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ON ADDITIVE OPERATORS

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1. Introduction. Representation theorems for additive functionals have been obtained in [**2**, **4**; **6**–**8**; **10**–**13**]. Our aim in this paper is to study the representation of additive operators.

Let S be a compact Hausdorff space and let $C(S)$ be the space of real-valued continuous functions defined on S. Let X be an arbitrary Banach space and let T be an additive operator (see § 2) mapping $C(S)$ into X. We will show (see Lemma 3.4) that additive operators may be represented in terms of a family of "measures" $\{\mu_h\}$ which take their values in X^{**}. If X is weakly sequentially complete, then $\{\mu_h\}$ can be shown to take their values in X and are vector-valued measures (i.e., countably additive in the norm) (see Lemma 3.7). And, if X^* is separable in the weak-* topology, T may be represented in terms of a kernel representation satisfying the Carathéordory conditions (see [**9**; **11**; § 4]):

$$(x^*, T(f)) = \int_S K(x^*, f(s), s)\, \mu(ds) \quad \text{for each } x^* \in X^*.$$

While these results are proved by a procedure different from the bounded linear operator case, corresponding results for this case are included in the generalization, such as the following (reformulated from [**5**, pp 492–494]).

THEOREM. *Let X be a weakly sequentially complete Banach space and $T\colon C(S) \to X$ a bounded linear operator. Then there is a vector-valued measure μ (on the Borel sets) taking values in X so that:*

$$T(f) = \int_S f(s)\, \mu(ds) \quad \textit{for each } f \in C(S).$$

2. Preliminaries. The dual of a Banach space X will be denoted by X^*. If $x \in X$ and $x^* \in X^*$, then the evaluation of x^* at x will be denoted by (x, x^*), $x^*(x)$, or $x(x^*)$ depending on the context. If two Banach spaces X_1 and X_2 are in duality, then the weak topology induced on X_1 by X_2 is denoted by $\sigma(X_1, X_2)$.

$\mathscr{B}$ denotes the class of Borel sets of a compact Hausdorff space S. $M(S)$ denotes the Banach space of all regular real-valued measures defined on $\mathscr{B}$

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with the norm of a measure given by $||\mu|| = |\mu|(S)$, where $|\mu|$ is the total variation of μ. The Banach space of all bounded measurable functions on S under the sup norm, $||-||_\infty$, will be denoted by $B(S)$.

2.1. *Definition.* Let $f \in C(S)$. The carrier of f is the open set where f does not vanish and is denoted by $c(f)$. The support of f is the closure of $c(f)$ and is denoted by $s(f)$. Given $A \subset S$, we say that f is carried (supported) in A if $c(f) \subset A$ $(s(f) \subset A)$.

2.2. *Definition.* Let $T: C(S) \to X$. T is β-uniform if T is uniformly continuous on bounded sets. That is, for every bounded set D and $\epsilon > 0$, there exists $\delta > 0$ such that $||T(f) - T(g)|| < \epsilon$ when $f, g \in D$ and $||f - g|| < \delta$. T is additive if for each $g \in C(S)$, the mapping $T_g: C(S) \to X$ defined by $T_g(f) = T_g(f + g) - T(g)$ satisfies $T_g(f_1 + f_2) = T_g(f_1) + T_g(f_2)$ when $f_1 f_2 = 0$. This condition is suggested by the measure-theoretic identity

$$\mu(F_1 \cup F_2 \cup G) = \mu(F_1 \cup G) + \mu(F_2 \cup G) - \mu(G),$$

where F_1 and F_2 are disjoint sets. If T is additive and $T(0) = 0$, then $f_1 f_2 \equiv 0$ implies $T(f_1 + f_2) = T(f_1) + T(f_2)$. T is bounded if T maps bounded sets into bounded sets.

2.3. *Remark.* If T is β-uniform, then T is bounded. Let D be bounded, where $||f|| \leqq b$, $f \in D$. Choose $\delta > 0$ so that $f_1, f_2 \in D$ and $||f_1 - f_2|| < \delta b$ imply $||T(f_1) - T(f_2)|| < 1$. Hence, for any $f \in D$, if n and r satisfy $\delta/2 < r = 1/n < \delta$, then

$$||T(f) - T(0)|| \leqq \left\| \sum_{1 \leqq k \leqq n} T(krf) - T((k-1)rf) \right\|$$
$$\leqq \sum_{1 \leqq k \leqq n} ||T(krf) - T((k-1)rf)|| \leqq n < 2/\delta.$$

Thus $f \in D$ implies $||T(f)|| < 2/\delta + ||T(0)||$.

2.4. *Definition.* Let $T: C(S) \to X$. T is an additive operator if T is β-uniform and additive. An additive functional is a real-valued additive operator.

Clearly, bounded linear operators are examples of additive operators. However, an additive operator is generally non-linear. For example, $T(f) = f^2$ is an additive operator mapping $C(S)$ into $C(S)$.

