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SOME WEAKER FORMS OF THE CHAIN (F) CONDITION FOR METACOMPACTNESS

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Abstract

We define, in a slightly unusual way, the rank of a partially ordered set. Then we prove that if X is a topological space and $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ satisfies condition (F) and, for every $x \in X$, $\mathcal{W}(x)$ is of the form $\bigcup_{i \in n(x)} \mathcal{W}_i(x)$, where $\mathcal{W}_0(x)$ is Noetherian of finite rank, and every other $\mathcal{W}_i(x)$ is a chain (with respect to inclusion) of neighbourhoods of x, then X is metacompact. We also obtain a cardinal extension of the above. In addition, we give a new proof of the theorem 'if the space X has a base $\mathcal{B}$ of point-finite rank, then X is metacompact', which was proved by Gruenhage and Nyikos.

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1. Introduction and terminology

The aim of this paper is to weaken the hypotheses of some results in [1, 4, 6] which are related to condition (F) and covering properties.

Recall that a T_1 topological space X has a $\mathcal{W}$ satisfying (F) if $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ where each $\mathcal{W}(x)$ consists of subsets of X containing x and the following condition is satisfied:

(F) if $x \in U$ and U is open, then there exists an open set $V = V(x, U)$ containing x such that $x \in W \subseteq U$ for some $W \in \mathcal{W}(y)$ whenever $y \in V$.

Any topological space clearly has such a family of open sets satisfying (F). If $\mathcal{W}$ satisfies (F) then $\mathcal{W}$ is said to satisfy *chain* (F), or *well-ordered* (F), if each $\mathcal{W}(x)$ is a chain with respect to inclusion, or each $\mathcal{W}(x)$ is well ordered by $\supseteq$.

In [1], it was established that if the space X has a $\mathcal{W}$ satisfying chain (F), then it is necessarily monotonically normal and hence it is collectionwise normal. The following results were also obtained in [1]:

(i) if the space X has a $\mathcal{W}$ satisfying well-ordered (F), then X is paracompact;

(ii) if the space X has a $\mathcal{W}$ satisfying chain (F) and each $\mathcal{W}(x)$ consists of neighbourhoods of x, then X is paracompact.

Furthermore, the following result was shown in [6]:

(iii) if the space X has a $\mathcal{W}$ satisfying chain (F) and, for each x,

$$\mathcal{W}(x) = \mathcal{W}_1(x) \cup \mathcal{W}_2(x),$$

where $\mathcal{W}_1(x)$ consists of neighbourhoods of x and $\mathcal{W}_2(x)$ is well ordered by $\supseteq$, then X is paracompact.

Throughout this paper, let X be a T_1 topological space, κ be an infinite cardinal number, and α, β, γ, λ, μ, ρ, τ denote cardinal or ordinal numbers, ω being the first infinite ordinal and cardinal. The interior of a subset A of X is denoted by $\text{int}(A)$ and the cardinality of a set B is denoted by $|B|$.

A family $\mathcal{A}$ of subsets of X is called *point-$<\kappa$*, if $|\{A \in \mathcal{A} : x \in A\}| < \kappa$, for each x in X.

Let $(P, \leq)$ be a partially ordered set. Two members a, b of P are said to be *independent* if $a \nleq b$ and $b \nleq a$. Also P is said to be *independent* if any two distinct members of P are independent. If P is not independent, then P is said to be *dependent*.

We define the *rank* of a partially ordered set P, denoted by $\text{rank}(P)$, in a slightly unusual way as the smallest cardinal number κ such that, for each subset B of P with $|B| \geq \kappa$, B is dependent. This definition is more distinctive than the usual definition of the rank of a partially ordered set P.

Let P be a partially ordered set. Let us say that P is *of sub-κ-rank* (*of finite rank*) if $\text{rank}(P) \leq \kappa$ (respectively, $\text{rank}(P) < \omega$).

