbol{\phi}, \boldsymbol{\eta}' \rangle. \tag{A.3}$$

A.3. Fourier multipliers

Let $\widetilde{\mathcal{M}} \colon \mathbb{R} \to \mathbb{C}^{m \times n}$ be measurable. A bounded linear operator $\mathcal{M} \colon H^r_{\mathrm{per}}(\mathbb{C}^n) \to H^s_{\mathrm{per}}(\mathbb{C}^m)$ is a Fourier multiplier with symbol $\widetilde{\mathcal{M}}$ if the identity

$$\widehat{\mathcal{M}\boldsymbol{\phi}}(k) = \widetilde{\mathcal{M}}(k) \widehat{\boldsymbol{\phi}}(k)$$

holds for all $\boldsymbol{\phi} \in H^r_{\mathrm{per}}(\mathbb{C}^n)$ and $k \in \mathbb{Z}$. In this case, the operator norm of $\mathcal{M}$ is

$$\|\mathcal{M}\|_{H^r_{\mathrm{per}}(\mathbb{C}^n) \to H^s_{\mathrm{per}}(\mathbb{C}^m)} = \sup_{k \in \mathbb{Z}} (1 + k^2)^{(s-r)/2} |\widetilde{\mathcal{M}}(k)|_2. \tag{A.4}$$

Conversely, if $\widetilde{\mathcal{M}} \colon \mathbb{R} \to \mathbb{C}^{m \times n}$ is such that the supremum in (A.4) is finite, then defining

$$(\mathcal{M}\boldsymbol{\phi})(x) := \sum_{k=-\infty}^{\infty} e^{ikx} \widetilde{\mathcal{M}}(k) \widehat{\boldsymbol{\phi}}(k)$$

gives a Fourier multiplier $\mathcal{M} \in \mathbf{B}(H^r_{\mathrm{per}}(\mathbb{C}^n), H^s_{\mathrm{per}}(\mathbb{C}^m))$ with symbol $\widetilde{\mathcal{M}}$. This and (A.4) are proved in [15, Lem. D.2.1].

The adjoint of $\mathcal{M}$ is the bounded linear operator $\mathcal{M}^* \colon H^s_{\mathrm{per}}(\mathbb{C}^m) \to H^r_{\mathrm{per}}(\mathbb{C}^n)$ satisfying

$$\langle \mathcal{M}\boldsymbol{\phi}, \boldsymbol{\eta} \rangle_{H^s_{\mathrm{per}}(\mathbb{C}^m)} = \langle \boldsymbol{\phi}, \mathcal{M}^* \boldsymbol{\eta} \rangle_{H^r_{\mathrm{per}}(\mathbb{C}^n)}$$

for all $\boldsymbol{\phi} \in H^r_{\mathrm{per}}(\mathbb{C}^n)$ and $\boldsymbol{\eta} \in H^s_{\mathrm{per}}(\mathbb{C}^m)$. We can calculate $\mathcal{M}^*$ explicitly via the formula

$$\widehat{\mathcal{M}^*\boldsymbol{\eta}}(k) := (1 + k^2)^{s-r} \widetilde{\mathcal{M}}(k)^* \widehat{\boldsymbol{\eta}}(k), \tag{A.5}$$

where $\widetilde{\mathcal{M}}(k)^*$ is the conjugate transpose of $\widetilde{\mathcal{M}}(k)$.

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A.4. Differentiating the shift operator

We prove that the map

$$\mathbb{R} \to \mathbf{B}(H^{r+2}_{\text{per}}(\mathbb{C}^n), H^r_{\text{per}}(\mathbb{C}^n)) : \omega \mapsto S^\omega$$

is differentiable and that its derivative is Lipschitz continuous on $\mathbb{R}$. This is proved more generally in [15, Thm. D.3.1] for a 'scaled' Fourier multiplier, but we include the calculation here for completeness and because all Fourier multipliers that we consider ultimately boil down to shifts. The derivative at $\omega \in \mathbb{R}$ is the operator $(S^\omega)'$ given by

$$\widehat{(S^\omega)'\boldsymbol{\phi}}(k) := k(ie^{i\omega k})\widehat{\boldsymbol{\phi}}(k). \tag{A.6}$$

For $\boldsymbol{\phi} \in H^{r+2}_{\text{per}}(\mathbb{C}^n)$, $\omega \in \mathbb{R}$, and $h \neq 0$, we compute

$$\left\| \left(\frac{S^{\omega+h} - S^\omega - h(S^\omega)'}{h} \right) \boldsymbol{\phi} \right\|^2_{H^r_{\text{per}}(\mathbb{C}^n)} = \sum_{k=-\infty}^{\infty} (1+k^2)^{-2} \left| \frac{e^{ihk} - 1 - ihk}{h} \right|^2 (1+k^2)^{r+2} |\widehat{\boldsymbol{\phi}}(k)|_2^2.$$

Two applications of the fundamental theorem of calculus yield

$$e^{ihk} - 1 - ihk = (ihk)^2 \int_0^1 \int_0^1 te^{ihkts}\, ds\, dt, \tag{A.7}$$

from which we bound

$$(1+k^2)^{-2} \left| \frac{e^{ihk} - 1 - ihk}{h} \right|^2 \leq \frac{(1+k^2)^{-2} h^4 k^4}{h^2} \int_0^1 t\, dt = h^2 \left(\frac{k^4}{2(1+k^2)^2} \right). \tag{A.8}$$

It follows that

$$\left\| \left(\frac{S^{\omega+h} - S^\omega - h(S^\omega)'}{h} \right) \boldsymbol{\phi} \right\|^2_{H^r_{\text{per}}(\mathbb{C}^n)} \leq Ch^2 \|\boldsymbol{\phi}\|^2_{H^{r+2}_{\text{per}}(\mathbb{C}^n)},$$

from which we have differentiability. The mismatch in regularity between the domain and codomain ($H^{r+2}_{\text{per}}(\mathbb{C}^n)$ vs. $H^r_{\text{per}}(\mathbb{C}^n)$) arises because of the factor of k^2 in (A.7); squaring that k^2 in (A.8) requires us to introduce the factor of $(1+k^2)^{-2}$ to compensate. This agrees with the regularity requirements in [15, Thm. D.3.1].