Given a closed set F and real h, let $P(F, h)$ denote the class of continuous functions f satisfying $0 \leqq f \leqq h$ (or $h \leqq f \leqq 0$ if $h \leqq 0$) and $f(G) = h$, where G is an open set containing F. Briefly, $P(F, h)$ is the class of peaks over F of height h. An ordering on $P(F, h)$ is defined by $f_2 \leqq f_1$ if $s(f_2) \subset s(f_1)$. Thus $f_2 \leqq f_1$ if f_2 is a better fit for F. A limit taken with respect to this ordering is denoted by $\lim_f$.

The following lemma is obtained in [**8**]. A proof for the case where T is an additive operator and μ_h is a vector-valued measure is given in § 3.

2.5. LEMMA. *Let T be an additive functional on $C(S)$. Then there is a regular Borel measure μ_h for each real h, such that for each closed set F,*

$$\mu_h(F) = \lim_f T(f), \qquad f \in P(F, h).$$

Utilizing the family of measures $\{\mu_h\}$, the following representation theorem is obtained [**8**].

2.6. THEOREM. *T is an additive functional on $C(S)$ if and only if there is a measure μ and a kernel function $K(\cdot, \cdot)$ such that*

$$T(f) = \int_S K(f(s), s)\, \mu(ds),$$

where

(i) *μ is a real-valued measure of finite variation,*
(ii) *$K(h, s)$ is a measurable function of s for each real h,*
(iii) *$K(h, s)$ is a continuous function of h for all $s \in S \backslash N$, where $\mu(N) = 0$ (μ-a.e. s),*
(iv) *for each $H > 0$ there exists $M > 0$ such that $|h| \leqq H$ implies*

$$|K(h, s)| \leqq M \quad \textit{for} \quad \mu\text{-a.e. } s.$$

A proof of the following result is contained in [**6**, Lemma 18].

2.7. LEMMA. *Let Φ be an additive functional on $C(S)$ with corresponding height measures $\{\mu_h\}$. If s_n is a sequence of simple functions*

$$s_n = \sum_{i=1}^{k(n)} c_{n,i} \chi_{B_{n,i}}$$

and $f \in C(S)$ such that $||s_n - f||_\infty \to 0$, then

$$\lim_n \sum_{i=1}^{k(n)} \mu_{c_{n,i}}(B_{n,i}) = \Phi(f).$$

The following result can be found in [**3**, p. 60]. The family of all finite subsets σ of the positive integers is denoted by $\mathscr{F}$.

2.8. THEOREM (Orlicz-Pettis). *Let (x_k) be a sequence in a Banach space X. Then*

(1) *(x_k) is subseries Cauchy in the weak topology if and only if there exists $M > 0$ such that*

$$\sup\left\{\left\|\sum_{k\in\sigma} x_k\right\| : \sigma \in \mathscr{F}\right\} < M.$$

(2) *If X is weakly sequentially complete, then (x_k) is subseries Cauchy in the weak topology if and only if it is subseries Cauchy in the norm topology. Thus, if (x_k) is subseries Cauchy in the weak topology, then $\lim_k ||x_k|| = 0$.*

3. Height measures. In this section we shall represent an additive operator in terms of a family of measures $\{\mu_h\}$. The proofs of Lemmas 3.1–3.3 are based on methods in [**2**; **6**; **8**].

3.1. LEMMA. *Let* $T: C(S) \to X$ *be continuous. Fix* $g \in C(S)$ *and an open set* U. *Let* f *be carried in* U *and* $\epsilon > 0$. *Then there exists* f_ϵ *supported in* U *such that* $||f_\epsilon|| \leqq ||f||$ *and* $||T(f + g) - T(f_\epsilon + g)|| < \epsilon$.

Proof. Choose $\delta > 0$ such that $||f - f_\epsilon|| < \delta$ implies

$$||T(f + g) - T(f_\epsilon + g)|| < \epsilon.$$

Let $V = \{s: |f(s)| < \delta\}$; hence V^c (the complement of V) is closed and disjoint from U^c. Choose disjoint open sets G and W such that $V^c \subset G$ and $U^c \subset W$. By Urysohn's lemma there exists $w \in C(S)$, $0 \leqq w \leqq 1$, $w(V^c) = 1$, and $w(G^c) = 0$. Let $f_\epsilon = wf$; hence $f_\epsilon \in C(S)$. Since G is disjoint from W, f_ϵ is supported in U. Also, by definition of V, $||f - f_\epsilon|| = ||(1 - w)f|| < \delta$.

3.2. LEMMA. *Let* X *be a weakly sequentially complete Banach space. Let* $T: C(S) \to X$ *be an additive operator. Given* $g \in C(S)$, $h > 0$, $\epsilon > 0$, *and a closed set* $F \subset S$, *there exists an open set* $U \supset F$ *such that if* f *is carried in* $U - F$ *and* $||f|| \leqq h$, *then* $||T(f + g) - T(g)|| \leqq \epsilon$.

Proof. Suppose the contrary. Then given $U_1 \supset F$, there exists f_1^* carried in $U_1 - F$ such that

(1) $||T(f_1^* + g) - T(g)|| > \epsilon$ and $||f_1^*|| \leqq h$.

Thus Lemma 3.1 implies that f_1 can be chosen so as to be supported in $U_1 - F$ and so that

(2) $||T(f_1 + g) - T(g)|| > \epsilon$ and $||f_1|| \leqq h$.

