ENTIRE AND MEROMORPHIC FUNCTIONS WITH ASYMPTOTICALLY PRESCRIBED CHARACTERISTIC

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Introduction. If $f(z)$ is a non-constant, entire function, then Hadamard's three-circles theorem asserts that

$$\log M(r,f) = \sup_{|z|\leqslant r} \log|f(z)|$$

is a convex, increasing function of $\log r$. Hence, by well-known properties of logarithmically convex functions,

$$\log M(r,f) = \log M(r_0,f) + \int_{r_0}^{r} \frac{\psi(t)}{t}\,dt \qquad (r \geqslant r_0),$$

where $r_0 > 0$ and $\psi(t)$ is a non-negative, non-decreasing function of t.

Valiron (**6**, p. 130) considered the following problem:

Given a function

$$\Lambda(r) = \text{const.} + \int_{\alpha}^{r} \frac{\psi(t)}{t}\,dt \qquad (r \geqslant \alpha > 0), \tag{1}$$

where $\psi(t)$ is non-negative and non-decreasing, is it always possible to find an entire function $f(z)$ such that

$$\log M(r,f) \sim \Lambda(r),$$

as $r \to \infty$.

Valiron's results may be summarized by

THEOREM A. *Let $\Lambda(r)$ be given by* (1), *where $\psi(t)$ is non-negative, non-decreasing, and unbounded. Assume further that*

$$\Lambda(r) < r^K, \tag{2}$$

for some $K > 0$ and all sufficiently large r.

Then there exists an entire function $f(z)$, of finite order, such that

$$\log M(r,f) \sim \Lambda(r) \qquad (r \to +\infty). \tag{3}$$

If the hypothesis (2) *is omitted, it is still possible to find an entire function $f(z)$, satisfying* (3), *provided that, as $r \to +\infty$, it omits the values of an exceptional set $\mathfrak{E}$, of finite logarithmic measure.*

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In this note we consider the analogue of Valiron's problem for the characteristic of Nevanlinna, $T(r,f)$. We assume that the reader is familiar with the fundamental concepts of Nevanlinna's theory and with the most usual of its symbols: $\overset{+}{\log}$, $m(r,f)$, $n(r,f)$, $N(r,f)$, $T(r,f)$.

It is well known that, for any meromorphic function $f(z)$, $T(r,f)$ is logarithmically convex and hence may be represented as

$$T(r,f) = \text{const.} + \int_{\alpha}^{r} \frac{\psi(t)}{t}\,dt \qquad (r \geqslant \alpha > 0), \tag{4}$$

where $\psi(t)$ is non-negative and non-decreasing. Moreover, if $f(z)$ is not rational,

$$\log r = o(T(r,f)) \qquad (r \to +\infty), \tag{5}$$

which is equivalent to saying that, in (4), $\psi(t)$ is unbounded.

Our main result is

THEOREM 1. I. *If $\Lambda(r)$ is of the form* (1), *where $\psi(t)$ is non-negative, non-decreasing, and unbounded and if*

$$\Lambda(r) < r^{K}, \tag{2}$$

for some $K > 0$ and all sufficiently large r, then there exists an entire function $f(z)$, of finite order, such that

$$T(r,f) \sim \Lambda(r) \qquad (r \to \infty). \tag{6}$$

II. *If Condition* (2) *is omitted, then* (6) *still holds provided r tends to infinity avoiding an exceptional set E of finite measure.*

III. *If $E(r, \infty)$ denotes the portion of E in (r, ∞), then*

$$\text{meas}\, E(r, \infty) = O(1/\Lambda(r)) \qquad (r \to \infty). \tag{7}$$

IV. *If $\Lambda(r)$ satisfies the relation*

$$\Lambda(r + 1/\Lambda(r)) < \exp\,(\Lambda^{\eta}(r))$$

for some η in $0 < \eta < \frac{1}{2}$, then (6) *holds without reference to an exceptional set.*

Our proof depends on the construction of entire functions of the form

$$f(z) = \prod_{j=1}^{\infty} \left(1 + \left\{\frac{z}{t_j}\right\}^{q_j}\right), \tag{8}$$

where the q_j are suitable positive integers.