Now we check Lipschitz continuity for the derivative and calculate

$$\|((S^\omega)' - (S^{\grave{\omega}})')\boldsymbol{\phi}\|^2_{H^r_{\text{per}}(\mathbb{C}^n)} = \sum_{k=-\infty}^{\infty} (1+k^2)^{-2} \left| k(ie^{i\omega k}) - k(ie^{i\grave{\omega}k}) \right|^2 (1+k^2)^{r+2} |\widehat{\boldsymbol{\phi}}(k)|_2^2.$$

Since

$$e^{i\omega k} - e^{i\grave{\omega}k} = ik(\omega - \grave{\omega}) \int_0^1 e^{i\grave{\omega}k + ik(\omega-\grave{\omega})t}\, dt, \tag{A.9}$$

we bound

$$(1+k^2)^{-2} \left| k(ie^{i\omega k}) - k(ie^{i\grave{\omega}k}) \right|^2 \leq |\omega - \grave{\omega}|^2 \left(\frac{k^2}{(1+k^2)^2} \right),$$

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and this yields

$$\|\big((S^{\omega})' - (S^{\grave{\omega}})'\big)\boldsymbol{\phi}\|^2_{H^r_{\mathrm{per}}(\mathbb{C}^n)} \le C|\omega - \grave{\omega}|^2\|\boldsymbol{\phi}\|^2_{H^{r+2}_{\mathrm{per}}(\mathbb{C}^n)}.$$

This is the Lipschitz continuity for the derivative. Here we did not strictly need the domain to be $H^{r+2}_{\mathrm{per}}(\mathbb{C}^n)$ and could have viewed S^{ω} as an operator from $H^{r+1}_{\mathrm{per}}(\mathbb{C}^n)$ to $H^r_{\mathrm{per}}(\mathbb{C}^n)$, as we only have one power of k emerging from (A.9). This too agrees with the regularity requirements in [15, Thm. D.3.1].

A.5. *Composition operators in periodic Sobolev spaces*

Let $\mathcal{V} \in \mathcal{C}^7(\mathbb{R})$ with $\mathcal{V}'(0) = 0$. We briefly sketch the argument that the composition operator

$$\mathcal{N}: H^2_{\mathrm{per}}(\mathbb{R}) \to H^2_{\mathrm{per}}(\mathbb{R}): \phi \mapsto \mathcal{V}' \circ \phi$$

is well-defined and twice-differentiable, and its second derivative is (locally) Lipschitz continuous. First, since $\mathcal{V}'(0) = 0$, we have

$$\mathcal{V}'(r) = r\int_0^1 \mathcal{V}''(tr)\,dt \qquad \text{and therefore} \qquad (\mathcal{N}(\phi))(x) = \phi(x)\int_0^1 \mathcal{V}''(t\phi(x))\,dt.$$

Next, differentiating under the integral, we can express $\partial_x^2[\mathcal{N}(\phi)]$ as a sum of products of derivatives of ϕ up to second order and, by the periodic Sobolev embedding [28, Thm. 7.9], continuous and periodic functions (involving integrals of the form $\int_0^1 \mathcal{V}^{(k)}(t\phi)\,dt$ for $k = 2, 3, 4$). It follows from [33, Cor. 8.8] that $\mathcal{N}(\phi) \in H^2_{\mathrm{per}}(\mathbb{R})$. Last, differentiability of $\mathcal{N}$ is straightforward to establish using the fundamental theorem of calculus; the proof is similar to the composition operator work in [17, Lem. A.2]. We obtain $D_\phi\mathcal{N}(\phi)\eta = (\mathcal{V}'' \circ \phi)\eta$ and $D^2_{\phi\phi}\mathcal{N}(\phi)[\eta, \grave{\eta}] = (\mathcal{V}''' \circ \phi)\eta\grave{\eta}$, and (local) Lipschitz continuity follows from the fundamental theorem again. For that, using $(\mathcal{V}''' \circ \phi)\eta\grave{\eta} = \phi\eta\grave{\eta}\int_0^1 \mathcal{V}^{(4)}(t\phi)\,dt$ and estimating in the $H^2_{\mathrm{per}}(\mathbb{R})$-norm, we need up to *seven* continuous derivatives on $\mathcal{V}$.

If we assume $\mathcal{V} \in \mathcal{C}^\infty(\mathbb{R})$, then the composition operator $\mathcal{N}$ is also infinitely differentiable on $H^2_{\mathrm{per}}(\mathbb{R})$ and so (more importantly, for the purposes of Theorem 6.2) by the Sobolev embedding all of its derivatives are locally bounded and locally Lipschitz. This can be proved using the composition operator techniques in [16, App. B], and we omit the details.

Appendix B. Proofs for linear analysis

B.1. *The proof of Corollary 2.2*

If $\mathcal{L}_c[\omega_c]\boldsymbol{\phi} = 0$ and $\widehat{\boldsymbol{\phi}}(k) \neq 0$, then by the arguments preceding the statement of Theorem 2.1, the scalar $c^2(\omega_c k)^2$ must be an eigenvalue of $M^{-1}\widetilde{\mathcal{D}}(\omega_c k)$, and so $c^2(\omega_c k)^2 = \widetilde{\lambda}_\pm(\omega_c k)$. By Theorem 2.1, this can happen only if $k = 0$ or $k \pm 1$, and so

$$\boldsymbol{\phi}(x) = e^{-ix}\widehat{\boldsymbol{\phi}}(-1) + \widehat{\boldsymbol{\phi}}(0) + e^{ix}\widehat{\boldsymbol{\phi}}(1).$$

We study each of these Fourier modes separately. Throughout, we are assuming that at least one of $w = m^{-1}$ or κ is greater than 1.

B.1.1. *The eigenfunction at* k = 0

We solve $M^{-1}\widetilde{\mathcal{D}}(0)\mathbf{v} = 0$ for $\mathbf{v} = (v_1, v_2)$. By definition of $\widetilde{\mathcal{D}}$ in (2.5), we have

$$M^{-1}\widetilde{\mathcal{D}}(0) = \begin{bmatrix} (1+\kappa) & -(1+\kappa) \\ -w(\kappa+1) & w(1+\kappa) \end{bmatrix},$$

and so the vector $\mathbf{v}$ must be a scalar multiple of

$$\boldsymbol{\nu}_0 := \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}.$$

Then $\widehat{\boldsymbol{\phi}}(0) = a_0\boldsymbol{\nu}_0$ for some $a_0 \in \mathbb{R}$ (since $\boldsymbol{\phi}$ is real-valued, $\widehat{\boldsymbol{\phi}}(0)$ must be real, too).