Let $U_2 = [c(f_1)]^c \cap U_1$; hence $U_2 \supset F$. Choose f_2^* carried in $U_2 - F$ such that (1) holds for f_2^*. Thus Lemma 3.1 implies that there exists f_2 supported in $U_2 - F$ and that (2) holds for f_2. Proceeding inductively, we obtain a sequence of disjointly supported functions (f_k) satisfying

(3) $||T(f_k + g) - T(g)|| > \epsilon$, $k = 1, 2, \ldots,$ and $||f_k|| \leqq h$.

However, T is additive; hence

(4) $T_g(\sum_{k \in \sigma} f_k) = \sum_{k \in \sigma} T_g(f_k)$, $\sigma \in \mathscr{F}$.

The class $\{\sum_{k \in \sigma} f_k: \sigma \in \mathscr{F}\}$ is bounded in $C(S)$ because the functions (f_k) are disjointly supported and $||f_k|| \leqq h$ for all k. By Remark 2.3, the class

$$\left\{T_g\left(\sum_{k \in \sigma} f_k\right) = \sum_{k \in \sigma} T_g(f_k): \sigma \in \mathscr{F}\right\}$$

is also bounded. By Theorem 2.8 (1), this class is subseries Cauchy in the weak topology. By Theorem 2.8 (2), we have $\lim_k ||T_g(f_k)|| = 0$, which contradicts (3).

3.3. LEMMA. *Let* X *be a weakly sequentially complete Banach space. Let* $T: C(S) \to X$ *be an additive operator and let* F *be closed. Then for each real* h,

$\lim_f T(f)$ *exists and is denoted by* $\lambda_h(F)$. *Moreover, if* $M_h > 0$ *satisfies* $\|T(f)\| \leqq M_h$ *for all* $\|f\| \leqq h$, *then* $\|\lambda_h(F)\| \leqq M_h$.

Proof. Let $\epsilon > 0$. By Lemma 3.2, we can choose an open set $U \supset F$ such that if g is carried in $U - F$, then

(1) $\|T(g)\| < \epsilon/6$.

Let f_1 and f_2 be in $P(F, h)$ and supported in U. It suffices to show that

$$\|T(f_1) - T(f_2)\| < \epsilon.$$

We have $f_i = h$ on $U_i \supset F$, $i = 1, 2$. Let $G_1 = U_1 \cap U_2$. By Lemma 3.2 we can choose $G_2 \supset F$ such that if v is carried in $G_2 - F$, then

(2) $\|T(f_i - v) - T(f_i)\| < \epsilon/3$, $i = 1, 2$.

Also assume that $G_2 \subset G_1$. Utilizing normality, choose open sets G_3 and G_4 such that

$$F \subset G_4 \subset \bar{G}_4 \subset G_3 \subset \bar{G}_3 \subset G_2,$$

where $\bar{G}$ denotes the closure of G. By Urysohn's lemma we can choose u_1 such that $u_1(\bar{G}_4) = 1$ and $u_1(G_3^c) = 0$. Also choose u_2 such that $u_2(G_2^c) = 1$ and $u_2(\bar{G}_3) = 0$. Since $G_2 \subset G_1$, we have $z = u_1 f_i = hu_1$, $i = 1, 2$. Let $g_i = u_2 F_i$, $i = 1, 2$, and $v_i = f_i - (z + g_i)$. Since z and g_i have disjoint carriers, $T(z + g_i) = T(z) + T(g_i)$. Also g_i is carried in $U - F$ and v_i is carried in $G_2 - F$. Thus (1) and (2) imply

$$\begin{aligned}\|T(f_1) - T(f_2)\| &\leqq \|T(f_1) - T(f_1 - v_1)\| + \|T(z + g_1) - T(z + g_2)\| \\ &\qquad + \|T(f_2 - v_2) - T(f_2)\| \\ &< \epsilon/3 + \|T(g_1)\| + \|T(g_2)\| + \epsilon/3 \\ &< \epsilon.\end{aligned}$$

Finally, let M_h be as in the statement of the lemma. Then,

$$\|\lambda_h(F)\| \leqq \sup\{\|T(f)\|: \|f\| \leqq h\} \leqq M_h.$$

We shall now assume that $T(0) = 0$; hence $T(f_1 + f_2) = T(f_1) + T(f_2)$ when f_1 and f_2 have disjoint supports. This is no loss of generality since $T(f) - T(0)$ satisfies this property in the general case.

3.4. LEMMA. *Let X be an arbitrary Banach space. Let T be an additive operator mapping $C(S)$ into X. For each $h \in R$ (R the set of reals) there is a vector-valued function μ_h: $\mathscr{B} \to X^{**}$ such that:*

(1) *For each $x^* \in X^*$, the mapping $(x^*, \mu_h(\cdot))$: $\mathscr{B} \to R$ is countably additive,*

(2) *If $M_h > 0$ satisfies $\|T(f)\| \leqq M_h$ when $\|f\| \leqq h$, then $\|\mu_h\| \leqq M_h$;*