A partially ordered set P is said to be *Noetherian* if every increasing subset of P is finite.

Since the family $\mathcal{W}(x)$ is partially ordered by inclusion for each x, we can mention rank and Noetherianness of $\mathcal{W}(x)$. Then $\mathcal{W}$ is said to be *Noetherian of sub-κ-rank* (F) if $\mathcal{W}$ satisfies (F), and each $\mathcal{W}(x)$ is Noetherian and of sub-κ-rank. Similarly, one can define $\mathcal{W}$ to be *Noetherian of finite rank* (F).

In [4], it was shown that if the space X has a $\mathcal{W}$ which is Noetherian of finite rank (F), then X is hereditarily metacompact.

The notation and terminology not explained above can be found in [3, 7].

2. Main results

It is clear that any family consisting of subsets of X, which is well ordered by reverse inclusion, is a chain (so it is of finite rank) and Noetherian. In [6], to assert that X is metacompact, the authors required the hypothesis that the union of $\mathcal{W}_1(x)$ and $\mathcal{W}_2(x)$ is a chain ($\mathcal{W}_1(x)$ and $\mathcal{W}_2(x)$ are also defined above). Our approach differs so that this condition is unnecessary (that is, the union of $\mathcal{W}_1(x)$ and $\mathcal{W}_2(x)$ need not be a chain). The proof in [6] would not hold if $\mathcal{W}_1(x)$, $\mathcal{W}_2(x)$ are considered as

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different chains so that their elements are not comparable. In Theorem 2.2, the families $\mathcal{W}(x)$ are given in a finite union of sets and $\mathcal{W}(x)$ is not a chain. The difficulties arising from the fact that $\mathcal{W}(x)$ is not chain are surpassed by employing the Erdös–Dushnik–Miller theorem.

We utilize the following lemma.

LEMMA 2.1. *Let n be a finite ordinal (that is, a non-negative integer), let $(P_i, \leq)$ be a partially ordered set for each $i \in n$, and let $P = \bigcup_{i\in n} P_i$. Let $\{a_\alpha : \alpha < \tau\}$ be a subset of P such that $a_\rho \nleq a_\alpha$ for each ρ, α in τ with $\alpha < \rho$, where τ is a cardinal number with $\tau \geq \kappa$. If P_0 is Noetherian of sub-κ-rank and P_i is a chain for each i with $1 \leq i \leq n$, then there exist a subset J of τ with $|J| = \kappa$ and an i_0 with $1 \leq i_0 \leq n$ such that $\{a_\alpha : \alpha \in J\}$ is an increasing subset of P_{i_0}.*

THEOREM 2.2. *If the space X has a $\mathcal{W}$ satisfying (F) and for each x there exists a finite ordinal $n(x)$ such that $\mathcal{W}(x) = \bigcup_{i\in n(x)} \mathcal{W}_i(x)$, where $\mathcal{W}_0(x)$ is Noetherian of sub-κ-rank and, for each $i \in n(x)\backslash\{0\}$, $\mathcal{W}_i(x)$ is a chain of neighbourhoods of x with respect to inclusion, then each open cover of X has a point-$<\kappa$ open refinement.*

PROOF. Let $\mathcal{O} = \{O_\alpha : \alpha < \tau\}$ be an open cover for X and $P_\alpha = O_\alpha \backslash \bigcup_{\beta<\alpha} O_\beta$ for each $\alpha < \tau$. Define

$$X_i = \{x \in X : 1 \leq i \leq n(x) \text{ and } \exists W \in \mathcal{W}_i(x), \exists \alpha < \tau, W \subseteq O_\alpha\},$$

for each $i \in \omega$,

$$\gamma(x, i) = \min\{\alpha < \tau : \exists W \in \mathcal{W}_i(x), W \subseteq O_\alpha\},$$