B.1.2. *The eigenfunction at $k = 1$*

We solve $M^{-1}\widetilde{\mathcal{D}}(\omega_c)\mathbf{v} = c^2\omega_c^2\mathbf{v}$ for $\mathbf{v} = (v_1, v_2)$. Then, using again the definition of $\widetilde{\mathcal{D}}$ in (2.5), we need

$$\begin{bmatrix}(1+\kappa) & -(e^{i\omega_c}+\kappa e^{-i\omega_c})\\ -w(\kappa e^{i\omega_c}+e^{-i\omega_c}) & w(1+\kappa)\end{bmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = c^2\omega_c^2\begin{pmatrix}v_1\\v_2\end{pmatrix}.$$

The first component here reads

$$(1+\kappa-c^2\omega_c^2)v_1 - (e^{i\omega_c}+\kappa e^{-i\omega_c})v_2 = 0.$$

Assume for the moment that $1+\kappa-c^2\omega_c^2 \neq 0$; we prove this below in Appendix B.1.3.

$$v_1 = \frac{e^{i\omega_c}+\kappa e^{-i\omega_c}}{1+\kappa-c^2\omega_c^2}v_1,$$

so $\mathbf{v}$ must be a scalar multiple of

$$\boldsymbol{\mu}_c := \begin{pmatrix}e^{i\omega_c}+\kappa e^{-i\omega_c}\\ 1+\kappa-c^2\omega_c^2\end{pmatrix}. \tag{B.1}$$

Then $\widehat{\boldsymbol{\phi}}(1) = a_1\boldsymbol{\mu}_c$ for some $a_1 \in \mathbb{C}$. (Here we are not guaranteed $a_1 \in \mathbb{R}$.)

B.1.3. *The proof that $1+\kappa-c^2\omega_c^2 \neq 0$*

We use the identity $c^2\omega_c^2 = \widetilde{\lambda}_+(\omega_c)$ and the definition of $\widetilde{\lambda}_+$ in (2.6) to compute

$$1+\kappa-c^2\omega_c^2 = 1+\kappa-\widetilde{\lambda}_+(\omega_c) = -\left(\frac{(1+\kappa)(w-1)+\widetilde{\varrho}(\omega_c)}{2}\right), \tag{B.2}$$

where $\widetilde{\varrho}$ is defined in (2.7). In particular, $\widetilde{\varrho}(\omega_c) \geq 0$. Thus for $w > 1$, we have $1+\kappa-c^2\omega_c^2 < 0$. When $w = 1$, and consequently $\kappa > 1$, (B.2) simplifies to

$$1+\kappa-c^2\omega_c^2 - \frac{\widetilde{\varrho}(\omega_c)}{2} - \sqrt{(1-\kappa)^2+4\kappa\cos^2(\omega_c)} < 1-\kappa.$$

The resulting estimate

$$|1+\kappa-c^2\omega_c^2| \geq \begin{cases}(1+\kappa)(w-1)/2, & w > 1\\ \kappa-1, & w = 1\end{cases} \tag{B.3}$$

will be useful in subsequent proofs, since it is uniform in c.

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B.1.4. A basis for the kernel of $\mathcal{L}_c[\omega_c]$

We are assuming $\mathcal{L}_c[\omega_c]\boldsymbol{\phi} = 0$ and so far know that

$$\boldsymbol{\phi}(x) = e^{-ix}\widehat{\boldsymbol{\phi}}(-1) + \widehat{\boldsymbol{\phi}}(0) + e^{ix}\widehat{\boldsymbol{\phi}}(1).$$

Since we always assume that $\boldsymbol{\phi}$ is real-valued, we have

$$\widehat{\boldsymbol{\phi}}(-1) = \overline{\widehat{\boldsymbol{\phi}}(1)},$$

and then

$$\boldsymbol{\phi}(x) = \widehat{\boldsymbol{\phi}}(0) + 2\,\mathrm{Re}[e^{ix}\widehat{\boldsymbol{\phi}}(1)].$$

Write $\widehat{\boldsymbol{\phi}}(1) = a_1\boldsymbol{\mu}_c$ with $\boldsymbol{\mu}_c$ defined in (B.1) and suppose $a_1 = a_r + ia_i$ for $a_r, a_i \in \mathbb{R}$. Then

$$\mathrm{Re}\left[e^{ix}\widehat{\boldsymbol{\phi}}(1)\right] = \mathrm{Re}\left[(a_r + ia_i)\left(\mathrm{Re}[e^{ix}\boldsymbol{\mu}_c] + i\,\mathrm{Im}[e^{ix}\boldsymbol{\mu}_c]\right)\right] = a_r\,\mathrm{Re}[e^{ix}\boldsymbol{\mu}_c] - a_i\,\mathrm{Im}[e^{ix}\boldsymbol{\mu}_c].$$

Thus

$$\boldsymbol{\phi}(x) = \widehat{\boldsymbol{\phi}}(0) + 2\,\mathrm{Re}[e^{ix}\widehat{\boldsymbol{\phi}}(1)] = a_0\boldsymbol{\nu}_0 + a_r\,\mathrm{Re}[e^{ix}\boldsymbol{\mu}_c] - a_i\,\mathrm{Im}[e^{ix}\boldsymbol{\mu}_c].$$

It follows that the vectors

$$\boldsymbol{\nu}_0, \qquad \boldsymbol{\nu}_1^c(x) := \frac{1}{\mathrm{N}_c}\mathrm{Re}[e^{ix}\boldsymbol{\mu}_c], \qquad \text{and} \qquad \boldsymbol{\nu}_2^c(x) := \frac{1}{\mathrm{N}_c}\mathrm{Im}[e^{ix}\boldsymbol{\mu}_c], \qquad \mathrm{N}_c := |\boldsymbol{\mu}_c|, \tag{B.4}$$

span the kernel of $\mathcal{L}_c[\omega_c]$. We check orthonormality as follows and obtain linear independence, so they are a basis for the kernel. First, that $\langle\boldsymbol{\nu}_0, \boldsymbol{\nu}_1^c\rangle = \langle\boldsymbol{\nu}_0, \boldsymbol{\nu}_2^c\rangle = 0$ follows directly from the formulas above and the identity

$$\int_{-\pi}^{\pi} e^{\pm ix}\,dx = 0.$$

Next, for any $\boldsymbol{\phi} \in L^2_{\mathrm{per}}(\mathbb{R}^2)$, we compute

$$\langle\boldsymbol{\phi}, \boldsymbol{\nu}_1^c\rangle = 2\,\mathrm{Re}\left[\widehat{\boldsymbol{\phi}}(1)\cdot\widehat{\boldsymbol{\nu}_1^c}(1)\right], \tag{B.5}$$

$$(S^{-\pi/2}\boldsymbol{\nu}_1^c)(x) = \frac{1}{N_c}\mathrm{Re}[-ie^{ix}\boldsymbol{\mu}_c] = \frac{1}{N_c}\mathrm{Im}[e^{ix}\boldsymbol{\mu}_c] = \boldsymbol{\nu}_2^c(x)$$

and

$$\langle\boldsymbol{\phi}, \boldsymbol{\nu}_2^c\rangle = \langle\boldsymbol{\phi}, S^{-\pi/2}\boldsymbol{\nu}_1^c\rangle = 2\,\mathrm{Re}\left[\widehat{\boldsymbol{\phi}}(1)\cdot(-i\widehat{\boldsymbol{\nu}_1^c}(1))\right] = 2\,\mathrm{Im}\left[\widehat{\boldsymbol{\phi}}(1)\cdot\widehat{\boldsymbol{\nu}_1^c}(1)\right]. \tag{B.6}$$

Combining (B.5) and (B.6), incidentally, proves the orthogonality equivalence condition (2.17). From (B.6), we have

$$\langle\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_2^c\rangle = 2\,\mathrm{Im}[\widehat{\boldsymbol{\nu}_1^c}(1)\cdot\widehat{\boldsymbol{\nu}_1^c}(1)] = 2\,\mathrm{Im}[|\boldsymbol{\nu}_1^c|_2^2] = 0.$$

This concludes the orthonormality proof. Last, the derivative identities (2.15) follow directly from the formulas (B.4).