(3) *Let $\epsilon > 0$ and $b > 0$. Let $D = \{f: \|f\| \leqq b\}$ and let δ be as in Definition 2.2. If B_i are disjoint Borel sets, h_i and $k_i \in (-b, b)$, $|h_i - k_i| < \delta$, $i = 1, 2, \ldots,$ then*

$$\left\|\sum_{i=1}^{\infty} \mu_{h_i}(B_i) - \sum_{i=1}^{\infty} \mu_{k_i}(B_i)\right\| < \epsilon.$$

(*We will show that $\sum_{i=1}^{\infty} \mu_{h_i}(B_i)$ and $\sum_{i=1}^{\infty} \mu_{k_i}(B_i)$ are in X^{**}.*)

(4) *Let* $f \in C(S)$ *satisfy* $||f|| \leqq b$ *and let* ϵ, δ *be as in* (3). *Let* $\{B_i\}$ *be a finite sequence of disjoint Borel sets such that*

$$\left\| f - \sum_{i=1}^{n} h_i \chi_{B_i} \right\| < \delta,$$

where $\{h_i\}$ *is a sequence in* $(-b, b)$. *Then*

$$\left\| T(f) - \sum_{i=1}^{n} \mu_{h_i}(B_i) \right\| \leqq \epsilon.$$

Proof. (1) Since T is an additive operator, setting $x^*T(f) = (T(f), x^*)$ defines an additive functional for each $x^* \in X^*$. Hence, by Lemma 3.3, there exists a family of regular contents $x^*\lambda_h$, where

$$x^*\lambda_h(F) = \lim_f\{x^*T(f): f \in \mathrm{P}(F, h)\}.$$

As in [**6**], [**1**, p. 209, Theorem 3], can be utilized to extend $x^*\lambda_h$ uniquely to a regular Borel measure $x^*\mu_h$. Given $x^* \in X^*$, we define $\mu_h(B)$ by setting

$$(3.4.1) \qquad (\mu_h(B), x^*) = (x^*\mu_h)(B).$$

If h and B are fixed, we verify that $\mu_h(B)$ defines a bounded linear functional on X^*. Boundedness is immediate: if $||T(f)|| \leqq M_h$ for all f of norm less than or equal to h, then

$$(3.4.2) \qquad \begin{aligned} |(x^*\mu_h)(B)| &= \sup\{|(x^*\mu_h)(F)|: F \text{ is a closed subset of } B\} \\ &\leqq \sup\{|(x^*T)(f): f \in \mathrm{P}(F, h), \text{ where } F \text{ is a closed subset of } B\} \\ &\leqq ||x^*||\, M_h. \end{aligned}$$

To verify linearity, we have, for closed sets F:

$$\begin{aligned} \mu_h(F)(c_1x_1^* + c_2x_2^*) &= \lim_f(c_1x_1^* + c_2x_2^*)T(f) \\ &= \lim_f((c_1x_1^*)T + (c_2x_2^*)T)(f) \\ &= \lim_f(c_1x_1^*)T(f) + \lim_f(c_2x_2^*)T(f) \\ &= c_1(x_1^*\mu_h)(F) + c_2(x_2^*\mu_h)(F) \\ &= c_1\mu_h(F)(x_1^*) + c_2\mu_h(F)(x_2^*). \end{aligned}$$

Thus,

$$(c_1x_1^* + c_2x_2^*)\mu_h(F) = c_1\mu_h(F)(x_1^*) + c_2\mu_h(F)(x_2^*)$$

Since $x^*\mu_h$ is regular, linearity holds also for all Borel sets.

(2) It is immediate from (3.4.2) that the total variation of $x^*\mu_h$ is less than $||x^*||M_h$. Hence, $||\mu_h|| = \sup\{||\mu_h(B)||: B \in \mathscr{B}\} \leqq M_h$.

(3) We first show that $\sum_i \mu_{h_i}(B_i) \in X^{**}$. Let $M > 0$ satisfy $||T(f)|| \leqq M$ whenever $||f|| \leqq 1$. It suffices to show that:

$$\sum_i |(\mu_{h_i}(B_i), x^*)| \leqq 2M||x^*||.$$

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Clearly, $\sum_i |(\mu_{h_i}(B_i), x^*)| = a + b$, where

$$a = \sup\left\{\left(\sum_{i\in\sigma} \mu_{h_i}(B_i), x^*\right): \sigma \in \mathscr{F}, \quad \text{where } (\mu_{h_i}(B_i), x^*) > 0 \text{ if } i \in \sigma\right\},$$

$$b = \sup\left\{\left(-\sum_{i\in\sigma} \mu_{h_i}(B_i), x^*\right): \sigma \in \mathscr{F}, \quad \text{where } (\mu_{h_i}(B_i), x^*) < 0 \text{ if } i \in \sigma\right\}.$$

