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REVERSE ITERATED FUNCTION SYSTEM AND DIMENSION OF DISCRETE FRACTALS

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Abstract

A reverse iterated function system is defined as a family of expansive maps $\{T_1, T_2, \ldots, T_m\}$ on a uniformly discrete set $M \subset \mathbb{R}^d$. An invariant set is defined to be a nonempty set $F \subseteq M$ satisfying $F = \bigcup_{j=1}^m T_j(F)$. A computation method for the dimension of the invariant set is given and some questions asked by Strichartz are answered.

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1. Introduction

Fractal structure is characterized by the repetition of detail at small scales. Why not large scales as well? Motivated by this, Strichartz [4] explored two ways to carry this out. He considered a set of expansive mappings $\{T_j\}_{j=1}^m$ defined on a discrete complete metric space $M \subset \mathbb{R}^d$ and called it a *reverse iterated function system* (RIFS). He defined the notion of an invariant set which is a union of forward orbits of fixed points of the iterated mappings from the RIFS in [4]. A simple example is the integer Cantor set (all positive integers which are expressible in base three using only zeros and twos as digits) discussed by Bedford and Fisher in [1]. For the case $M = \mathbb{Z}$, Strichartz defined the dimension of invariant sets and gave a method to compute the dimension of invariant sets for the nonoverlapping case, as well as asking some basic questions about the case when nonoverlapping occurs.

If we extend the mappings $\{T_j\}_{j=1}^m$ to be defined on $\mathbb{R}^d$, then $\{T_j^{-1}\}_{j=1}^m$ is a contractive *iterated function system* (IFS). It can be expected that an invariant set of the RIFS $\{T_j\}_{j=1}^m$ is related to the attractor of the IFS $\{T_j^{-1}\}_{j=1}^m$. By this observation, we obtain a method to compute the dimension of the invariant set of the RIFS $\{T_j\}_{j=1}^m$ for overlapping cases, and three questions asked by Strichartz [4] are also answered.

We first introduce some definitions given in [4]; a small modification is made.

DEFINITION 1.1. Let $M \subseteq \mathbb{R}^d$ be a uniformly discrete set and $\{T_j\}_{j=1}^m$ be a family of expansive (under some metric of $\mathbb{R}^d$) transforms defined on $\mathbb{R}^d$. Assume $T_j(M) \subseteq M$ for all j:

(i) $\{T_j\}_{j=1}^m$ is called an RIFS;
(ii) $P = \{x \in M \mid T_J(x) = x$ for some J with positive length$\}$;
(iii) a nonempty compact set $F \subseteq M$ is called an invariant set if

$$F = \bigcup_{j=1}^{m} T_j(F);$$

(iv) if the union in (iii) is disjoint, F is said to be nonoverlapping;
(v) for any $x \in M$, $F_x = \{T_J(x) \mid J = j_1 \ldots j_n$ with $1 \le j_i \le m$ and $n > 0\}$ is called a forward orbit of x for $\{T_j\}_{j=1}^m$.

For any invariant set F, a basic aspect of F is how dense it is in M. Strichartz [4] gave the following definition for the case $d = 1$, $M = \mathbb{Z}$.

DEFINITION 1.2. For an RIFS $\{T_j\}_{j=1}^m$ on a uniformly discrete set $M \subseteq \mathbb{R}^d$, let F be an invariant set. We define its dimension as

$$\dim F = \lim_{r\to+\infty} \log(\#\{F \cap B_r\})/\log r;$$

if the limit exists, where B_r is the ball centered at the origin with radius $r > 0$.

For the case that $\{T_j(x) = Rx + b_j\}_{j=1}^m$ and F is nonoverlapping, Strichartz [4] obtained a method to compute the dimension of F, that is, $\dim F = -\ln m/\ln \gamma$.

In general, however, invariant sets are overlapping. Natural questions are: how can we deal with the overlapping cases and when does overlapping occur? Our goal in this paper is to consider these questions. We prove the following result.

THEOREM 1.3. *Let $a \in P$; if the RIFS $\{T_j(x) = R^{k_j}x + b_j\}_{j=1}^m$ is a family of similitudes on $\mathbb{R}^d$ and the IFS $\{S_j(x) = R^{-k_j}(x - b_j)\}_{j=1}^m$ satisfies the finite type condition (FTC), then*

$$\dim F_a = \dim_{\mathrm{H}}(K),$$

where K is the attractor of the IFS $\{S_j\}_{j=1}^m$.

