Canad. Math. Bull. Vol. **17** (4), 1974

ON PURE-HIGH SUBGROUPS OF ABELIAN GROUPS

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Introduction. L. Fuchs, in [3], problem 14, proposes the study of pure-high subgroups of an abelian group. In this paper we show that in abelian torsion groups, pure-high subgroups are also high. A natural problem arises, that of characterizing the pure-absolute summands. We show that this concept is the same as absolute summands in torsion groups, but that it is more general in mixed abelian groups. There is a definite connection between the existence of pure N-high subgroups and the splitting of mixed groups. The notation is that of [3].

1. **Pure-high Subgroups.** Let N be a subgroup of a group G. We say that a subgroup H of G is N-pure-high if it is maximal among the pure subgroups disjoint from N. Zorn's Lemma guarantees the existence of N-pure-high subgroups. The following theorem establishes the connection between N-pure-high and N-high subgroups of a torsion group.

THEOREM 1. *N-pure-high subgroups are N-high subgroups in torsion groups.*

Proof. Clearly, it is sufficient to establish the result for primary groups. We need the following fact: If N is a subgroup of a p-group G such that $N[p]\neq G[p]$ then there exists a non-zero pure subgroup of G disjoint from N. Indeed, two cases may occur: all elements of $G[p]$ are of infinite p-height or there exists $x \in G[p]$ such that $h(x)<\infty$. In the first case, G must be a divisible group (see [5], lemma 8) and any N-high subgroup of G is pure and non-zero. In the second case, there exists an element of $G[p]$ which is not in $N[p]$ and which is of finite height, for if $y \in G[p]\backslash N[p]$ and $h_p(y)=\infty$ then $y+x \in G[p]\backslash N[p]$ and $h_p(x+y)=h_p(x)<\infty$. This element generates the socle of a pure subgroup which will clearly be non-zero and disjoint from N (see [5], proof of theorem 9). Now we are ready to prove the theorem. Let K be an N-pure-high subgroup of G. To see that K is N-high we need only show that $K[p]\oplus N[p]=G[p]$ (see [4]). In G/K, we have

$$((N\oplus K)/K)[p] = (G/K)[p]. \tag{1}$$

Otherwise, by the result established above, there exists a subgroup H/K which is non-zero, pure and disjoint from $(N\oplus K)/K$. It follows from ([5], lemma 2) that H is pure in G, which contradicts the maximality of K.

From (i) we conclude $N[p]\oplus K[p]=G[p]$, since K is pure. Therefore K is N-high.

Received by the editors September 22, 1972 and, in revised form, May 14, 1973.

https://doi.org/10.4153/CMB-1974-084-3 Published online by Cambridge University Press

COROLLARY. *Every pure subgroup disjoint from a subgroup N of a torsion group G can be extended to a pure N-high subgroup of G.*

The result in theorem 1 is trivially true in torsion free groups because all N-high subgroups are pure. However, in mixed groups G which do not split over their torsion subgroups G_t, all G_t-high subgroups are not pure and thus G_t-pure-high subgroups are not G_t-high.

For abelian groups in general we have the following partial result:

THEOREM 2. *Let N be a subgroup of a group G. If one N-high subgroup is torsion, all N-high subgroups are torsion and N-pure-high subgroups are N-high.*

2. **Pure-Absolute Summands.** In this section we ask the following question. Which subgroups A of a group G have the property that $G=A\oplus H$ for every pure A-high subgroup H of G? We call such subgroups pure-absolute summands. Clearly absolute summands are pure-absolute summands but the converse is not true. Indeed, it is easy to show that the torsion subgroup of a splitting group is a pure-absolute summand which is not in general an absolute summand. Absolute summands were characterized for the first time by L. Fuchs and are also described in [1].

THEOREM 3. *In torsion groups, pure-absolute summands are absolute summands.*

Proof. It is sufficient to show that the statement is true for p-groups. Let A be a pure-absolute summand of a p-group G. Then by theorem 1, A is a summand of G. If $A[p]\subset G^1$, then A is divisible and therefore it is an absolute summand. If $A[p]\not\subset G^1$ then we must show (see [1], theorem 4.4, p. 343) that there exists $n\in Z^+$, such that

$$(p^{n+1}G)[p] \subset A[p] \subset (p^nG)[p].$$

Let $n\in Z^+$, such that $A[p]\subset(p^nG)[p]$ but $A[p]\not\subset(p^{n+1}G)[p]$. Such an n exists since $A[p]\not\subset G^1$. Thus there exists $a\in A[p]$ such that $h_p(a)=n$. Let $x\in(p^{n+1}G)[p]$ and suppose $x\notin A[p]$. Then $x+a\notin A[p]$ and $h_p(x+a)=n$. Therefore there exists a pure subgroup K containing $x+a$ and A-high. We have $G=A\oplus K$.

Since $x+a\in K$ and $a\in A$ and $h_p(x+a)=h_p(a)=n$, we have $h_p(x+a-a)=n$, but this is a contradiction because $h_p(x)\geqq n+1$. Therefore $x\in A[p]$ and

$$(p^{n+1}G)[p] \subset A[p] \subset (p^nG)[p].$$

Therefore A is an absolute summand.

The next result exhibits a family of subgroups of a mixed group which are pure-absolute summands, but not in general absolute summands.

THEOREM 4. *Let G be a mixed group and let P be the set of positive prime numbers. Then the subgroups* $G_S=\oplus_{p\in S}G_p$ *are pure absolute summands of G for every* $S\subset P$.