It might be interesting to point out that the functions of Theorem 1 satisfy the inequality

$$T(r,f) \leqslant \log M(r,f) \leqslant T(r,f) + KT^{2\eta}(r,f)\log^{1-2\eta} r \qquad (r \notin E), \tag{9}$$

where $K(>0)$ and $\eta(0 < \eta < \frac{1}{2})$ are suitable constants and E has the same meaning as in Theorem 1. (Throughout this note the symbols K and r_0 denote positive constants which are not necessarily the same ones each time they occur. The symbols $(r > r_0)$ following some relation simply mean that the relation holds for all sufficiently large values of r.)

If $f(z)$ is of finite order, (9) holds for all sufficiently large values of r, without reference to an exceptional set.

Combining (9) and Theorem 1, we obtain Valiron's Theorem A, with the additional information that the exceptional set $\mathfrak{E}$ may be taken of finite measure.

An inequality such as (9) makes it impossible for $f(z)$ to have exceptional values. More precisely, we prove, in Section 5, that (9) implies the existence of a set $\mathfrak{E}$, of finite logarithmic measure, such that, for any finite value of c,

$$(10) \qquad \lim_{\substack{r\to\infty \\ r\notin\mathfrak{E}}} \frac{N(r, 1/(f-c))}{T(r,f)} = 1.$$

The method which enables us to deduce (10) from (9) is of some independent interest and readily yields other results of the same type. For instance:

If $f(z)$ is entire, of finite lower order, and if

$$\log M(r,f) \sim T(r,f) \qquad (r \to \infty),$$

then $f(z)$ has no finite deficient values.

In a recent note, Alpár and Turán **(1)** have shown, by an ingenious use of gap series, that, *given any strictly positive, decreasing function $h(x)$, it is always possible to find an entire function $f(z)$ such that, for every finite c, the sequence*

$$z_1(c), \qquad z_2(c), \qquad z_3(c), \ldots$$

of the zeros of $f(z) - c$ has the property

$$(11) \qquad \sum_{k=1}^{\infty} h(|z_k(c)|) = +\infty.$$

Alpár and Turán posed the problem of constructing such functions by other methods. Theorem 1 provides such a method since, by choosing suitably $\Lambda(r)$, it is easy to deduce (11) from (10).

The problem of constructing functions with prescribed asymptotic behaviour of the Nevanlinna characteristic becomes much easier if we select our examples from the wider class of meromorphic functions. The construction may then be based on the following

THEOREM 2. *Let $z_1, z_2, z_3, \ldots$ $(|z_1| \leqslant |z_2| \leqslant |z_3| \leqslant \ldots)$ be a given sequence of distinct complex numbers having no finite point of accumulation and let $\mu_1, \mu_2, \mu_3, \ldots$ be a given sequence of positive integers. Finally, let $\xi(r)$ be a given function of r (>0), decreasing, strictly positive, otherwise arbitrary.*

Then, it is possible to find a meromorphic function $f(z)$, of the form

$$(12) \qquad f(z) = \sum_{k=1}^{\infty} \frac{\alpha_k}{(z-z_k)^{\mu_k}} \qquad \left(\alpha_k > 0, \sum \alpha_k < +\infty\right),$$

and a set E, of finite measure, such that

$$T(r,f) = N(r,f) \qquad (r \notin E)$$

and

$$0 \leqslant T(r,f) - N(r,f) < \xi(r) \qquad (r > r_0,\ r \in E).$$

Moreover

$$N(r, 1/f) = T(r,f) + O(\log r) \qquad (r \to +\infty),$$

and for any $c \neq 0$,

$$N(r, 1/(f-c)) = T(r,f) + O(1) \qquad (r \to +\infty).$$

The proof of Theorem 2 is too elementary to be included here. It may be supplied readily by noticing that, if

$$p = \inf_k \{\mu_k\},$$

and if the positive sequence $\{\alpha_k\}$ decreases very rapidly, the function in (12) satisfies, as $r \to +\infty$, the asymptotic relation

$$f(z) = \frac{\gamma}{z^p} + O\left(\frac{1}{r^{p+1}}\right) \qquad \left(\gamma = \sum_{\mu_k = p} \alpha_k\right),$$

outside narrow rings

$$|z_j| - \epsilon_j < |z| < |z_j| + \epsilon_j \qquad (\epsilon_j > 0,\ \sum \epsilon_j < +\infty).$$

It is obvious, in view of Theorem 2, that given $h(x)$ as in the problem of Alpár and Turán, it is possible to find a meromorphic function $f(z)$ such that (11) holds for $c = \infty$ as well as for every finite c.