Ngai and Wang [2] have given a method for the computation of $\dim_{\mathrm{H}}(K)$, so our theorem means that we have been able to compute the dimension of any invariant set of an RIFS satisfying the conditions stated in our theorem.

We arrange the paper as follows. In Section 2, we introduce some results of Strichartz [4]. Then we discuss the questions asked by Strichartz by connecting the related IFS. Section 3 is devoted to considering the computation of the dimensions of invariant sets of the RIFS $\{T_j\}_{j=1}^m$ for overlapping cases. Theorem 1.3 is proved there.

2. Nonoverlapping and the open set condition

The next two lemmas are easy to see.

LEMMA 2.1. *For any $x \in M$, F_x is a finite set if and only if $F_x = \{x\}$, and so $T_j(x) = x$ for all j.*

LEMMA 2.2 [4]. *The system $\{T_j\}_{j=1}^m$ is an RIFS defined on a uniformly discrete set $M \subset \mathbb{R}^d$.*

(i) *There exists an invariant set $F \subseteq M$ if and only if $P \neq \emptyset$.*
(ii) *The set P is finite.*
(iii) *Any invariant set F is a union of forward orbits of points in P.*

In the following, we consider a self-similar RIFS defined on some uniformly discrete set $M \subset \mathbb{R}^d$:

$$T_j(x) = R_j x + b_j, \quad j = 1, 2, \ldots, m, \tag{2.1}$$

where $\{R_j\}_{j=1}^m$ are similar matrices with expansive ratios $\{\gamma_j > 1\}_{j=1}^m$, and a self-affine (or self-similar) RIFS:

$$T_j(x) = Rx + b_j, \quad j = 1, 2, \ldots, m, \tag{2.2}$$

where R is an expanding matrix (which may be an affine or a similar matrix), $b_j \in \mathbb{R}^d$.

DEFINITION 2.3. The following two iterated function systems

$$S_j(x) = R_j^{-1}(x - b_j), \quad j = 1, 2, \ldots, m, \quad x \in \mathbb{R}^d, \tag{2.3}$$

$$S_j(x) = R^{-1}(x - b_j), \quad j = 1, 2, \ldots, m, \quad x \in \mathbb{R}^d, \tag{2.4}$$

are called the dual IFSs of (2.1) and (2.2), respectively.

(i) We say that an RIFS satisfies the open set condition (OSC) if its dual IFS does.
(ii) We say that an RIFS satisfies the FTC (see [2]) if its dual IFS does.

REMARK. Since it is required that $\{T_j\}_{j=1}^m$ satisfies $\bigcup_{j=1}^m T_j(M) \subseteq M$ for some uniformly discrete set $M \subseteq \mathbb{R}^d$, so not all families of expansive transforms on $\mathbb{R}^d$ belong to the family of RIFSs.

Both (2.3) and (2.4) are families of contractive transforms (IFS) defined on $\mathbb{R}^d$. Let K be the attractor of these IFSs [2].

It is easy to see the following.

LEMMA 2.4. *For the RIFSs (2.1) and (2.2), $P \subset K$.*

If $R_j \in M_d(\mathbb{Z})$, $b_j \in \mathbb{Z}^d$ and $M = \mathbb{Z}^d$, Strichartz asked the following questions in [4] for the case that $d = 1$.

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(1) Does there exist an RIFS of the form (2.1) with a nonoverlapping invariant set but such that the images of $\mathbb{Z}^d$ overlap?
(2) Is it possible for such RIFSs to have both overlapping and nonoverlapping invariant sets?
(3) Is it possible for such RIFSs to have an overlapping invariant set but with just a finite number of overlaps?

For (3), we have the following result which answers one aspect of the question.