Proof. Let $A=G_S$ and $B=G_{P\setminus S}$. If there are no pure A-high subgroups then A is pure-absolute vacuously. Suppose then that H is a pure A-high subgroup of G. We know that $G_t=A\oplus B$. We show first that H contains B. We do this by showing that $G_p\subset H$, $\forall p\in P\setminus S$. Indeed, let $b\in G_p$. Then if $b\notin H$, we have $\langle H, b\rangle \cap A$ is not zero so there exists $m\in Z^+$, $h\in H$, and $a\in A$ such that $a\neq 0$ and $mb+h=a$. Clearly $mb\neq 0$. Let the order of mb be p^α, $\alpha\geqq 1$. Then we have $p^\alpha mb+p^\alpha h=p^\alpha a\in A\cap H=0$. Therefore $p^\alpha a=0$ and so $a=0$, a contradiction. Hence we have $b\in H$ and $G_p\subseteq H$ for all $p\in P\setminus S$.

Now suppose $g\in G$ and $g\notin H$. Then $\langle g, H\rangle \cap A\neq 0$ and as before there exists $n\in Z^+$, $h'\in H$ and $a'\in A$, so that $ng+h'=a'\neq 0$. Let the order of a' be r. Then $rng+rh'=ra'=0$, so $rng=-rh'\in H$. Since H is pure there exists $h''\in H$ such that $rn(g-h'')=0$. Hence $g-h''$ is torsion and thus $g-h''=a''+b''$ where $a''\in A$ and $b''\in B$. Thus we have $g=(h''+b'')+a''\in H\oplus A$, and A is a pure-absolute summand.

COROLLARY. *A mixed group G splits over its torsion subgroups G_t if and only if there exists a pure G_t-high subgroup of G.*

3. **Some related results.** We give next a necessary and sufficient condition for a group to contain a subgroup N for which no N-high subgroup is pure.

THEOREM 5. *Let G be a group. There exists a subgroup N of G for which no N-high subgroup is pure if and only if G does not split over its torsion subgroup G_t.*

Proof. If G does not split over G_t, take $N=G_t$ and use the corollary to theorem 4. Suppose now that N is a subgroup of G for which no N-high subgroup is pure. Let R/N_t be N/N_t-high in G/N_t. Then R is a pure subgroup of G containing G_t (see [1], lemma 3.2). Clearly if $G=G_t\oplus K$ then $R=G_t\oplus(K\cap R)$ and if we show that R does not split it will follow that G does not either. If $R=G_t\oplus H$ then H is G_t-high in R and pure in G. Now by the corollary to theorem 1, there exists a pure N_t-high subgroup M in G_t, and the subgroup $M\oplus H$ is a pure subgroup of G. It is easy to verify that $M\oplus H$ is an N-high subgroup of G and we are led to a contradiction. Therefore R does not split and thus G does not either.

COROLLARY. *If K is an N-pure-high subgroup of a group G and G/K splits over its torsion subgroup then K is N-high.*

Finally, the subgroups introduced in theorem 4 of the preceding section have a curious property embodied in the next result. The notation is the same as in theorem 4.

THEOREM 6. *Let G be a group, N a subgroup of G. Then there exists a unique N-high subgroup if and only if $N=0$ or N is an essential subgroup of a $(\oplus_{p\in S}G_p)$-high subgroup of G.*

Proof. Let $A=\oplus_{p\in S} G_p$ and let H be an A-high subgroup of G. If K is an essential subgroup of H then K-high subgroups are precisely H-high subgroups. We

https://doi.org/10.4153/CMB-1974-084-3 Published online by Cambridge University Press

will therefore show that if M is H-high then $M=A$. Note first that since A is H-high (see ex. 41 p. 95 in [2]) it suffices to show that $A\subset M$. We show this by showing that $G_p\subset M$ for each $p\in S$. Let $x\in G_p$ and suppose $x\notin M$, then there would exist $n\in Z^+$, $m\in M$, $h\in H$ such that $nx+m=h\neq 0$; but if $0(nx)=p^\alpha$ then $p^\alpha m=p^\alpha h=0$. However $(0(h), p)=1$ which implies $h=0$. This is a contradiction. Therefore $M=A$. Suppose now that N has a unique N-high subgroup K and $N\neq 0$ and N is not essential in G. We know that for each p, $N[p]\oplus K[p]=G[p]$. We show that for each p, either $N[p]=0$ or $K[p]=0$. Indeed if $N[p]$ and $K[p]\neq 0$ then there exists $x\in N[p]$ and $y\in K[p]$ and since $\langle y\rangle\cap N=0=N\cap\langle x+y\rangle$, y and $x+y$ are both in K which is a contradiction. We show now that $K[p]\neq 0$ implies $G_p\subset K$. Note that K is necessarily a torsion subgroup of G and from theorem 2, K is a pure subgroup of G. Let $K[p]=G[p]$ and let $g\in G_p$. We use induction on the order of g. Suppose $0(g)=p^n$ then $p^{n-1}g\in G[p]=K[p]$ and by the purity of K there exists $k\in K$ such that $p^{n-1}g=p^{n-1}k$. Therefore $g-k\in K$, by induction, and $g\in K$. Now let $S=\{p\mid K[p]=G[p]\}$, then $K\supset\bigoplus_{p\in S} G_p=A$ and since K is torsion $K=A$. If we let H be a K-high subgroup of G containing N we see that N is an essential subgroup of H.

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