Finally, we observe that a very simple solution of the problem of Alpár and Turán may be derived from the following immediate consequence of the fundamental relations of R. Nevanlinna:

Let $F(z)$ be a given function, entire or meromorphic, but not rational. Then, if the constant a is suitably chosen, the function

$$f(z) = F(z) - az \tag{13}$$

satisfies, for every finite c, the relation

$$\lim_{\substack{r \to \infty \\ r \notin E}} \frac{N(r, 1/(f-c))}{T(r,f)} = 1. \tag{14}$$

The exceptional set E which appears in (14) *is now of finite measure; it is independent of c and may be omitted altogether if $F(z)$ is of finite order.*

To prove this proposition, we first choose a, in (13), so that

$$N(r, 1/f') = N(r, 1/(F'-a)) \sim T(r, F') \qquad (r \to \infty); \tag{15}$$

this is possible by a classical result (**5**, p. 280).

By Nevanlinna's first fundamental theorem,

(16) $$T(r, F') = T(r,f') + O(1) = m(r, 1/f') + N(r, 1/f') + O(1) \qquad (r \to \infty).$$

Combining (15) and (16), we find that

(17) $$m(r, 1/f') = o(T(r,f')) \qquad (r \to \infty).$$

By the well-known properties of the derivative (**4**, p. 104)

(18) $$T(r,f') < KT(r,f) \qquad (r \notin E),$$

and hence, combining (17) and (18), we find that

(19) $$m(r, 1/f') = o(T(r,f)) \qquad (r \to \infty, r \notin E).$$

Consider now the identity

$$\frac{1}{f-c} = \frac{f'}{f-c}\frac{1}{f'},$$

where c is finite, otherwise arbitrary.

Using elementary inequalities, (19), and the theorem on the logarithmic derivative, we find that

(20) $$m(r, 1/(f-c)) \leqslant m(r, f'/(f-c)) + m(r, 1/f') = o(T(r,f)) \qquad (r \to \infty, r \notin E).$$

(The fact that, in (20), E is independent of c may be seen by using the first fundamental theorem in Nevanlinna's estimate for $m(r, f'/(f-c))$ (**4**, p. 61).)

Since

$$m(r, 1/(f-c)) + N(r, 1/(f-c)) = T(r,f) + O(1),$$

(20) yields (14).

1. Preliminary constructions. In the following proof, we replace (1) by

(1.1) $$\Lambda(r) = \int_1^r \frac{\phi(t)}{t}\,dt,$$

and assume that

(i) $\phi(t)$ is continuous,

(ii) $\phi(t)$ is strictly increasing and unbounded,

(iii) $\phi(1) = 0$.

These regularity assumptions do not restrict the generality of Theorem 1 because, given $\psi(t)$, as in Theorem 1, it is easily seen that there exists a function $\phi(t)$, satisfying the conditions (i), (ii), (iii), and such that

$$\int_1^r \frac{\phi(t)}{t}\,dt \sim \text{const.} + \int_\alpha^r \frac{\psi(t)}{t}\,dt \qquad (r \to \infty).$$

Starting from (1.1), we define a function $B(r)$ by the conditions

(1.2) $$B(r) = \frac{\Lambda(r)}{\log r} \qquad (r > 1, B(1) = 0).$$

By the mean-value theorem, it is obvious that $B(r)$ is continuous for $r \geqslant 1$, strictly increasing, and unbounded.

Let

(1.3) $$r_1, r_2, r_3, \ldots$$

be the sequence defined by the conditions

(1.4) $$j = B^{2\eta}(r_j)\log r_j \qquad (j = 1, 2, 3, \ldots),$$

where η $(0 < \eta < \frac{1}{2})$ is fixed. Since $B^{2\eta}(x)\log x$ is continuous, strictly increasing, and unbounded and since $B^{2\eta}(1)\log 1 = 0$, the sequence (1.3) is uniquely determined, strictly increasing, and unbounded.