PROPOSITION 2.5. *For the RIFS (2.2) defined on a uniformly discrete set $M \subset \mathbb{R}^d$, assume that $a \in P$ is a fixed point of some T_j $(1 \le j \le m)$ and F_a is an infinite set.*

(i) *The set F_a is nonoverlapping if and only if the dual IFS (2.4) satisfies the OSC.*
(ii) *If $T_i(F_a) \cap T_j(F_a) \neq \emptyset$, then $T_i(F_b) \cap T_j(F_b)$ is infinite for any $b \in P$.*

PROOF. (i) Assume $z \in T_i(F_a) \cap T_j(F_a) (i \neq j)$, then there exist I, J so that $z = T_i(T_I(a)) = T_j(T_J(a))$. Since $a = T_l(a)$ for some T_l, without loss of generality, we assume $|I| = |J| = n$. Note that

$$T_i(T_I(x)) = R^{n+1}(x - a) + T_j(T_J(a)), \quad T_j(T_J(x)) = R^{n+1}(x - a) + T_j(T_J(a)),$$

we have $T_{iI} = T_{jJ}$, so the OSC implies $iI = jJ$, a contradiction.

Conversely, assume that F_a is nonoverlapping. Let $I = i_1 i_2 \dots i_n$, $J = j_1 j_2 \dots j_n$. If $I \neq J$, then $i_n i_{n-1} \dots i_1 \neq j_n j_{n-1} \dots j_1$.

Since $S_I = T^{-1}_{i_n i_{n-1} \dots i_1}$, $S_J = T^{-1}_{j_n j_{n-1} \dots j_1}$ and

$$S_I(x) = R^{-n}(x - b_{i_n} - A b_{i_{n-1}} - \cdots - A^{n-1} b_{i_1}),$$
$$S_J(x) = R^{-n}(x - b_{j_n} - A b_{j_{n-1}} - \cdots - A^{n-1} b_{j_1}).$$

Note that F_a is nonoverlapping, and $T_{i_n i_{n-1} \dots i_1}(a)$ and $T_{j_n j_{n-1} \dots j_1}(a)$ belong to a uniformly discrete set M, hence there exists a constant $c > 0$ such that

$$\begin{gathered} \|x - S^{-1}_{i_1 i_2 \dots i_n} \circ S_{j_1 j_2 \dots j_n}(x)\| = \|T_{i_n i_{n-1} \dots i_1}(a) - T_{j_n j_{n-1} \dots j_1}(a)\| \ge c, \\ \forall I \neq J \in \Sigma^n, \ n > 0, \ x \in \mathbb{R}^d. \end{gathered} \tag{2.5}$$

Choose a bounded invariant open set O of $\{S_i\}_{i=1}^m$, let

$$N = \sup_{n>0} \sup_{I \in \Sigma^n} \#\{J \in \Sigma^n \mid S_I(O) \cap S_J(O) \neq \emptyset\}.$$

Since

$$\begin{aligned} S_I(O) \cap S_J(O) \neq \emptyset &\iff O \cap S_I^{-1} S_J(O) \neq \emptyset \\ &\iff O \cap (O - T_{i_n i_{n-1} \dots i_1}(a) + T_{j_n j_{n-1} \dots j_1}(a)) \neq \emptyset. \end{aligned}$$

Let $B_\delta(x_0)$ be a ball contained in O with radius $\delta < c/2$, then all of the balls of

$$\{B_\delta(x_0) - T_{i_n i_{n-1} \dots i_1}(a) + T_{j_n j_{n-1} \dots j_1}(a) \mid S_I(O) \cap S_J(O) \neq \emptyset,$$
$$J = j_1 j_2 \dots j_n \in \Sigma^n\}$$

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are disjoint for any given n and $I = i_1 i_2 \ldots i_n \in \Sigma^n$, and contained in

$$O_1 = \{x \in \mathbb{R}^d \mid \|x - y\| \leq |O| \text{ for some } y \in O\},$$

where $|O|$ is the diameter of O. Using (2.5) shows that $N < +\infty$. Hence, there is an integer n_0 and $I \in \Sigma^{n_0}$ such that

$$N = \#\{J \in \Sigma^{n_0} \mid S_I(O) \cap S_J(O) \neq \emptyset\}. \tag{2.6}$$

Fix this I, let

$$V = \bigcup_{n=1}^{+\infty} \bigcup_{J \in \Sigma^n} S_J(S_I(O)).$$

It is easy to see that V is a bounded nonempty open set. Furthermore, it is an invariant open set of the dual IFS.