for each x in X_i, and

$$I(x) = \{i \in n(x)\backslash\{0\} : x \in X_i\}.$$

For each $x \in X_i$ and $i \in I(x)$ choose a $W(x, i) \in \mathcal{W}_i(x)$ with $W(x, i) \subseteq O_{\gamma(x,i)}$. Let

$$W_x = \bigcap_{i\in I(x)} W(x, i) \quad \text{and} \quad Y = \bigcup_{i\in\omega\backslash\{0\}} X_i,$$

where Y is indexed by some ordinal λ: $Y = \{x_\beta : \beta < \lambda\}$. In the manner of the proof of Theorem 5 in [1], we shall construct a subset Y_β of Y for each $\beta < \lambda$. Suppose that Y_γ has been constructed for each $\gamma < \beta$. Then define

$$Y_\beta = \begin{cases} \emptyset & \text{if } x_\beta \in \bigcup_{\gamma<\beta} Y_\gamma \\ \{z \in Y : x_\beta \in W_z,\ z \notin \bigcup_{\gamma<\beta} Y_\gamma\} & \text{otherwise.} \end{cases}$$

It is clear that $Y = \bigcup_{\beta<\lambda} Y_\beta$. Take any element x of X. There exists a unique $\alpha < \tau$ such that $x \in P_\alpha$. If x belongs to Y, then there exists a unique $\beta < \lambda$ such that $x \in Y_\beta$.

Define an open neighbourhood T_x of x as

$$T_x = \begin{cases} O_\alpha & x \in X \setminus Y \\ \operatorname{int}(W_x) \cap O_\alpha & x \in Y_\beta \text{ and } x = x_\beta \\ (\operatorname{int}(W_x) \setminus \{x_\beta\}) \cap O_\alpha & x \in Y_\beta \text{ and } x \neq x_\beta. \end{cases}$$

Then, for each $\alpha < \tau$, define $V_\alpha = \bigcup\{V(x, V(x, T_x)) : x \in P_\alpha\}$ where $V(x, T_x)$ is an open set arising from condition (F). Let $\mathcal{V} = \{V_\alpha : \alpha < \tau\}$. It is easy to see that $\mathcal{V}$ is an open refinement of $\mathcal{O}$. Suppose that $\mathcal{V}$ is not point-$<\kappa$. It follows that there exist an $x \in X$ and a subset I of κ such that the order type of I is equal to κ and $x \in V_\alpha$ for each $\alpha \in I$. From the definition of V_α, there exists a $y_\alpha \in P_\alpha$ such that $x \in V(y_\alpha, V(y_\alpha, T_{y_\alpha}))$ and so there exists an $S_\alpha \in \mathcal{W}(x)$ such that $y_\alpha \in S_\alpha \subseteq V(y_\alpha, T_{y_\alpha})$. Since $y_\alpha \in P_\alpha$ and $S_\alpha \subseteq T_{y_\alpha} \subseteq O_\alpha$ for each $\alpha \in I$, we have $y_\rho \notin S_\alpha$ for each α, ρ in I with $\alpha < \rho$. This leads us to the fact that $S_\rho \nsubseteq S_\alpha$ for $\alpha < \rho$. Since

$$\{S_\alpha : \alpha \in I\} \subset \mathcal{W}(x) \quad \text{and} \quad \mathcal{W}(x) = \bigcup_{i \in n(x)} \mathcal{W}_i(x),$$

there exist a subset J of I and an i_0 such that $|J| = \kappa$, $1 \leq i_0 \leq n(x)$ and the family $\{S_\alpha : \alpha \in J\}$ is an increasing subfamily of $\mathcal{W}_{i_0}(x)$, by Lemma 2.1. It follows that $y_\alpha \in V(y_\rho, T_{y_\rho})$ for each α, ρ in J with $\alpha < \rho$, and therefore there exists a $W_\rho^\alpha \in \mathcal{W}(y_\alpha)$ such that $y_\rho \in W_\rho^\alpha \subseteq T_{y_\rho}$. Thus, these facts and Lemma 2.1 lead us to the fact that $\{y_\alpha : \alpha \in J\} \subseteq Y$.