We now set

(1.5) $$k_j^{\,j} = \exp\left(\frac{j}{\{B(r_j)\}^\eta}\right) = \exp(\{B(r_j)\}^\eta \log r_j),$$

and notice that the sequence $\{k_j^{\,j}\}_{j=1}^\infty$ is increasing and unbounded, whereas the sequence $\{k_j\}_{j=1}^\infty$ is decreasing and

$$\lim_{j\to\infty} k_j = 1.$$

Denoting by $[X]$ the greatest integer contained in X, it is easily seen that the sequence $\{q_j\}$ defined by

$$q_j = [2jk_1k_2k_3\ldots k_j] + 1 \qquad (j = 1, 2, 3, \ldots)$$

satisfies simultaneously the four following relations:

(1.6) $$q_j > k_j^{\,j} \qquad (j \geqslant 1),$$

(1.7) $$q_{j+1} > q_j \qquad (j \geqslant 1),$$

(1.8) $$\lim_{j\to\infty} (q_{j+1}/q_j) = 1,$$

(1.9) $$\lim_{j\to\infty} (q_j/\{q_1 + q_2 + q_3 + \ldots + q_j\}) = 0.$$

2. Construction of an entire function f such that $N(r, 1/f) \sim \Lambda(r)$. Put

(2.1) $$Q_j = q_1 + q_2 + q_3 + \ldots + q_j \qquad (j = 1, 2, 3, \ldots)$$

and define an increasing unbounded sequence $\{t_j\}_{j=1}^\infty$ by the conditions

(2.2) $$\phi(t_j) = Q_j \qquad (j = 1, 2, 3, \ldots).$$

The existence and uniqueness of $\{t_j\}$ are obvious since, by assumption, $\phi(1) = 0$ and $\phi(t)$ is continuous, strictly increasing, and unbounded.

We now set

$$\begin{cases} n(t) = 0 & (0 \leqslant t < t_1), \\ n(t) = Q_j & (t_j \leqslant t < t_{j+1}, j = 1, 2, 3, \ldots), \end{cases} \tag{2.3}$$

and notice that, in view of (2.2), (1.9), and (1.8),

$$1 \leqslant \frac{\phi(t)}{n(t)} < 1 + \frac{q_{j+1}}{Q_j} \qquad (t_j \leqslant t < t_{j+1}, j \geqslant 1),$$

$$\lim_{t\to\infty} \frac{\phi(t)}{n(t)} = 1. \tag{2.4}$$

We consider next the infinite product

$$\prod_{j=1}^{\infty} \left(1 + \left\{\frac{z}{t_j}\right\}^{q_j}\right) = f(z), \tag{2.5}$$

where, by (1.7),

$$q_m - q_j \geqslant m - j \qquad (m \geqslant j).$$

Hence, if $|z| = r < R$ and if $p = p(R)$ is defined by

$$t_p \leqslant R < t_{p+1}, \tag{2.6}$$

we have

$$\begin{aligned} \sum_{t_j > R} \left|\frac{z}{t_j}\right|^{q_j} &\leqslant \sum_{t_j > R} \left\{\frac{r}{R}\right\}^{q_j} < \sum_{j=p}^{\infty} \left\{\frac{r}{R}\right\}^{q_j} \\ &= \left\{\frac{r}{R}\right\}^{q_p} \sum_{j=p}^{\infty} \left\{\frac{r}{R}\right\}^{q_j - q_p} \leqslant \left\{\frac{r}{R}\right\}^{q_p} \frac{R}{R-r}. \end{aligned} \tag{2.7}$$

These inequalities show that the product in (2.5) converges uniformly in every bounded region. Hence $f(z)$ is an entire function. Comparing (2.3) and (2.5), we see that

$$n(r, 1/f) = n(r),$$

and hence, by (2.4) and (1.1),

$$N(r, 1/f) = \int_1^r \frac{n(t)}{t}\, dt \sim \Lambda(r) \qquad (r \to \infty). \tag{2.8}$$

Further, for $r < R$, and p defined by (2.6),

$$\begin{aligned} \log M(r, f) &= \sum_{t_j \leqslant r} q_j \log(r/t_j) + \sum_{t_j \leqslant r} \log\left(1 + \left\{\frac{t_j}{r}\right\}^{q_j}\right) \\ &\quad + \sum_{r < t_j \leqslant R} \log\left(1 + \left\{\frac{r}{t_j}\right\}^{q_j}\right) + \sum_{R < t_j} \log\left(1 + \left\{\frac{r}{t_j}\right\}^{q_j}\right) \\ &\leqslant N(r, 1/f) + p \log 2 + \sum_{t_j > R} \left\{\frac{r}{t_j}\right\}^{q_j}. \end{aligned} \tag{2.9}$$