Without loss of generality, assume that σ satisfies $(\mu_{h_i}(B_i), x^*) > 0$ for all $i \in \sigma$. We will show that

$$\sum_{i\in\sigma} (\mu_{h_i}(B_i), x^*) \leqq M||x^*||. \tag{3.4.3}$$

For the fixed x^* and σ, choose closed subsets F_i of B_i so that

$$\sum_{i\in\sigma} |(\mu_{h_i}(B_i\backslash F_i), x^*)| < \epsilon/2$$

and so that $(\mu_{h_i}(F_i), x^*) > 0$. Choose disjointly supported functions

$$f_i \in \mathrm{P}(F_i, h_i)$$

so that $\sum_{i\in\sigma} |(\mu_{h_i}(F_i) - T(f_i), x^*)| < \epsilon/2$ and so that $(T(f_i), x^*) \geqq 0$ for all $i \in \sigma$. Let $f = \sum_{i\in\sigma} f_i$. Since T is additive, $T(f) = \sum_{i\in\sigma} T(f_i)$. We have:

$$\begin{aligned}\sum_{i\in\sigma} (\mu_{h_i}(B_i), x^*) &\leqq \sum_{i\in\sigma} |(\mu_{h_i}(B_i\backslash F_i), x^*)| + \sum_{i\in\sigma} |(\mu_{h_i}(F_i), x^*)| \\ &\leqq \epsilon/2 + \sum_{i\in\sigma} |(\mu_{h_i}(F_i) - T(f_i), x^*)| + \sum_{i\in\sigma} |(T(f_i), x^*)| \\ &\leqq \epsilon + \left(\sum_{i\in\sigma} T(f_i), x^*\right) \\ &\leqq \epsilon + T(f)||x^*|| \\ &\leqq \epsilon + M||x^*||.\end{aligned}$$

Since ϵ is arbitrary, this proves (3.4.3).

We now show that $||\sum_i \mu_{h_i}(B_i) - \mu_{k_i}(B_i)|| < \epsilon$. It suffices to verify that if σ is a finite index set and $x^* \in X^*$, then

$$\left|\left(\sum_{i\in\sigma} \mu_{h_i}(B_i) - \mu_{k_i}(B_i), x^*\right)\right| < \epsilon||x^*||. \tag{3.4.4}$$

Let $\epsilon' > 0$ be arbitrary. As before, we choose disjoint closed subsets $F_i \subset B_i$ so that

$$\sum_{i\in\sigma} |(\mu_{h_i}(B_i\backslash F_i), x^*)| < \epsilon'/4 \quad \text{and} \quad \sum_{i\in\sigma} |(\mu_{k_i}(B_i\backslash F_i), x^*)| < \epsilon'/4.$$

Choose disjointly supported functions $f_i \in \mathrm{P}(F_i, h_i)$ and $g_i \in \mathrm{P}(F_i, k_i)$ so that:

$$||f_i - g_i|| < \delta,$$

$$\sum_{i\in\sigma} |(\mu_{h_i}(F_i), x^*) - (T(f_i), x^*)| < \epsilon'/4,$$

$$\sum_{i\in\sigma} |(\mu_{k_i}(F_i), x^*) - (T(g_i), x^*)| < \epsilon'/4.$$

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By the triangle inequality, we have:

$$(3.4.5)\qquad \left|\left(\sum_{i\in\sigma} \mu_{h_i}(B_i) - \mu_{k_i}(B_i), x^*\right)\right| < \epsilon' + \left|\left(\sum_{i\in\sigma} T(f_i) - T(g_i), x^*\right)\right|.$$

Write $f = \sum_{i\in\sigma} f_i$ and $g = \sum_{i\in\sigma} g_i$. Then, $||f - g|| < \delta$ so that

$$\left\|\sum_{i\in\sigma} T(f_i) - \sum_{i\in\sigma} T(g_i)\right\| = ||T(f) - T(g)|| < \epsilon.$$

Thus,

$$\left|\left(\sum_{i\in\sigma} T(f_i) - T(g_i), x^*\right)\right| \leqq \epsilon||x^*||.$$

Applying this to (3.4.5) and observing that ϵ' is arbitrary, we obtain (3.4.4).

(4) Let f_n be a sequence of step functions converging in the uniform norm to f. For any $x^* \in X^*$, Theorem 2.6 yields $\lim_n x^*T(f_n) = x^*T(f)$ so that $T(f)$ is the limit of $T(f_n)$ in the weak topology. By (3) above, the sequence $T(f_n)$ is also Cauchy in the norm topology and so must converge to $T(f)$ in the norm. And, if g is any step function such that $||f - g|| \leqq \delta$, then $\lim_n ||f_n - g|| \leqq \delta$ and so by (3) above, $\lim_n ||T(f_n) - T(g)|| \leqq \epsilon$. Thus $||T(f) - T(g)|| \leqq \epsilon$, as required.

Lemma 3.4 suggests the following definition of a non-linear integral.

3.5. *Definition.* Let Y be a Banach space and $Z \subset Y^*$. Let μ_h: $\mathscr{B} \to Z$ such that $(y, \mu_h(\cdot))$ is countably additive for each $y \in Y$. For each $\epsilon > 0$ and $b > 0$ there exists $\delta > 0$ such that if B_i are disjoint, $h_i, k_i \in (-b, b)$, $|h_i - k_i| < \delta$, $1 \leqq i \leqq n$, then

$$(3.5.1)\qquad \left\|\sum_{i=1}^{n} \mu_{h_i}(B_i) - \sum_{i=1}^{n} \mu_{k_i}(B_i)\right\| < \epsilon.$$

Given a simple function $f = \sum_{i=1}^{n} h_i\chi_{B_i}$, define

$$\int f\,d\mu = \sum_{i=1}^{n} \mu_{h_i}(B_i).$$

Given $f \in \mathrm{B}(S)$, let f_n be a sequence of simple functions such that

$$||f - f_n|| \to 0.$$

By (3.5.1) we may define

$$\int f\,d\mu = \lim_n \int f_n\,d\mu.$$

We may regard $\int f\,d\mu$ as a non-linear integral with respect to the family of measures, $\mu = \{\mu_h: h \in R\}$.