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B.1.5. The kernel of $\mathcal{L}_c[\omega_c]^*$

As discussed in Appendix A.3, the adjoint operator $\mathcal{L}_c[\omega_c]^* \colon \mathcal{L}^2_{\text{per}}(\mathbb{R}^2) \to H^2_{\text{per}}(\mathbb{R}^2)$ satisfies

$$\widehat{\mathcal{L}_c[\omega_c]^*\boldsymbol{\eta}}(k) = (1+k^2)^{-2}(-c^2(\omega_c k)^2 M + \widetilde{\mathcal{D}}(\omega_c k)^*)\widehat{\boldsymbol{\eta}}(k).$$

Here $\widetilde{\mathcal{D}}(K)^*$ is the conjugate transpose of the matrix $\widetilde{\mathcal{D}}(K)$ defined in (2.5). Happily, $\widetilde{\mathcal{D}}(K)$ is a symmetric matrix, so $\widetilde{\mathcal{D}}(K)^* = \widetilde{\mathcal{D}}(K)$, and therefore

$$\widehat{\mathcal{L}_c[\omega_c]^*\boldsymbol{\eta}}(k) = (1+k^2)^{-2}(-c^2(\omega_c k)^2 M + \widetilde{\mathcal{D}}(\omega_c k))\widehat{\boldsymbol{\eta}}(k) = (1+k^2)^{-2}\widehat{\mathcal{L}_c[\omega_c]\boldsymbol{\eta}}(k). \tag{B.7}$$

Thus if $\mathcal{L}_c[\omega_c]\boldsymbol{\eta} = 0$, then $\widehat{\mathcal{L}_c[\omega_c]\boldsymbol{\eta}}(k) = 0$ for all k, and so $\mathcal{L}_c[\omega_c]\boldsymbol{\eta} = 0$. Consequently, the kernel of $\mathcal{L}_c[\omega_c]^*$ is contained in the span of $\boldsymbol{\nu}_0$, $\boldsymbol{\nu}_1^c$ and $\boldsymbol{\nu}_2^c$, and the reverse containment is also obvious from (B.7).

B.2. The proof of Corollary 2.3

We compute the exact value of $\langle \mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_1^c\rangle$, where, from the definitions of M in (1.10) and $\widetilde{\mathcal{D}}$ in (2.5), the symbol of $\mathcal{L}_c'[\omega_c]$ is

$$\widetilde{\mathcal{L}}_c'(\omega_c k) = -2c^2\omega_c k M + \widetilde{\mathcal{D}}'(\omega_c k) = \begin{bmatrix} -2c^2\omega_c k & -i(e^{i\omega_c k} - \kappa e^{-i\omega_c k}) \\ -i(\kappa e^{i\omega_c k} - e^{-i\omega_c k}) & -2c^2 w^{-1}\omega_c k \end{bmatrix}.$$

Our goal is to use the inequality

$$\inf_{|c|>c_\star} 2c^2\omega_c - \widetilde{\lambda}_+'(\omega_c) > 0 \tag{B.8}$$

from Theorem 2.1 and recognize $\langle \mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_1^c\rangle$ as the product of $2c^2\omega_c - \widetilde{\lambda}_+'(\omega_c)$ and a quantity that is uniformly bounded in c away from 0.

By (B.5), we have

$$\langle \mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_1^c\rangle = 2\,\text{Re}\left[\widehat{\mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c}1 \cdot \widehat{\boldsymbol{\nu}_1^c}(1)\right] = 2\,\text{Re}\left[\widetilde{\mathcal{L}}_c(\omega_c)\widehat{\boldsymbol{\nu}_1^c}(1) \cdot \widehat{\boldsymbol{\nu}_1^c}(1)\right]. \tag{B.9}$$

The formula (2.12) for $\boldsymbol{\nu}_1^c$ gives

$$\widetilde{\mathcal{L}}_c'(\omega_c)\widehat{\boldsymbol{\nu}_1^c}(1) \cdot \widehat{\boldsymbol{\nu}_1^c}(1) = \frac{1}{\mathrm{N}_c^2}\begin{bmatrix} -2c^2\omega_c & -i(e^{i\omega_c} - \kappa e^{-i\omega_c}) \\ -i(\kappa e^{i\omega_c} - e^{-i\omega_c}) & -2c^2 w^{-1}\omega_c \end{bmatrix}\begin{pmatrix} e^{i\omega_c} + \kappa e^{-i\omega_c} \\ 1+\kappa - c^2\omega_c^2 \end{pmatrix} \cdot \begin{pmatrix} e^{i\omega_c} + \kappa e^{-i\omega_c} \\ 1+\kappa - c^2\omega_c^2 \end{pmatrix}. \tag{B.10}$$

Some preparation and attention to detail will simplify what would otherwise be a burdensome calculation into a slightly less burdensome calculation. Suppressing dependence on c, we put

$$z_1 = -2c^2\omega_c, \quad z_2 = e^{i\omega_c} - \kappa e^{-i\omega_c}, \quad v_1 = e^{i\omega_c} + \kappa e^{-i\omega_c} \quad \text{and} \quad v_2 = 1+\kappa - c^2\omega_c^2. \tag{B.11}$$

In particular,

$$\overline{z_2} = e^{-i\omega_c} - \kappa e^{i\omega_c} \qquad \text{and so} \qquad -i(\kappa e^{i\omega_c} - e^{-i\omega_c}) = i(e^{-i\omega_c} - \kappa e^{i\omega_c}) = i(\overline{z_2}).$$