If $S_i(V) \cap S_j(V) \neq \emptyset$ for some $i \neq j$, then there exist τ, σ such that

$$S_i(S_\tau(S_I(O))) \cap S_j(S_\sigma(S_I(O))) \neq \emptyset. \tag{2.7}$$

Let $|\tau| = n$, $|\sigma| = k$. Without loss of generality, we can assume $n \geq k$, then τI can be written as $\tau I = \tau_1 \tau_2$ with $|\tau_1| = k + n_0$. Since O is invariant, so $S_i(S_\tau(S_I(O))) \subseteq S_i(S_{\tau_1}(O))$ (see (2.7)) implies

$$S_i(S_{\tau_1}(O)) \cap S_j(S_\sigma(S_I(O))) \neq \emptyset.$$

Hence,

$$\begin{aligned} &\{J \in \Sigma^{n_0+k+1} \mid S_j(S_\sigma(S_I(O))) \cap S_J(O) \neq \emptyset\} \\ &\quad \supseteq \{(i\tau_1)\} \cup \{(j\sigma J) \mid J \in \Sigma^{n_0},\ S_I(O) \cap S_J(O) \neq \emptyset\}. \end{aligned}$$

Hence, $\#\{J \in \Sigma^{n_0+k+1} \mid S_i(S_\tau(S_I(O))) \cap S_J(O) \neq \emptyset\} > N$ by (2.6) and $i \neq j$, it contradicts the definition of N.

Therefore, $S_i(V) \cap S_j(V) = \emptyset$ for all $i \neq j$, that is, the dual IFS (equivalently the RIFS) satisfies the OSC.

(ii) Assume $z \in T_i(F_a) \cap T_j(F_a)$, $i \neq j$. Note that a is a fixed point of some T_l, the above proof implies that $T_i(T_I(a)) = T_j(T_J(a))$ for some $I = i_1 i_2 \ldots i_p$, $J = j_1 j_2 \ldots j_p$. Since $T_i T_I(x) = R^{p+1}(x - a) + T_{iI}(a)$, $T_j T_J(x) = R^{p+1}(x - a) + T_{jJ}(a)$. Hence, $T_{iI} = T_{jJ}$ and so $T_i(F_b) \cap T_j(F_b) \supseteq T_i T_I(F_b)$ for all $b \in P$. Since F_a is infinite, by Lemma 2.1, F_b is also infinite, so $T_i(F_b) \cap T_j(F_b)$ is an infinite set. □

The following example answers question (3) in another aspect.

EXAMPLE 2.6. Let $T_1(z) = 9z + 2$, $T_2(z) = 9z - 8$, $T_3(z) = 9z - 18$, and $T_4(z) = 9z - 36$.

Then the dual IFS is

$$\{S_1(x) = \tfrac{1}{9}(x - 2),\ S_2(x) = \tfrac{1}{9}(x + 8),\ S_3(x) = \tfrac{1}{9}(x + 18),\ S_4(x) = \tfrac{1}{9}(x + 36)\}.$$

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It is easy to see that the dual IFS $\{S_j\}_{j=1}^4$ satisfies the OSC with respect to the invariant open interval $V = (-\frac{1}{4}, \frac{9}{2})$.

For this RIFS, 1 is the fixed point of T_2, so F_1 is nonoverlapping by Proposition 2.5(i).

However, for 0, the fixed point of T_{31}, we have $T_3(F_0) \cap T_4(F_0) = \{-18\}$ and $T_i(F_0)$
$\cap\, T_j(F_0) = \emptyset$ for the other cases of $i \neq j$.

This shows that, if $a \in P$ is not a fixed point of some T_i $(i = 1, 2, \ldots, m)$, F_a may be overlapping even if the RIFS satisfies the OSC, that is, the conclusion of Proposition 2.5(i) does not hold without the assumption that $a \in P$ is a fixed point of some T_i $(1 \le i \le m)$.

PROOF. Let $b_1 = 2$, $b_2 = -8$, $b_3 = -18$ and $b_4 = -36$.