Now, let μ be any element of J. There exists a $W_\alpha^\mu \in \mathcal{W}(y_\mu)$ for each $\alpha \in J$ with $\mu < \alpha$ such that $y_\alpha \in W_\alpha^\mu \subseteq T_{y_\alpha}$. Since

$$\{W_\alpha^\mu : \alpha \in J,\ \alpha > \mu\} \subseteq \mathcal{W}(y_\mu) \quad \text{and} \quad \mathcal{W}(y_\mu) = \bigcup_{i \in n(y_\mu)} \mathcal{W}_i(y_\mu),$$

there exist a subset M of J and a j_0 such that $|M| = \kappa$, $1 \leq j_0 \leq n(y_\mu)$ and

$$\{W_\alpha^\mu : \alpha \in M,\ \alpha > \mu\} \subseteq \mathcal{W}_{j_0}(y_\mu),$$

from Lemma 2.1. Since

$$\{y_\alpha : \alpha \in J\} \subseteq Y \quad \text{and} \quad Y = \bigcup_{\beta < \lambda} Y_\beta,$$

there exists a $\beta_\alpha < \lambda$ with $y_\alpha \in Y_{\beta_\alpha}$ for each $\alpha \in M$. We can assume that $\beta_\alpha \leq \beta_\rho$ for each α, ρ in M with $\alpha \leq \rho$.

Let $\alpha_1 \in M$ such that $\alpha_1 > \mu$. So $W_{\alpha_1}^\mu \subseteq T_{y_{\alpha_1}} \subseteq O_{\alpha_1}$, $W_{\alpha_1}^\mu \in \mathcal{W}_{j_0}(y_\mu)$ and minimalities of $\gamma(y_\mu, j_0)$ lead us to the fact that $\gamma(y_\mu, j_0) \leq \alpha_1$. Choose an $\alpha_2 \in M$ such that $\alpha_2 > \alpha_1$ and $y_{\alpha_2} \neq x_{\beta_\mu}$. Since $\gamma(y_\mu, j_0) \leq \alpha_1 < \alpha_2$ and $y_{\alpha_2} \in P_{\alpha_2}$, we have that $y_{\alpha_2} \notin W(y_\mu, j_0)$, and, since $\mathcal{W}_{j_0}(y_\mu)$ is a chain with respect to inclusion, we have $W(y_\mu, j_0) \subseteq W_{\alpha_2}^\mu$. So $y_\mu \in Y_{\beta_\mu}$ and the definition of the set Y_{β_μ} lead us to the fact that $x_{\beta_\mu} \in W_{y_\mu}$, and we know that $W_{y_\mu} \subseteq W(y_\mu, j_0) \subseteq W^{\mu}_{\alpha_2} \subseteq T_{y_{\alpha_2}}$. Therefore $x_{\beta_\mu} \in T_{y_{\alpha_2}}$. Since $\mu < \alpha_2$, we have $\beta_\mu \leq \beta_{\alpha_2}$.

Suppose that $\beta_{\alpha_2} = \beta_\mu$. Then $T_{y_{\alpha_2}} \subseteq W_{y_{\alpha_2}} \backslash \{x_{\beta_\mu}\}$ by the definition of the set $T_{y_{\alpha_2}}$. But this contradicts the fact that $x_{\beta_\mu} \in T_{y_{\alpha_2}}$. Suppose that $\beta_\mu < \beta_{\alpha_2}$. Since $x_{\beta_\mu} \in T_{y_{\alpha_2}}$ and $T_{y_{\alpha_2}} \subseteq W_{y_{\alpha_2}}$, we have $x_{\beta_\mu} \in W_{y_{\alpha_2}}$. So $\beta_\mu < \beta_{\alpha_2}$, $y_{\alpha_2} \in Y_{\beta_{\alpha_2}}$ and the definition of the set $Y_{\beta_{\alpha_2}}$ lead us to the fact that $y_{\alpha_2} \notin \bigcup\{Y_\rho : \rho < \beta_\mu\}$. At the same time, since $x_{\beta_\mu} \in W_{y_{\alpha_2}}$, y_{α_2} has to belong to Y_{β_μ} by the definition of the set Y_{β_μ}. But this contradicts the fact that $\beta_\mu < \beta_{\alpha_2}$. So the family $\mathcal{V}$ is point-$< \kappa$. □