Then (B.10) is equivalent to

$$\begin{aligned}
\mathrm{N}_c^2\, \widetilde{\mathcal{L}}_c'(\omega_c)\widehat{\boldsymbol{v}}_1^c(1)\cdot\widehat{\boldsymbol{v}}_1^c(1) &= \begin{bmatrix} z_1 & -iz_2 \\ i(\overline{z_2}) & w^{-1}z_1 \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\cdot\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
&= (z_1v_1 - iz_2v_2)\overline{v_1} + (i(\overline{z_2})v_1 + w^{-1}z_1v_2)\overline{v_2} \\
&= z_1|v_1|^2 - iz_2\overline{v_1}v_2 + i(\overline{z_2})v_1\overline{v_2} + w^{-1}z_1v_2^2 \\
&= z_1\left(|v_1|^2 + w^{-1}v_2^2\right) + \overline{i(\overline{z_2})v_1\overline{v_2}} + i(\overline{z_2})v_1v_2 \\
&= z_1\left(|v_1|^2 + w^{-1}v_2^2\right) + 2\,\mathrm{Re}\left[i(\overline{z_2})v_1v_2\right].
\end{aligned} \tag{B.12}$$

This immediately shows that $\widetilde{\mathcal{L}}_c'(\omega_c)\widehat{\boldsymbol{v}}_1^c(1)\cdot\widehat{\boldsymbol{v}}_1^c(1)$ is real, and so, after (re)introducing what turns out to be a helpful factor of w, (B.9) reads

$$\frac{w\,\mathrm{N}_c^2\langle\mathcal{L}_c'[\omega_c]\boldsymbol{v}_1^c, \boldsymbol{v}_1^c\rangle}{2} = w\,\mathrm{Re}\left[\widetilde{\mathcal{L}}_c'(\omega_c)\widehat{\boldsymbol{v}}_1^c(1)\cdot\widehat{\boldsymbol{v}}_1^c(1)\right] = -2c^2\omega_c\left(w|v_1|^2+v_2^2\right)+2w\,\mathrm{Re}\left[i(\overline{z_2})v_1v_2\right]. \tag{B.13}$$

The first term on the right in (B.13) contains a factor of $2c^2\omega_c$, which appears in our favourite estimate (B.8). Now we work on the second term in (B.13) to expose a factor of $\widetilde{\lambda}_+'(\omega_c)$, which also appears in that estimate. We have

$$\begin{aligned}
\overline{z_2}v_1 &= (e^{-i\omega_c} - \kappa e^{i\omega_c})(e^{i\omega_c} + \kappa e^{-i\omega_c}) \\
&= 1 + \kappa e^{-2i\omega_c} - \kappa e^{2i\omega_c} - \kappa^2 \\
&= 1 - \kappa^2 - 2i\kappa\sin(2\omega_c),
\end{aligned}$$

and so

$$i(\overline{z_2})v_1v_2 = i(1-\kappa^2 - 2i\kappa\sin(2\omega_c))v_2 = i(1-\kappa^2)v_2 + 2\kappa\sin(2\omega_c)v_2.$$

Thus

$$2\,\mathrm{Re}\left[i(\overline{z_2})v_1v_2\right] = 2\,\mathrm{Re}\left[i(1-\kappa^2)v_2 + 2\kappa\sin(2\omega_c)v_2\right] = 4\kappa\sin(2\omega_c)v_2$$

since $v_2 \in \mathbb{R}$. By definition of $\widetilde{\lambda}_+$ in (2.6), we compute

$$\widetilde{\lambda}_+'(\omega_c) = -\frac{4\kappa w\sin(2\omega_c)}{\widetilde{\varrho}(\omega_c)},$$

where $\widetilde{\varrho}$ is defined in (2.7). Thus

$$4\kappa w\sin(2\omega_c)v_2 = -\widetilde{\varrho}(\omega_c)v_2\widetilde{\lambda}_+'(\omega_c),$$

and so (B.13) becomes

$$\frac{\mathrm{N}_c^2\, w\langle\mathcal{L}_c'[\omega_c]\boldsymbol{v}_1^c, \boldsymbol{v}_1^c\rangle}{2} = -2c^2\omega_c\left(w|v_1|^2 + v_2^2\right) - \widetilde{\varrho}(\omega_c)v_2\widetilde{\lambda}_+'(\omega_c). \tag{B.14}$$

The first term in (B.14) now needs our attention. We compute

$$|v_1|^2 = |e^{i\omega_c} + \kappa e^{-i\omega_c}|^2 = (1-\kappa)^2 + 4\kappa\cos^2(\omega_c). \tag{B.15}$$

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Next, in (B.2), we calculated

$$v_2 = 1 + \kappa - c^2\omega_c^2 = \frac{(1+\kappa)(1-w) - \widetilde{\varrho}(\omega_c)}{2}. \tag{B.16}$$

We use these expansions for $|v_1|^2$ and v_2 as well as the expansion

$$\widetilde{\varrho}(\omega_c)^2 = (1+w)^2(1-\kappa)^2 + 4\kappa(1-w)^2 + 16\kappa w\cos^2(\omega_c)$$

from the definition of $\widetilde{\varrho}$ in (2.7) to compute, laboriously,

$$w|v_1|^2 + v_2^2 = w(1-\kappa)^2 + 4\kappa w\cos^2(\omega_c) + \left(\frac{(1+\kappa)(1-w) - \widetilde{\varrho}(\omega_c)}{2}\right)^2 \tag{B.17}$$

$$= -\frac{\widetilde{\varrho}(\omega_c)\big[(1+\kappa)(1-w) - \widetilde{\varrho}(\omega_c)\big]}{2} = -\widetilde{\varrho}(\omega_c)v_2.$$

Back to (B.14), we now see that

$$\frac{w\,\mathrm{N}_c^2\langle \mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_1^c\rangle}{2} = -2c^2\omega_c\big(-\widetilde{\varrho}(\omega_c)v_2\big) - \widetilde{\varrho}(\omega_c)v_2\widetilde{\lambda}_+'(\omega_c) = \widetilde{\varrho}(\omega_c)v_2\big(2c^2\omega_c - \lambda_+'(\omega_c)\big).$$

That is,

$$\langle \mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_1^c\rangle = \frac{2\widetilde{\varrho}(\omega_c)v_2}{w\,\mathrm{N}_c^2}\big(2c^2\omega_c - \lambda_+'(\omega_c)\big)$$

All that remains is to check that the product $\rho(\omega_c)v_2/\mathrm{N}_c$ is uniformly bounded in c away from 0. First, the definition of $\widetilde{\varrho}$ in (2.7) implies

$$|\widetilde{\varrho}(\omega_c)| \geq \sqrt{(1+w)^2(1-\kappa)^2 + 4\kappa(1-w)^2}.$$

Since at least one of κ or w is greater than 1, this quantity is positive. Next, the estimate (B.3) gives a positive lower bound on v_2 that is independent of c. Finally, the definition (2.14) of N_c gives

$$|\,\mathrm{N}_c\,| \geq \begin{cases} \sqrt{2}(\kappa - 1), & \kappa > 1 \\ \sqrt{2}(w-1), & w > 1. \end{cases}$$

We conclude that $\widetilde{\varrho}(\omega_c)v_2/\mathrm{N}_c$ is uniformly bounded in c away from 0.