CLAIM 1. If $T_{i_1 i_2 \ldots i_n}(0) - 2 = T_{j_1 j_2 \ldots j_p}(0)$ and $p > 0$, then $i_1 = 1$, $j_1 = 3$ and

$$T_{i_2 \ldots i_n}(0) = T_{j_2 \ldots j_p}(0) - 2. \tag{2.8}$$

Since $p > 0$, so $T_{i_1 i_2 \ldots i_n}(0) - 2 = T_{j_1 j_2 \ldots j_p}(0)$ implies that $n > 0$ and

$$9T_{i_2 \ldots i_n}(0) + b_{i_1} - 2 = 9T_{j_2 \ldots j_p}(0) + b_{j_1}.$$

Hence, $9|(b_{i_1} - 2 - b_{j_1})$, this means that $i_1 = 1$ and $j_1 \in \{3, 4\}$. If $j_1 = 4$, then the above equality implies $T_{i_2 \ldots i_n}(0) = T_{j_2 \ldots j_p}(0) - 4$, so $9|(b_{i_2} + 4 - b_{j_2})$, which is impossible. Hence, $j_1 = 3$ and (2.8) hold.

CLAIM 2. If $T_{i_1 i_2 \ldots i_n}(0) - 2 = T_{j_1 j_2 \ldots j_p}(0)$, then $T_{j_1 j_2 \ldots j_p}(0) = 0$.

Assume $T_{i_1 i_2 \ldots i_n}(0) - 2 = T_{j_1 j_2 \ldots j_p}(0)$ and $p > 0$. If $n = 1$, Claim 1 implies that $i_1 = 1$, $j_1 = 3$ and so $T_{j_2 \ldots j_p}(0) = 2$, hence $T_{j_1 j_2 \ldots j_p}(0) = 0$. Assume that the conclusion is true for $n < k$. For the case $n = k > 1$, Claim 1 implies that $i_1 = 1$, $j_1 = 3$ and

$$T_{j_2 \ldots j_p}(0) - 2 = T_{i_2 \ldots i_k}(0).$$

Use Claim 1 repeatedly, we have $j_2 = 1$, $i_2 = 3$ and

$$T_{i_3 \ldots i_n}(0) - 2 = T_{j_3 \ldots j_p}(0).$$

Hence, $T_{j_3 \ldots j_p}(0) = 0$ by induction. Therefore, $T_{j_1 j_2 \ldots j_p}(0) = 0$ by $j_1 = 3$, $j_2 = 1$.

We now turn to prove the conclusion of the example. Assume $z \in T_i(F_0) \cap T_j(F_0)$. Without loss of generality, we can assume $i > j$. Then there exist I, J such that $T_i(T_I(0)) = T_j(T_J(0))$. Hence,

$$9T_I(0) + b_i = 9T_J(0) + b_j, \tag{2.9}$$

so $9|(b_i - b_j)$. Note that $i > j$, we have $i = 4$ and $j = 3$. Therefore, $T_i(F_0)$ $\cap\, T_j(F_0) = \emptyset$ for other cases of $i \neq j$.

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If $i = 4$ and $j = 3$, then (2.9) implies

$$T_I(0) - 2 = T_J(0).$$

The result of Claim 2 implies $T_J(0) = 0$, so $T_3(T_J(0)) = -18$. Therefore $T_3(F_0) \cap T_4(F_0) = \{-18\}$ by noting that $T_3(0) = T_{41}(0) = -18$. The proof is complete. □

For general case, we do not know whether a general conclusion as in the above example is true. We have the following conjecture.

CONJECTURE. If the RIFS (2.2) satisfies the OSC, then $T_i(F_a) \cap T_j(F_a)$ is finite for any distinct i, j and $a \in P$.

For questions (1) and (2), we have the following two examples.

EXAMPLE 2.7. Suppose $T_1(z) = 4z$, $T_2(z) = 4z - 3$, $T_3(z) = 4z - 24$, and $T_4(z) = 4z - 27$. Then the images of $\mathbb{Z}$ overlap. It is easy to see that $P = \{0, 1, 8, 9\}$ and F_0, F_1, F_8 and F_9 are nonoverlapping by Proposition 2.5(i), since the dual IFS

$$\{S_1(x) = \tfrac{1}{4}x,\ S_2(x) = \tfrac{1}{4}(x + 3),\ S_3(x) = \tfrac{1}{4}(x + 24),\ S_4(x) = \tfrac{1}{4}(x + 27)\}$$

generates a tile $[0, 3] \cup [6, 9]$ and so satisfies the OSC. However, the invariant set $F_1 \cup F_9$ is overlapping, since $T_{13}(9) = 48 = T_{221}(1)$.