3.6. Theorem. *Let T: $\mathrm{C}(S) \to X$, where T is additive and X is an arbitrary Banach space. Then there exists $\mu = \{\mu_h\}$ as in Definition* 3.5 *with $Z = X^{**}$ such that*

$$(3.6.1)\qquad T(f) = \int f\,d\mu, \qquad f \in \mathrm{C}(S).$$

Proof. Let μ be the family as in Lemma 3.4. Then (1)-(3) of Lemma 3.4 imply that μ satisfies Definition 3.5 and (3.6.1) follows from (4).

3.7. LEMMA. *Let T: $C(S) \to X$, where T is additive and X is a weakly sequentially complete Banach space. Then μ_h: $\mathscr{B} \to X$ and μ_h is countably additive in the norm of X.*

Proof. Since $(x^*, \mu_h(F)) = (x^*, \lambda_h(F))$ for every $x^* \in X^*$, we have $\mu_h(F) = \lambda_h(F)$. By Lemma 3.3, $\lambda_h(F) \in X$, and so $\mu_h(F) \in X$. It remains to verify that $\mu_h(B) \in X$ for every Borel set B. It is sufficient to show that

$$\mu_h(B) = \lim\{\mu_h(F): F \text{ is a closed subset of } B\}$$

in the norm topology (we order the net $\{\mu_h(F)\}$ by setting $\mu_h(F_1) < \mu_h(F_2)$ if and only if $F_2 \subset F_1$).

Suppose the contrary. Then, there is an $\epsilon > 0$ such that

(3.7.1) $$||\mu_h(B\backslash F)|| > \epsilon \text{ for any closed subset } F \subset B.$$

We construct, inductively, a sequence of disjoint closed sets $\{F_i\}$ so that $||\mu(F_i)|| > \epsilon/2$ for all i.

Since (3.7.1) holds when $F = \emptyset$, we have $||\mu_h(B)|| > \epsilon$. Choose a unit vector $x^* \in X^*$ so that $(x^*, \mu_h(B)) > \epsilon/2$. Since $(x^*, \mu_h(\cdot))$ is a regular Borel measure, we can find a closed subset $F_1 \subset B$ so that $(x^*, \mu_h(F_1)) > \epsilon/2$. Thus

$$||\mu_h(F_1)|| > \epsilon/2.$$

Assume now that disjoint closed subsets $F_1, \ldots, F_n$ of B have been chosen so that $||\mu_h(F_i)|| > \epsilon/2$ for $i = 1, 2, \ldots, n$. Set

$$F = \bigcup_{1 \leqq i \leqq n} F_n.$$

Then F is closed and (3.7.1) applies, so that $(x^*, \mu_h(B\backslash F)) > \epsilon/2$ for some unit vector x^*. Since $(x^*, \mu_h(\cdot))$ is a regular Borel measure, choose a closed subset $F_{n+1} \subset B\backslash F$ such that $(x^*, \mu_h(F_{n+1})) > \epsilon/2$. Thus, $||\mu_h(F_{n+1})|| > \epsilon/2$. This completes the induction. However, the set

$$\left\{\sum_{i \in \sigma} \mu_h(F_i) : \sigma \text{ is any finite set}\right\}$$

is bounded in the norm by $||\mu_h||$. Since $||\mu_h(F_i)|| > \epsilon/2$, Theorem 2.8 is contradicted.

Finally, to show that μ_h is countably additive in the norm, we observe that since μ_h is X-valued, part (1) of Lemma 3.4 proves that whenever $\{B_i\}$ is a sequence of disjoint Borel sets, then

$$\mu_h\left(\bigcup_{1 \leqq i < \infty} B_i\right) = \sum_{1 \leqq i < \infty} \mu_h(B_i),$$

where convergence is taken in the weak topology on X. By Theorem 2.8, the series $\sum_{1 \leqq i < \infty} \mu_h(B_i)$ converges in the norm.

3.8. THEOREM. *Let T: $C(S) \to X$ be an additive operator and X weakly sequentially complete. Then there exists $\mu = \{\mu_h\}$ as in Definition* 3.5 *with $Z = X$ such that*

$$T(f) = \int f\, d\mu, \qquad f \in C(S).$$

The theorem follows by combining Lemma 3.7 with Theorem 3.6.