B.3. The proof of Corollary 2.4

Assume that $\mathcal{L}_c[\omega_c]\boldsymbol{\psi} = \boldsymbol{\eta}$, where $\boldsymbol{\psi} = (\psi_1, \psi_2) \in H_{\mathrm{per}}^{r+2}(\mathbb{R}^2)$ and $\boldsymbol{\eta} = (\eta_1, \eta_2) \in H_{\mathrm{per}}^r(\mathbb{R}^2)$ with

$$\langle \boldsymbol{\psi}, \boldsymbol{\nu}_0\rangle = \langle \boldsymbol{\psi}, \boldsymbol{\nu}_1^c\rangle = \langle \boldsymbol{\psi}, \boldsymbol{\nu}_2^c\rangle = 0 \qquad \text{and} \qquad \langle \boldsymbol{\eta}, \boldsymbol{\nu}_0\rangle = \langle \boldsymbol{\eta}, \boldsymbol{\nu}_1^c\rangle = \langle \boldsymbol{\eta}, \boldsymbol{\nu}_2^c\rangle = 0.$$

We will solve for $\boldsymbol{\psi}$ in terms of $\boldsymbol{\eta}$ and uniformly estimate $\|\boldsymbol{\psi}\|_{H_{\mathrm{per}}^{r+2}}$ in terms of c and $\|\boldsymbol{\eta}\|_{H_{\mathrm{per}}^r}$. Since $\mathcal{L}_c[\omega_c]\boldsymbol{\psi} = \boldsymbol{\eta}$, we have $\widetilde{\mathcal{L}}_c(\omega_c k)\widehat{\boldsymbol{\psi}}(k) = \widehat{\boldsymbol{\eta}}(k)$ for each $k \in \mathbb{Z}$, and so we really need to solve

$$\big(-c^2\omega_c^2k^2M + \widetilde{\mathcal{D}}(\omega_c k)\big)\widehat{\boldsymbol{\phi}}(k) = \widehat{\boldsymbol{\eta}}(k) \tag{B.18}$$

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for each $k \in \mathbb{Z}$, where $\widetilde{\mathcal{D}}$ is defined in (2.5). We treat the cases $k=0$, $k = \pm 1$, and $|k| \geq 2$ separately. This is the same strategy as the proofs of [27, Lem. B.1] for the mass dimer small mass limit, [18, Lem. C.2] for the mass dimer equal mass limit and [14, Prop. 5] for the MiM small mass limit.

Before proceeding, we point out some consequences of the orthogonality conditions above for $k=0$ and $k=1$ that make the entire argument possible. (This is essentially an exercise in solving 2×2 linear systems, but we need to be careful with our material parameters w and κ and our wave speed c.) Suppose that $\boldsymbol{\phi} = (\phi_1, \phi_2) \in L^2_{\text{per}}(\mathbb{R}^2)$ with

$$\langle \boldsymbol{\phi}, \boldsymbol{\nu}_0 \rangle = \langle \boldsymbol{\phi}, \boldsymbol{\nu}_1^c \rangle = \langle \boldsymbol{\phi}, \boldsymbol{\nu}_2^c \rangle = 0.$$

We use these orthogonality conditions to derive formulas for $\widehat{\phi}_2(k)$ in terms of $\widehat{\phi}_1(k)$ for $k=0$ and $k=1$.

First, the condition $\langle \boldsymbol{\phi}, \boldsymbol{\nu}_0 \rangle = 0$ immediately implies

$$\widehat{\phi}_2(0) = -\widehat{\phi}_1(0). \tag{B.19}$$

Next, the orthogonality condition (2.17) implies

$$\widehat{\boldsymbol{\phi}}(1) \cdot \widehat{\boldsymbol{\nu}_1^c}(1) = 0,$$

and from the definition of $\boldsymbol{\nu}_1^c$ in (2.12), this reads

$$\widehat{\phi}_1(1)(e^{-i\omega_c} + \kappa e^{i\omega_c}) + (1 + \kappa - c^2\omega_c^2)\widehat{\phi}_2(1) = 0.$$

Since $1 + \kappa - c^2\omega_c^2 \neq 0$ by the work in Appendix B.1.3, we have

$$\widehat{\phi}_2(1) = -\frac{e^{-i\omega_c} + \kappa e^{i\omega_c}}{1 + \kappa - c^2\omega_c^2}\widehat{\phi}_1(1). \tag{B.20}$$

B.3.1. The case k = 0

Here the first component of (B.18) reads

$$\widehat{\psi}_1(0) - \widehat{\psi}_2(0) = \frac{\widehat{\eta}_1(0)}{1+\kappa},$$

and from (B.19) this is

$$2\widehat{\psi}_1(0) = \widehat{\eta}_1(0).$$

Thus

$$\widehat{\psi}_1(0) = \frac{\widehat{\eta}_1(0)}{2(1+\kappa)}.$$

It follows from this equality and (B.19) that

$$|\widehat{\boldsymbol{\psi}}(0)|_2 \leq \frac{|\widehat{\boldsymbol{\eta}}(0)|_2}{2(1+\kappa)}.$$

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B.3.2. The case $k = \pm 1$

We only need to estimate $|\widehat{\boldsymbol{\psi}}(1)|$, as $|\widehat{\boldsymbol{\psi}}(1)| = |\widehat{\boldsymbol{\psi}}(-1)|$ since $\boldsymbol{\psi}$ is $\mathbb{R}^2$-valued. At $k = 1$ the first component of (B.18) reads

$$(1 + \kappa - c^2\omega_c^2)\widehat{\psi}_1(1) - (e^{i\omega_c} + \kappa e^{-i\omega_c})\widehat{\psi}_2(1) = \widehat{\eta}_1(1).$$

We use the identity (B.20) to remove $\widehat{\psi}_2(1)$ from this equation and write it in terms of $\widehat{\psi}_1(1)$ alone. We find

$$\left(1 + \kappa - c^2\omega_c^2 + \frac{|e^{i\omega_c} + \kappa e^{-i\omega_c}|^2}{1 + \kappa - c^2\omega_c^2}\right)\widehat{\psi}_1(1) = \widehat{\eta}_1(1).$$

We use (B.15) and rearrange this into

$$\big((1 + \kappa - c^2\omega_c^2)^2 + (1 - \kappa)^2 + 4\kappa\cos^2(\omega_c)\big)\widehat{\psi}_1(1) = (1 + \kappa - c^2\omega_c^2)\widehat{\eta}_1(1). \tag{B.21}$$

Since

$$(1 + \kappa - c^2\omega_c^2)^2 + (1 - \kappa)^2 + 4\kappa\cos^2(\omega_c) \geq (1 + \kappa - c^2\omega_c^2)^2,$$

the uniform lower bound on $1 + \kappa - c^2\omega_c^2$ from (B.3) and the upper bound

$$|1 + \kappa - c^2\omega_c^2| \leq 1 + \kappa + (1 + \kappa)(1 + w)$$

from (2.9), we can derive from (B.21) the estimate

$$|\widehat{\psi}_1(1)|_2 f \leq C|\widehat{\eta}_1(1)|_2,$$

where C depends on κ and w but not on c, $\boldsymbol{\psi}$, or $\boldsymbol{\eta}$. The identity (B.20) and the uniform lower bound on $1 + \kappa - c^2\omega_c^2$ imply

$$|\widehat{\psi}_2(1)|_2 \leq C|\widehat{\eta}_1(1)|_2$$

as well. A final invocation of (B.20) allows us to estimate $|\widehat{\eta}_1(1)|_2 \leq C|\widehat{\boldsymbol{\eta}}(1)|_2$.