This example shows that all forward orbits of points in P are nonoverlapping does not imply that a union of some of them is nonoverlapping or that the images of $\mathbb{Z}^d$ under T_j are nonoverlapping. We have answered questions (1) and (2).

Furthermore, we have the following example.

EXAMPLE 2.8. Suppose $T_1(z) = 4z$, and $T_2(z) = 3z + 2$. Then $T_1(\mathbb{Z}) \cap T_2(\mathbb{Z})$ is infinite. It is easy to see that $P = \{0, -1\}$, F_{-1} is nonoverlapping, but $T_1(F_0) \cap T_2(F_0) = \{8\}$.

3. Dimensions of invariant sets

We consider the dimensions of invariant sets of the RIFS (2.1) in this section. By Lemma 2.2, we need only consider a forward orbit F_a with $a \in P$.

For the RIFS (2.1) and the dual IFS (2.3), let $\gamma_j > 0$ be the expansive ratio of T_j. Define

$$\begin{aligned}
&\gamma = \max_j \{\gamma_j\},\\
&\Lambda_k = \{I = i_1 i_2 \ldots i_n \in \Sigma^n \mid \gamma^{-1}_{i_1 i_2 \ldots i_n} \le \gamma^{-k} < \gamma^{-1}_{i_1 i_2 \ldots i_{n-1}},\ n > 0\}, \quad \forall k \in \mathbb{N},\\
&\widetilde{\Lambda}_k = \{I = i_1 i_2 \ldots i_n \in \Sigma^n \mid i_n \ldots i_2 i_1 \in \Lambda_k,\ n > 0\}, \quad \forall k \in \mathbb{N},\\
&N_k = \#\{S_J \mid J \in \Lambda_k\}, \quad \forall k \in \mathbb{N}.
\end{aligned}$$

We first consider nonoverlapping forward orbits.

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THEOREM 3.1. *For an RIFS defined in (2.1), let F_a be a forward orbit with $a \in P$. If F_a is nonoverlapping, then its dimension is the solution of*

$$\sum_{j=1}^{m} \gamma_j^{-\alpha} = 1, \tag{3.1}$$

that is, $\dim_{\mathrm{H}}(K) = \dim F_a$.

PROOF. Strichartz has proved the result when $R_j = R$ for all j and mentioned the general result in [4]. For completeness, we give a proof here.

Since F_a is nonoverlapping, let μ be the counting measure on F_a, then

$$\mu(E) = \sum_{i=1}^{m} \mu \circ T_j^{-1}(E), \quad \forall E \subseteq F_a. \tag{3.2}$$

Consider the dual IFS $\{S_j(x) = R_j^{-1}(x - b_j)\}_{j=1}^m$, there is a unique nonempty compact set K such that $K = \bigcup_{j=1}^m S_j(K)$. Since $a \in K$, so $R_J^{-1}(a - b_J) = T_J^{-1}(a) \in K$, $\|R_J^{-1} b_J\| \le 2|K|$ for all J. Hence, there exists a constant $c > 0$ such that $\|R_J^{-1} b_J\| < c$ for all J, where $R_J = R_{j_1} R_{j_2} \ldots R_{j_n}$ and

$$b_J = b_{j_1} + R_{j_1} b_{j_2} + R_{j_1} R_{j_2} b_{j_3} + \cdots + R_{j_1} R_{j_2} \ldots R_{j_{n-1}} b_{j_n}.$$

Since

$$T_J^{-1}(B_r) = R_J^{-1}(B_r - b_J),$$

so

$$B_{\gamma^{-k-1} r - c} \subseteq T_J^{-1}(B_r) \subseteq B_{\gamma^{-k} r + c}, \quad \forall J \in \Lambda_k. \tag{3.3}$$

By the definition of Λ_k, the identity (3.2) implies

$$\mu(B_r) = \sum_{J \in \Lambda_k} \mu \circ T_J^{-1}(B_r).$$

Use relation (3.3), then

$$\sum_{J \in \Lambda_k} \mu(B_{\gamma^{-k-1} r - c}) \le \mu(B_r) \le \sum_{J \in \Lambda_k} \mu(B_{\gamma^{-k} r + c}), \quad \forall k > 0. \tag{3.4}$$