Hence, by (2.7) and elementary inequalities of Nevanlinna's theory,

$$N(r, 1/f) \leqslant T(r, f) \leqslant \log M(r, f) \leqslant N(r, 1/f) + p \log 2 + \left\{\frac{r}{R}\right\}^{q_p} \frac{R}{R-r} \qquad (r < R), \tag{2.10}$$

where p is defined by (2.6).

3. Proof of Assertion I of Theorem 1. Choose, in (2.10),

$$R = 2r. \tag{3.1}$$

Then, by (2.1), (2.2), and (2.6),

$$q_p \leqslant Q_p \leqslant \phi(2r) < \int_{2r}^{2er} \frac{\phi(t)}{t}\, dt < \Lambda(2er)$$

and hence, by (2),

$$q_p = O(r^K) \qquad (p = p(2r),\ r \to \infty). \tag{3.2}$$

By (1.5) and (1.6),

$$\exp(B^{\eta}(r_p)\log r_p) < q_p,$$

which, combined with (3.2), yields

$$B^{\eta}(r_p)\log r_p = O(\log r), \tag{3.3}$$

and hence

$$r_p < r \qquad (p > p_0). \tag{3.4}$$

Returning to (1.4), using (3.3), (3.4), and the fact that $B(r)$ is increasing, we find that

$$p < B^{\eta}(r) K \log r \qquad (r > r_0).$$

Hence, we deduce from (2.10) that

$$N(r, 1/f) \leqslant T(r, f) \leqslant \log M(r, f) \leqslant N(r, 1/f) + KB^{\eta}(r)\log r \qquad (r \to \infty), \tag{3.5}$$

which, in view of (2.8) and (1.2), yields (6).

4. Proof of Assertion II of Theorem 1. Assume that r is sufficiently large and fixed, and that the variable ρ increases from r to $+\infty$.

Consider the increasing continuous function $H(\rho) = (\rho - r)/\rho$ and the non-increasing step function

$$L(\rho) = \frac{\log q_s}{q_s} \qquad (q_s \geqslant 3),$$

where $s = s(\rho)$ is defined by

$$Q_s \leqslant \phi(\rho) < Q_{s+1}. \tag{4.1}$$

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The function

$$G(\rho) = L(\rho) - H(\rho)$$

is strictly decreasing and continuous on the right. For ρ sufficiently large, we have $G(\rho) < 0$, whereas

$$G(r) = L(r) > 0.$$

Denote by R the least upper bound of those values of $\rho(\geqslant r)$ such that $G(\rho) \geqslant 0$ and let

$$p = s(R). \tag{4.2}$$

If $\epsilon > 0$, we have $G(R + \epsilon) < 0$ and so

$$\lim_{\epsilon \to 0+} G(R + \epsilon) = G(R) \leqslant 0,$$

which, in view of (4.2), may be rewritten as

$$\frac{\log q_p}{q_p} \leqslant \frac{R - r}{R}. \tag{4.3}$$

Hence $R > r$ and, if ϵ (>0) is sufficiently small

$$p - 1 \leqslant s(R - \epsilon).$$

Then, by definition of R,

$$\frac{R - \epsilon - r}{R - \epsilon} \leqslant \frac{\log q_{p-1}}{q_{p-1}}$$

and hence, letting $\epsilon \to 0+$, we find that

$$\frac{R - r}{R} \leqslant \frac{\log q_{p-1}}{q_{p-1}}. \tag{4.4}$$

Now (1.7) and (1.8) imply that

$$\lim_{p \to \infty} \left\{ \frac{\log q_{p-1}}{q_{p-1}} \Big/ \frac{\log q_p}{q_p} \right\} = 1,$$

and hence (4.3) and (4.4) yield

$$\frac{\log q_p}{q_p} \leqslant \frac{R - r}{R} < 2\,\frac{\log q_p}{q_p} \qquad (r > r_0, R = R(r), p = s(R)). \tag{4.5}$$

Using the first of the inequalities (4.5), we obtain

$$\left(\frac{r}{R}\right)^{q_p} \frac{R}{R - r} < \exp\left(-q_p \frac{R - r}{R}\right) \frac{R}{R - r} \leqslant \frac{1}{\log q_p}. \tag{4.6}$$

Using (4.6) in (2.10), we find that

$$N(r, 1/f) \leqslant T(r, f) \leqslant \log M(r, f) \leqslant N(r, 1/f) + p \log 2 + O(1), \tag{4.7}$$

as $r \to +\infty$.