From Theorem 2.2, the following result can be concluded immediately.

COROLLARY 2.3. *If the space X has a $\mathcal{W}$ satisfying (F), and if, for each x, there exists a finite ordinal $n(x)$ such that $\mathcal{W}(x) = \bigcup_{i \in n(x)} \mathcal{W}_i(x)$, where $\mathcal{W}_0(x)$ is Noetherian of finite rank and, for each $i \in n(x)\backslash\{0\}$, $\mathcal{W}_i(x)$ is a chain of neighbourhoods of x with respect to inclusion, then X is metacompact.*

COROLLARY 2.4. *If the space X has a $\mathcal{W}$ satisfying (F), and if, for each x, there exists a finite ordinal $n(x)$ such that $\mathcal{W}(x) = \bigcup_{i \in n(x)} \mathcal{W}_i(x)$, where $\mathcal{W}_i(x)$ is a chain of neighbourhoods of x with respect to inclusion for each $i \in n(x)$, then X is metacompact.*

In [6], the authors pointed out that the Sorgenfrey line has a $\mathcal{W}$ satisfying chain (F) and, for each x, $\mathcal{W}(x) = \mathcal{W}_1(x) \cup \mathcal{W}_2(x)$ where $\mathcal{W}_1(x)$ consists of neighbourhoods of x and $\mathcal{W}_2(x)$ is well ordered by $\supseteq$. (Put $\mathcal{W}_1(x) = \{[x - \delta, x + \delta] : \delta > 0\}$ and $\mathcal{W}_2(x) = \{\{x\}\}$ for each x.)

The Sorgenfrey line is also an example of a space which satisfies the hypotheses of Corollary 2.4: one just puts, for each x,

$$\mathcal{W}(x) = \{[x, x + \delta] : \delta > 0\} \cup \{[x - \delta, x + \delta] : \delta > 0\}.$$

(Note that here $\mathcal{W}(x)$ is not a chain for each x.)

Dilworth's lemma, mentioned in [2, 7], says that 'if P is a partially ordered set such that every subset of $n + 1$ elements of P is dependent while at least one subset of n elements is independent, then P can be expressed as the sum of n disjoint totally ordered sets'. So, if $\mathcal{W}(x)$ is of finite rank for each x in X, then there exists a finite ordinal $n(x)$ for each x such that $\operatorname{rank}(\mathcal{W}(x)) = n(x) + 1$. Therefore, there exists an independent subset $\mathcal{A}_x$ of $n(x)$ elements of $\mathcal{W}(x)$ and we have that each subset of $n(x) + 1$ elements of $\mathcal{W}(x)$ is dependent. Hence, from Dilworth's lemma, $\mathcal{W}(x)$ can be expressed as the union of $n(x)$ chains. So, Corollary 2.4 and Dilworth's lemma give us the following result.

COROLLARY 2.5. *If the space X has a $\mathcal{W}$ which is of finite rank (F), and each $\mathcal{W}(x)$ consists of neighbourhoods of x, then X is metacompact.*

By means of the above corollary, the following result proved by Gruenhage and Nyikos [5, 7] is obtained in a different manner.

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COROLLARY 2.6. *If the space X has a base $\mathcal{B}$ of point-finite rank (that is, for each x, the family $\{B \in \mathcal{B} : x \in B\}$ is of finite rank), then X is metacompact.*

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