Since $\sum_{j=1}^m \gamma_j^{-\alpha} = 1$, by the definition of Λ_k, it is easy to show that $\sum_{J \in \Lambda_k} \gamma_J^{-\alpha} = 1$. Since $\gamma^k \le \gamma_J < \gamma^{k+1}$ when $J \in \Lambda_k$, we have

$$\gamma^{k\alpha} \le \#\Lambda_k < \gamma^{(k+1)\alpha}. \tag{3.5}$$

For each large $r > 0$, let $k = k(r) \in \mathbb{N}$ be such that $\gamma^{k+l} \le r < \gamma^{k+1+l}$, where $l > 0$ is a fixed integer satisfying $\gamma^{l-1} > c$, then (3.4) and (3.5) implies

$$\alpha \le \liminf_{r \to +\infty} \frac{\log \mu(B_r)}{\log r} \le \limsup_{r \to +\infty} \frac{\log \mu(B_r)}{\log r} \le \alpha.$$

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Therefore,

$$\lim_{r\to+\infty}\frac{\log \mu(B_r)}{\log r}=\alpha. \qquad \square$$

We now consider overlapping forward orbits.

LEMMA 3.2. *If the dual IFS (2.3) satisfies the FTC, then*

$$\dim_{\mathrm{H}}(K)=\lim_{k\to+\infty}\frac{\log N_k}{\log \gamma^k},$$

where $\dim_{\mathrm{H}}$ *is the Hausdorff dimension and* K *is the attractor of the dual IFS (2.3).*

PROOF. Note that N_k is equal to $|\mathcal{V}_k|$ defined in [2, p. 2]; by [2, Lemma 3.2] and the proof of [2, Theorem 1.1], we see that the limit

$$\lim_{k\to+\infty}\frac{\log N_k}{\log \gamma^k}$$

exists and

$$\dim_{\mathrm{H}}(K)=\lim_{k\to+\infty}\frac{\log N_k}{\log \gamma^k}. \qquad \square$$

THEOREM 3.3. *Let* $a\in P$, *if the RIFS (2.1) satisfies the FTC and there is a matrix* R *and integers* $k_j>0$ *such that* $R_j=R^{k_j}$ *for all* j, *then*

$$\dim F_a=\dim_{\mathrm{H}}(K)=\lim_{k\to+\infty}\frac{\log N_k}{\log \gamma^k}.$$

PROOF. Without loss of generality, we assume that there exist $a_1, a_2, \ldots, a_d\in F_a$ to be a linearly independent set.

By Lemma 3.2, we only need to show that

$$\begin{aligned}\liminf_{r\to+\infty}\frac{\log N_k}{\log \gamma^k}&\le \liminf_{r\to+\infty}\frac{\log(\#\{F_a\cap B_r\})}{\log r}\\ &\le \limsup_{r\to+\infty}\frac{\log(\#\{F_a\cap B_r\})}{\log r}\le \limsup_{k\to+\infty}\frac{\log N_k}{\log \gamma^k}.\end{aligned}\tag{3.6}$$

Assume that the dual IFS (2.3) satisfies the FTC with respect to bounded invariant open set V. Assume, without loss of generality, $\gamma=\max\{\gamma_j\}=\vartheta^{k_1}$, where $\vartheta>1$ is the expansive ratio of the matrix R.

Similar to (3.3), there is a constant $c>0$ independent of k such that

$$T_J(a)\in B_{\gamma^k c},\quad \forall J\in\widetilde{\Lambda}_k.$$

Since $R_j=R^{k_j}$ and $T_J(x)=R_J(x-a)+T_J(a)$, by the assumption

$$\gamma=\max\{\gamma_j\}=\vartheta^{k_1},\quad k_1=\max_{1\le i\le m}\{k_i\}.$$

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If $J \in \widetilde{\Lambda}_k$, then $T_J(x) = R^{kk_1+r}(x) + T_J(a)$ for some integral r satisfying $0 \le r < k_1$, so

$$\#\{T_J \mid J \in \widetilde{\Lambda}_k,\ T_J(a) = z\} \le k_1, \quad \forall z \in \mathbb{Z}^d.$$