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If

$$p < B^{2\eta}(r)\log r, \tag{4.8}$$

then, by (2.8) and (1.2),

$$p < 2N^{2\eta}(r, 1/f)\log^{1-2\eta}r \leqslant 2T^{2\eta}(r, f)\log^{1-2\eta}r. \tag{4.9}$$

The assertion (6) of Theorem **1** now follows (under the assumption (4.8)) from (4.7), (4.9), (2.8), and (5).

To complete the proof of Theorem 1, we show first that

$$p \geqslant B^{2\eta}(r)\log r \tag{4.10}$$

implies that

$$\Lambda(r + 1/\Lambda(r)) > \exp\{\Lambda^{\eta}(r)\} \qquad (r > r_0). \tag{4.11}$$

By (1.4) and (4.10)

$$B^{2\eta}(r)\log r < B^{2\eta}(r_p)\log r_p \tag{4.12}$$

and therefore, since $B(r)$ is increasing,

$$r < r_p. \tag{4.13}$$

By (1.6), (1.5), and (4.13),

$$q_p > \exp\{B^{\eta}(r_p)\log r_p\} > \exp\{B^{\eta}(r)\log r\}, \tag{4.14}$$

and since $(\log x)/x$ is decreasing for $x > e$, (4.14) and the second inequality in (4.5) yield

$$\begin{aligned} \frac{R-r}{R} &< 2\,\frac{\log q_p}{q_p} \\ &< 2B^{\eta}(r)\log r\,\exp\{-B^{\eta}(r)\log r\} \\ &< 2B^{\eta}(r)\exp\{-\tfrac{1}{2}B^{\eta}(r)\log r\}r^{-1}\log r \qquad (r > r_0). \end{aligned}$$

Hence, since

$$\tfrac{1}{2}B^{\eta}(r)\log r > (B(r)\log r)^{\eta} = \Lambda^{\eta}(r) \qquad (r > r_0),$$

we have

$$\begin{aligned} \frac{R-r}{R} &< 4\,\Lambda^{\eta}(r)\,\exp\{-\,\Lambda^{\eta}(r)\}\cdot\frac{1}{r} \\ &< \frac{1}{4r\,\Lambda(r)} \qquad (r > r_0), \end{aligned}$$

which implies

$$R < r + \frac{1}{2\,\Lambda(r)} \qquad (r > r_0). \tag{4.15}$$

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By (4.14) and the definitions (4.1) and (4.2),

$$\exp\{B^{\eta}(r)\log r\} < \exp\{B^{\eta}(r_p)\log r_p\} < q_p < Q_p \leqslant \phi(R). \tag{4.16}$$

On the other hand, by (4.15) and (1.1),

$$\begin{aligned}\phi(R) &< \frac{1}{r + \Lambda^{-1}(r) - R}\int_R^{r+\Lambda^{-1}(r)} \frac{r + \Lambda^{-1}(r)}{t}\,\phi(t)dt \\ &< (2r\Lambda(r) + 2)\ \Lambda\left(r + \frac{1}{\Lambda(r)}\right).\end{aligned} \tag{4.17}$$

Now in view of (1.2)

$$\Lambda^{\eta}(r) < \tfrac{1}{4}B^{\eta}(r)\log r \qquad (r > r_0),$$

$$\begin{aligned}2r\Lambda(r) + 2 < 3r\Lambda(r) &< \exp\{\tfrac{1}{4}B^{\eta}(r)\log r + \Lambda^{\eta}(r)\} \\ &< \exp\{\tfrac{1}{2}B^{\eta}(r)\log r\} \qquad (r > r_0).\end{aligned}$$

Therefore, by (4.16) and (4.17),

$$\Lambda(r + 1/\Lambda(r)) > \exp\{\tfrac{1}{2}B^{\eta}(r)\log r\} > \exp(\Lambda^{\eta}(r)) \qquad (r > r_0). \tag{4.18}$$

We have thus shown that (4.10) implies (4.11).