B.3.3. The case $|k| \geq 2$

Since $k \neq 0$, we may rewrite (B.18) as

$$\left(I_2 - \frac{1}{c^2\omega_c^2k^2}M^{-1}\widetilde{\mathcal{D}}(\omega_c k)\right)\widehat{\boldsymbol{\psi}}(k) = -\frac{1}{c^2\omega_c^2k^2}M^{-1}\widehat{\boldsymbol{\eta}}(k). \tag{B.22}$$

Here I_2 is the 2×2 identity matrix. We will use the Neumann series to solve (B.22) for $\widehat{\boldsymbol{\psi}}(k)$ in terms of $\widehat{\boldsymbol{\eta}}(k)$ with uniform estimates in c.

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The following estimates use our conventions for matrix norms from Appendix A.1. First, the definition of $\widetilde{\mathcal{D}}$ in (2.5) yields the estimate

$$|M^{-1}\widetilde{\mathcal{D}}(\omega_c k)|_\infty \le (1+\kappa)w.$$

Since $|k| \ge 2$, we have

$$\frac{1}{c^2\omega_c^2 k^2}|M^{-1}\widetilde{\mathcal{D}}(\omega_c k)|_2 \le \frac{2}{c^2\omega_c^2 k^2}|M^{-1}\widetilde{\mathcal{D}}(\omega_c k)|_\infty \le \frac{2(1+\kappa)w}{4c^2\omega_c^2}.$$

Next, the inequality (2.9) on ω_c and the definition of $\widetilde{\lambda}_+$ in (2.6) imply

$$\frac{1}{c^2\omega_c^2} \le \frac{1}{\widetilde{\lambda}_+(\pi/2)} \le \frac{2}{(1+\kappa)(1+w)}. \tag{B.23}$$

Thus

$$\frac{1}{c^2\omega_c^2 k^2}|M^{-1}\widetilde{\mathcal{D}}(\omega_c k)|_2 \le \frac{4(1+\kappa)w}{4(1+\kappa)(1+w)} = \frac{w}{1+w} < 1.$$

We may therefore use the Neumann series to solve (B.22) for $\widehat{\boldsymbol{\psi}}(k)$ in terms of $\widehat{\boldsymbol{\eta}}(k)$, and we obtain

$$|\widehat{\boldsymbol{\psi}}(k)|_2 \le \frac{1}{1-w/(1+w)}\left(\frac{|M^{-1}|_2}{c^2\omega_c^2k^2}\right)|\widehat{\boldsymbol{\eta}}(k)|_2 \le \frac{1}{1-w/(1+w)}\left(\frac{4w}{(1+\kappa)(1+w)}\right)\frac{|\widehat{\boldsymbol{\eta}}(k)|_2}{k^2}.$$

The second inequality follows from (B.23) and the estimate $|M^{-1}|_2 \le 2|M^{-1}|_\infty = 2w$ from (A.1). This, along with the uniform estimates in c on $|\widehat{\boldsymbol{\psi}}(k)|_2$ for $k=0$, 1 from the previous sections, gives the coercive estimate $\|\boldsymbol{\psi}\|_{H^{r+2}_{\mathrm{per}}} \le C\|\boldsymbol{\eta}\|_{H^r_{\mathrm{per}}}$. The constant C depends on κ and w but is independent of r.

B.4. The proof of Lemma 3.6

Continuity and differentiability of $\boldsymbol{\Phi}_c$ in $\boldsymbol{\phi}$ follow from the composition operator calculus in Appendix A.5 and in ω from the shift operator calculus in Appendix A.4. A second appeal to these appendices gives the same results for $D_{\boldsymbol{\phi}}\boldsymbol{\Phi}_c$. In each case, we are only taking one derivative with respect to ω, and that is all that Appendix A.4 guarantees when we consider S^ω as a map from $H^2_{\mathrm{per}}(\mathbb{R})$ to $H^0_{\mathrm{per}} = L^2_{\mathrm{per}}$.

B.5. The proof of Lemma 4.3

B.5.1. A proof using the gradient formulation

We claim that

$$\langle \mathcal{L}_c[\omega]\boldsymbol{\phi}, \boldsymbol{\phi}'\rangle = 0 \tag{B.24}$$

for all $\boldsymbol{\phi} \in H^2_{\mathrm{per}}(\mathbb{R}^2)$ and $\omega \in \mathbb{R}$ and prove this claim below. Assuming this to be true, we differentiate (B.24) with respect to ω and obtain

$$\langle \mathcal{L}_c'[\omega]\boldsymbol{\phi}, \boldsymbol{\phi}'\rangle = 0$$

for all $\boldsymbol{\phi} \in H^2_{\mathrm{per}}(\mathbb{R}^2)$ and $\omega \in \mathbb{R}$. In particular,

$$\langle \mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_2^c\rangle = -\langle \mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c, \partial_x\boldsymbol{\nu}_1^c\rangle = 0.$$

Now we prove the claim (B.24). The proofs of the derivative orthogonality condition $\langle \mathbf{\Phi}_c(\boldsymbol{\phi}, \omega), \boldsymbol{\phi}' \rangle = 0$ in both part (iii) of Corollary 3.2 and in Lemma 4.1 did not rely on the precise structure of the spring potentials $\mathcal{V}_1$ and $\mathcal{V}_2$, provided that they were continuously differentiable. So, assume here that both are linear with $\mathcal{V}_1(r) = \mathcal{V}_1'(0)r$ and $\mathcal{V}_2(r) = \mathcal{V}_2'(0)r$. Then $\mathbf{\Phi}_c(\boldsymbol{\phi}, \omega) = D_{\boldsymbol{\phi}}\mathbf{\Phi}_c(0, \omega)\boldsymbol{\phi} = \mathcal{L}_c[\omega]\boldsymbol{\phi}$, and (B.24) follows from the original derivative orthogonality condition.