Therefore,

$$\#(F_a \cap B_{\gamma^k c}) \ge \frac{N_k}{k_1}.$$

This means that

$$\liminf_{k\to+\infty} \frac{\log N_k}{\log \gamma^k} \le \liminf_{r\to+\infty} \frac{\log(\#\{F_a \cap B_r\})}{\log r}. \tag{3.7}$$

For any fixed $\varepsilon \in (0, \vartheta - 1)$, let

$$M_\varepsilon = \varepsilon^{-1} \max\{\|b_j\| : 1 \le j \le m\},$$

then

$$\begin{aligned} \|T_j(x)\| &\ge \gamma_j \|x\| - \max\{\|b_j\| : 1 \le j \le m\} \\ &\ge (\vartheta^{k_j} - \varepsilon)\|x\| \ge (\vartheta - \varepsilon)^{k_j}\|x\| > \|x\|, \quad \text{if } \|x\| > M_\varepsilon. \end{aligned}$$

If $J = j_1 j_2 \ldots j_n \in \Lambda_k$, then $\gamma_{j_1 j_2 \ldots j_n} \ge \gamma^k$. Hence,

$$\|T_J(x)\| \ge (\vartheta - \varepsilon)^{k_1 k}\|x\| > \|x\|, \quad \text{if } \|x\| > M_\varepsilon.$$

Note that we have assumed that $\gamma = \max \gamma_j = \vartheta^{k_1}$, that any $z \in F_a \cap B_{M_\varepsilon(\vartheta-\varepsilon)^{kk_1}}$ can be written as $z = T_J(b)$ for some $b \in B_{M_\varepsilon} \cap F_a$ and that $J \in \Sigma^*$ with $\gamma_J < \gamma^{k+1}$. Hence, there exist $J_1 \in \bigcup_{j=0}^{\max\{k_i\}} \Sigma^j$, $J_2 \in \bigcup_{j=0}^{k} \Lambda_j$ such that $J = J_1 J_2$, where $\Lambda_0 = \Sigma^0 = \emptyset$. Let $Q = \#(\bigcup_{j=0}^{\max\{k_i\}} \Sigma^j)$, $N_0 = \#(B_{M_\varepsilon} \cap F_a)$, then

$$\#(\{F_a \cap B_{M_\varepsilon(\vartheta-\varepsilon)^k}\}) \le N_0 Q \sum_{j=1}^{k} N_j. \tag{3.8}$$

From Lemma 3.2, we see that

$$\limsup_{k\to+\infty} \frac{\log \sum_{j=1}^{k} N_j}{\log \gamma^k} = \limsup_{k\to+\infty} \frac{\log N_k}{\log \gamma^k}.$$

Therefore, (3.8) implies

$$\begin{aligned} &\limsup_{r\to+\infty} \log(\#\{F_a \cap B_r\})/\log r \\ &\quad = \limsup_{k\to+\infty} \frac{\log(\#\{F_a \cap B_{M_\varepsilon(\vartheta-\varepsilon)^{kk_1}}\})}{\log(M_\varepsilon(\vartheta-\varepsilon)^{kk_1})} \\ &\quad \le \frac{\log \vartheta}{\log(\vartheta - \varepsilon)} \limsup_{k\to+\infty} \frac{\log N_k}{\log \gamma^k}, \quad \forall \varepsilon \in (0, \vartheta - 1). \end{aligned}$$

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Let $\varepsilon \to 0$,

$$\limsup_{r\to+\infty} \frac{\log(\#\{F_a \cap B_r\})}{\log r} \le \limsup_{k\to+\infty} \frac{\log N_k}{\log \gamma^k}.$$

This relation and (3.7) imply (3.6), and we complete the proof. □

COROLLARY 3.4. *For the RIFS (2.2) and $a \in P$, assume that R is a similar expansive matrix with a similar ratio $\gamma > 1$. If the RIFS satisfies the OSC, then the dimension of F_a is $\log m/\log \gamma$.*

For an RIFS (2.2) satisfying the OSC, a forward orbit may overlap, but this corollary indicates that it will not have too many overlaps.

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