By a well-known lemma of E. Borel, the inequality

$$\Lambda(r + 1/\Lambda(r)) > 2\Lambda(r)$$

cannot hold outside a set E, of values of r, satisfying the condition (7). (This form of Borel's lemma will be found in (**2**, p. 18).)

Hence $r \notin E$ implies (4.9) and all the assertions of Theorem 1 become obvious.

The relation (9) follows from (3.5) if $f(z)$ is of finite order and from (4.7) and (4.9) if the condition (2) is omitted.

5. Study of $N(r, 1/(f-c))/T(r,f)$. We start from the following form of a lemma which has been proved elsewhere.

LEMMA (**3**, Lemma III). *Let $f(z)$ be meromorphic and let c be any finite complex quantity. Let $I(r)$ be the set of θ for which*

$$|f(re^{i\theta}) - c| < 1,$$

and let

$$\mu(r) = \text{meas}\ I(r).$$

Then, for $1 < r < R$,

$$m\left(r, \frac{1}{f-c}\right) \leqslant \frac{11R}{R-r}\, T\left(R, \frac{1}{f-c}\right)\mu(r)\left\{1 + \overset{+}{\log}\frac{1}{\mu(r)}\right\}. \tag{5.1}$$

In view of the first fundamental theorem, it is obvious that (5.1) may be replaced by

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$$(5.2) \quad m\left(r, \frac{1}{f-c}\right) \leqslant \frac{12R}{R-r} T(R,f)\mu(r)\left\{1 + \overset{+}{\log}\frac{1}{\mu(r)}\right\} \qquad (r_0 < r < R).$$

In view of the mean-value theorem, the definition of $T(r,f)$ implies that

$$2\pi T(r,f) \leqslant \int_{I(r)} \overset{+}{\log}|f(re^{i\theta})|d\theta + \{2\pi - \mu(r)\}\log M(r,f)$$
$$\leqslant \mu(r)\log(1+|c|) + (2\pi - \mu(r))\log M(r,f).$$

Hence, by (9),

$$(5.3) \qquad \mu(r) \leqslant K\frac{T^{2\eta}(r,f)\log^{1-2\eta}r}{\log M(r,f)} \leqslant K\left\{\frac{\log r}{T(r,f)}\right\}^{1-2\eta} \qquad (r \notin E).$$

Since $T(r,f)$ is logarithmically convex,

$$(5.4) \qquad T(r,f) = T(1,f) + D(r)\log r \qquad (T(1,f) \geqslant 0),$$

where $D(r)$ is non-decreasing. Hence (5.3) implies that

$$\mu(r) \leqslant K\left\{\frac{1}{D(r)}\right\}^{1-2\eta} \qquad (r \notin E).$$

Using this inequality in (5.2) and assuming that

$$r_0 < r < R < 2r,$$

we find that

$$(5.5) \qquad m\left(r, \frac{1}{f-c}\right) \leqslant \frac{K\, r(\log r)D(R)}{(R-r)\{D(r)\}^{\tau}} \qquad (r \notin E, r > r_0),$$

where $\tau(0 < \tau < 1)$ is a suitable constant.

By Borel's lemma (**2**, p. 19), we have

$$(5.6) \qquad D(R) = D\left(r + \frac{r}{\log D(r)}\right) < D^{1+\epsilon}(r) \qquad (r \notin \mathfrak{E}^*),$$

where ϵ is an arbitrary fixed positive quantity and $\mathfrak{E}^*$ is a set of finite logarithmic measure.

Using (5.6) and (5.4) in (5.5), we obtain

$$\frac{m(r, 1/(f-c))}{T(r,f)} \leqslant K\{\log D(r)\}D^{\epsilon-\tau}(r) = o(1),$$

provided that we have chosen $0 < \epsilon < \tau$ and that, as $r \to +\infty$ we assume that

$$r \notin \{E \cup \mathfrak{E}^*\} = \mathfrak{E}.$$

Since

$$N(r, 1/(f-c)) = T(r,f) - m(r, 1/(f-c)) + O(1),$$

the proof of (10) is complete.

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