We note that the measures μ_h: $B \to X^{**}$ determine linear operators T_h: $B(S) \to X^{**}$ as follows. If $f = \sum_{1 \leq i \leq n} c_i \chi_{B_i}$ is a step function, we set

$$T_h(f) = \sum_{1 \leq i \leq n} c_i \mu_h(B_i).$$

It is easy to check that $T_h(f)$ is well-defined. Moreover,

$$\|T_h(f)\| \leq \sum_{1 \leq i \leq n} |c_i|\, \|\mu_h(B_i)\| \leq \|f\|_\infty \|\mu_h\|.$$

Hence, we have defined T_h to be a bounded linear operator on the dense subspace of step functions. Since X^{**} is Banach, we may therefore uniquely extend T_h to the space $B(S)$ so that $\|T_h\| = \|\mu_h\|$. It is also easy to check that $(x^*, T_h(f)) = \int f(s) x^* \mu_h\,(ds)$ for $f \in B(S)$.

To summarize, we have the following.

3.9. THEOREM. *Let T: $C(S) \to X$ be an additive operator. Then there are bounded linear operators T_h: $C(S) \to X^{**}$ so that*

(1) *If M_h satisfies $\|T(f)\| \leq M_h$ whenever $\|f\| \leq h$, then $\|T_h\| \leq M_h$,*

(2) *For each $f \in C(S)$, $x^* \in X^*$,*

$$(x^*, T_h(f)) = \int f(s)\, x^* \mu_h(ds),$$

(3) *If X is weakly sequentially complete, then T_h is a weakly compact operator.*

Proof. (1) and (2) have been proven above.

(3) If X is weakly sequentially complete, Lemma 3.7 shows that μ_h: $\mathscr{B} \to X$ is countably additive in the norm. Applying [**5**, p. 493, Theorem 3] yields the result.

We note that if T: $C(S) \to X$ were a bounded linear operator, then it can be verified that

$$(x^*, T(f)) = \int f(s) x^* \mu_1(ds) \quad \text{for } f \in C(S).$$

Therefore, by (2) of Theorem 3.9, $T = T_1$. And, if X is weakly sequentially complete, then (3) of Theorem 3.9 yields the well-known result (see [**5**, p. 494, Theorem 6]) that T is weakly compact.

4. Kernel representation. Let T: $C(S) \to X$ be an additive operator. We shall extend Theorem 2.6 by constructing a kernel representation for T

for the case where X^* is separable in the $\sigma(X, X^*)$ topology and the family of measures $\{\mu_h\}$ corresponding to T is X-valued.

4.1. LEMMA. *There exists a finite positive measure m and a family of measurable functions* $\{K(x^*, h, s)\}$ *such that*

$$x^*\mu_h(B) = \int_B K(x^*, h, s)\, m(ds), \qquad B \in \mathscr{B}.$$

Proof. Let $\{x_n^*\}$ be a countable dense net in X^* under the $\sigma(X, X^*)$ topology. Given $x^* \in X^*$, there exists a subsequence $x_{n_i}^*$ such that for each h,

(1) $\lim_i x_{n_i}^*\mu_h(B) = x^*\mu_h(B), \quad B \in \mathscr{B}$.

Let $|x_n^*\mu_h|$ denote the variation of $x_n^*\mu_h$ and $||x_n^*\mu_h|| = |x_n^*\mu_h|(S)$. Define a finite measure m_h by setting

(2) $m_h(B) = \sum_{n=1}^{\infty} |x_n^*\mu_h|(B)/2^n||x_n^*\mu_h||$.

Choose a countable dense set of reals $\{h_k\}$ and define

(3) $m(B) = \sum_{k=1} m_{h_k}(B)/2^k$.

Thus m is a finite positive measure defined on $\mathscr{B}$. Suppose that $m(B) = 0$; hence $m_{h_k}(B) = 0$ for each k. Thus (2) implies that $|x_n^*\mu_{h_k}|(B) = 0$ for each k and n. By (1), we have $x^*\mu_{h_k}(B) = 0$ for each k. As in [**4**, Lemma 16], it can be shown that $x^*\mu_h(B)$ is a continuous function of h. Hence $\{h_k\}$ dense in R implies that $x^*\mu_h(B) = 0$ for each h and x^*.

Thus each measure $x^*\mu_h$ is absolutely continuous with respect to m; hence the conclusion follows by the Radon-Nikodym theorem.

We shall now show that the kernels can be chosen as to be continuous in h. The proof in [**2**, Lemma 11] only verified convergence in measure.

4.2. LEMMA. *There exist kernels* $K_1(x^*, h, s)$ *which are continuous in h for m-a.e. s such that*

$$x^*\mu_h(B) = \int_B K_1(x^*, h, s)\, dm.$$

Proof. Fix $a < b$ and x^*. We shall verify that $K(x^*, h, s)$ is uniformly continuous for rational $h \in [a, b]$ for a.e. s. Suppose the contrary. Then the set where $K(x^*, h, s)$ is not uniformly continuous may be written as

$$A = \bigcup_{n=1}^{\infty} \bigcap_{t=1}^{\infty} A_{n,t},$$

where

$$A_{n,t} = \bigcup_{\substack{0<h-k<1/t,\\ h,k \text{ rational}}} \{s: |K(x^*, h, s) - K(x^*, k, s)| > 1/n\}.$$

Now $m(A) > 0$ implies that there exists n such that $A_n = \bigcap_{t=1}^{\infty} A_{n,t}$ has positive measure. Let $r = m(A_n)$ and $\epsilon = r/2n$. Choose $\delta > 0$ such that $||f - g|| < \delta$ implies $|x^*T(f) - x^*T(g)| < \epsilon$. Choose t such that $1/t < \delta$. Now $A_{n,t} \supset A_n$; hence $m(A_{n,t}) \geqq r$.