B.5.2. A proof via direct calculation

The same reasoning that led to (B.9) implies

$$\langle \mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_2^c \rangle = 2\,\mathrm{Re}\left[\widetilde{\mathcal{L}}_c'(\omega_c)\widehat{\boldsymbol{\nu}_1^c}(1) \cdot \widehat{\boldsymbol{\nu}_2^c}(1)\right].$$

Now, from the definitions of $\boldsymbol{\nu}_1^c$ in (2.12) and $\boldsymbol{\nu}_2^c$ in (2.13), we have

$$\widehat{\boldsymbol{\nu}_2^c}(1) = -i\widehat{\boldsymbol{\nu}_1^c}(1),$$

and so

$$\widetilde{\mathcal{L}}_c'(\omega_c)\widehat{\boldsymbol{\nu}_1^c}(1) \cdot \widehat{\boldsymbol{\nu}_2^c}(1) = \widetilde{\mathcal{L}}_c'(\omega_c)\widehat{\boldsymbol{\nu}_1^c}(1) \cdot \big(-i\widehat{\boldsymbol{\nu}_1^c}(1)\big) = i\big(\widetilde{\mathcal{L}}_c'(\omega_c)\widehat{\boldsymbol{\nu}_1^c}(1) \cdot \widehat{\boldsymbol{\nu}_1^c}(1)\big),$$

thus

$$\langle \mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_2^c \rangle = 2\,\mathrm{Re}\left[i\big(\widetilde{\mathcal{L}}_c'(\omega_c)\widehat{\boldsymbol{\nu}_1^c}(1) \cdot \widehat{\boldsymbol{\nu}_1^c}(1)\big)\right].$$

But in (B.12), we calculated that $\widetilde{\mathcal{L}}_c'(\omega_c)\widehat{\boldsymbol{\nu}_1^c}(1) \cdot \widehat{\boldsymbol{\nu}_1^c}(1)$ is real, and so $\langle \mathcal{L}_c'[\omega_c]\boldsymbol{\nu}_1^c, \boldsymbol{\nu}_2^c \rangle = 0$.

B.6. The proof of Lemma 5.6

Recall that $\mathcal{S}_{\mathbf{K}}$ was defined in (5.8). We have $\mathcal{S}_{\mathbf{K}}\boldsymbol{\nu}_1^c = \pm\boldsymbol{\nu}_1^c$ if and only if $\widehat{\mathcal{S}_{\mathbf{K}}\boldsymbol{\nu}_1^c}(1) = \pm\widehat{\boldsymbol{\nu}_1^c}(1)$ and $\widehat{\mathcal{S}_{\mathbf{K}}\boldsymbol{\nu}_1^c}(-1) = \pm\widehat{\boldsymbol{\nu}_1^c}(-1)$. Since $\mathcal{S}_{\mathbf{K}}\boldsymbol{\nu}_1^c$ and $\boldsymbol{\nu}_1^c$ are real-valued, the second equality automatically holds if the first does. Thus $\mathcal{S}_{\mathbf{K}}\boldsymbol{\nu}_1^c = \pm\boldsymbol{\nu}_1^c$ if and only if $\widehat{\mathcal{S}_{\mathbf{K}}\boldsymbol{\nu}_1^c}(1) = \pm\widehat{\boldsymbol{\nu}_1^c}(1)$. We compute

$$\widehat{\mathcal{S}_{\mathbf{K}}\boldsymbol{\nu}_1^c}(1) = -J\widehat{R\boldsymbol{\nu}_1^c}(1) = -J\widehat{\boldsymbol{\nu}_1^c}(-1).$$

From the definition of $\boldsymbol{\nu}_1^c$ in (2.12), where it is not at this time at all apparent that taking $w = 1$ matters, we have

$$-\mathrm{N}_c J\widehat{\boldsymbol{\nu}_1^c}(-1) = -J\begin{pmatrix} e^{-i\omega_c} + \kappa e^{i\omega_c} \\ 1 + \kappa - c^2\omega_c^2 \end{pmatrix} = -\begin{pmatrix} 1 + \kappa - c^2\omega_c^2 \\ e^{-i\omega_c} + \kappa e^{i\omega_c} \end{pmatrix}$$

Thus $\widehat{\mathcal{S}_{\mathbf{K}}\boldsymbol{\nu}_1^c}(1) = \pm\widehat{\boldsymbol{\nu}_1^c}(1)$ if and only if

$$-\begin{pmatrix} 1 + \kappa - c^2\omega_c^2 \\ e^{-i\omega_c} + \kappa e^{i\omega_c} \end{pmatrix} = \pm\begin{pmatrix} e^{i\omega_c} + \kappa e^{-i\omega_c} \\ 1 + \kappa - c^2\omega_c^2 \end{pmatrix},$$

from which it follows that $\widehat{\mathcal{S}_{\mathbf{K}}\boldsymbol{\nu}_1^c}(1) = \pm\widehat{\boldsymbol{\nu}_1^c}(1)$ is equivalent to

$$e^{i\omega_c} + \kappa e^{-i\omega_c} = \pm(1 + \kappa - c^2\omega_c^2).$$

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Taking real and imaginary parts, we have $\widehat{\mathcal{S}_{\mathbf{K}}\boldsymbol{\nu}_1^c}(1) = \pm\widehat{\boldsymbol{\nu}_1^c}(1)$ if and only if

$$\begin{cases} (1+\kappa)\cos(\omega_c) = \pm(1+\kappa-c^2\omega_c^2) & (B.25a) \\ (1-\kappa)\sin(\omega_c) = 0. & (B.25b) \end{cases} \quad \text{(B.25)}$$

Since we are working with a spring dimer and $\kappa \neq 1$, (B.25b) is equivalent to $\omega_c = j\pi$ for some $j \in \mathbb{Z}$.

We use (B.2) with $w = 1$ and $\omega_c = j\pi$, $j \in \mathbb{Z}$, and the definition of $\widetilde{\varrho}$ in (2.7) to compute

$$1 + \kappa - c^2\omega_c^2 = -\frac{\widetilde{\varrho}(\omega_c)}{2} = 1 + \kappa,$$

and so (B.25a) is equivalent to

$$(1+\kappa)(-1)^j = \pm(1+\kappa).$$

Thus $\widehat{\mathcal{S}_{\mathbf{K}}\boldsymbol{\nu}_1^c}(1) = \pm\widehat{\boldsymbol{\nu}_1^c}(1)$ if and only if $(-1)^j = \pm 1$, so $\widehat{\mathcal{S}_{\mathbf{K}}\boldsymbol{\nu}_1^c}(1) = \widehat{\boldsymbol{\nu}_1^c}(1)$ if and only if j is even, while $\widehat{\mathcal{S}_{\mathbf{K}}\boldsymbol{\nu}_1^c}(1) = -\widehat{\boldsymbol{\nu}_1^c}(1)$ if and only if j is odd.

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