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$A_{n,t}$ can be expressed as a disjoint union of countably many sets B_j, where $s \in B_j$ implies that there exists rational h_j and k_j such that $0 < h_j - k_j < \delta$ and

(1) $|K(x^*, h_j, s) - K(x^*, k_j, s)| > 1/n$.

We may remove the absolute value sign in (1) by interchanging h_j and k_j, still having $0 < |h_j - k_j| < \delta$. Choose J so large that

(2) $m(\bigcup_{j=1}^{J} B_j) > r/2$.

Thus (1) and (2) imply that

(3) $\sum_{j=1}^{J} \{x^*\mu_{h_j}(B_j) - x^*\mu_{k_j}(B_j)\} = \sum_{j=1}^{J} \int_{B_j} (K(x^*, h_j, s) - K(x^*, k_j, s))\, dm > 1/n \cdot r/2 = \epsilon$.

Now we can approximate B_j by a closed subset F_j with respect to $x^*\mu_{h_j}$ and $x^*\mu_{k_j}$. We can then choose a peak $f_j \in \mathrm{P}(F_j, 1)$ so that $x^*T(h_jf_j)$ and $x^*T(k_jf_j)$ approximate $x^*\mu_{h_j}(F_j)$ and $x^*\mu_{k_j}(F_j)$. Since $F_j \subset B_j$ and the B_j are disjoint, it is possible to choose f_j with disjoint supports. Let

$$f = \sum_{j=1}^{J} h_jf_j \quad \text{and} \quad g = \sum_{j=1}^{J} k_jf_j.$$

Then $||f - g|| < 1/t < \delta$ and the left side of (3) is approximated by

$$x^*T(f) - x^*T(g).$$

This contradicts the choice of δ. Thus $K(x^*, h, s)$ is uniformly continuous for rational $h \in [a, b]$ for a.e. s.

Proceeding as in [**2**], we consider $a = n$, $b = n + 1$, $n = 0, \pm 1, \pm 2, \ldots$ to conclude that $K(x^*, h, s)$ is uniformly continuous for rational $h \in [n, n + 1]$ for all n for a.e. s. We now define $K_1(x^*, h, s) = K(x^*, h, s)$ for rational h. If h is irrational, then we choose rational $h_i \to h$ and define $K_1(x^*, h, s) = \lim_i K(x^*, h_i, s)$. An argument similar to the above implies that $K(x^*, h, s) = K_1(x^*, h, s)$ for a.e. s, when x^* and h are fixed.

4.3. THEOREM. *Let T: $\mathrm{C}(S) \to X$ be an additive operator. Assume that X^* is separable in the $\sigma(X, X^*)$ topology and the family of measures $\{\mu_h\}$ corresponding to T are X-valued. Then for each x^*,*

(1) $x^*T(f) = \int K(x^*, f(s), s)H(x^*, s)\, m(ds)$, *where*

(2) *m is a measure of finite variation defined on $\mathscr{B}$;*

(3) *$K(x^*, h, s)$ is a measurable function of s for each h;*

(4) *$K(x^*, h, s)$ is a continuous function of h for m-a.a. s;*

(5) *For each $b > 0$ there exists $B > 0$ such that $|h| \leqq b$ implies that*

$$|K(x^*, h, s)| \leqq B \quad \text{for} \quad m\text{-a.a. } s;$$

(6) *$H(x^*, s)$ is a measurable function of s and $d\mu = H(x^*, s)m(ds)$ defines a measure μ with finite variation;*

(7) *For each $f \in \mathrm{C}(S)$, the right side of* (1) *defines a continuous linear functional on X^* in X.*

Conversely, if (2)–(7) *hold, then there exists an additive operator T satisfying* (1).

Proof. As in [**2**], it follows from Lemma 4.2 that $K_1 = KH$, where K and H satisfy (3)–(6). As in [**2**; **4**], it is verified that (1) holds.

Conversely, fix $f \in C(S)$. By (7) there exists $T(f) \in X$ such that (1) holds for each x^*. It remains to verify that T is an additive operator from $C(S)$ into X. Let us fix x^*. Then (2)–(6) imply that $x^*T(f)$ is an additive functional on $C(S)$. This follows as in [**2**]. The Hahn-Banach theorem now implies that T is additive on functions with disjoint support. We now verify that T is β-uniform. Let $\epsilon > 0$, $b > 0$, and consider $||f|| \leqq b$ and $||g|| \leqq b$. By the Hahn-Banach theorem it suffices to show that there exists $\delta > 0$ such that

(8) $||f - g|| < \delta$ implies $|x^*(T(f) - T(g))| < \epsilon$, $||x^*|| = 1$.

Let $B_n = \{x^*: \text{(8) holds for } \delta = 1/n\}$. Then B_n is convex and (7) implies that B_n is closed. Since $x^*T(f)$ defines an additive functional, we have

$$\bigcup_{n=1}^{\infty} B_n = X^*.$$

The Baire category theorem now implies that some B_n has non-empty interior. The existence of δ follows by a standard argument.

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