SOME LOWER BOUNDS FOR DENSITY OF MULTIPLE PACKING

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1. Introduction. A system of circles is said to form a k-fold packing of the plane if every point of the plane is an interior point of at most k circles. In this paper we consider only lattice packings, that is, we suppose that the centers of the circles form a lattice. We further restrict our consideration to circles of equal radius. Without loss of generality, the circles may have unit radius.

The least upper bound for fixed k of the density of such packings taken over every lattice in the plane is denoted by d_k and is called the closest k-fold lattice packing of equal circles in the plane. The determination of a general formula for d_k seems to be a difficult problem. In a recent survey of current developments in discrete geometry, L. Fejes Tóth [3] has remarked that final results in this direction are not to be expected. Even finding d_k for individual small k has proved to be at the least quite tedious.

A more tractable problem is the search for good lower bounds for d_k. It is known that

(1) $$d_1 = \pi/2\sqrt{3} = 0.9069\ldots,$$

and it is easily proved [1] that

(2) $$d_k/kd_1 \geq 1 .$$

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Heppes [4] has proved that equality holds in (2) only for $k < 5$. Blundon [2] has proved that

$$d_k/kd_1 \geq (k^2 - 1)/k(k^2 - 4)^{1/2}, \quad k \geq 5 . \tag{3}$$

The central idea of the proof is essentially due to Heppes, and makes use of a lattice which is the union of k congruent lattices, each of these lattices having a rhombus of side 2 as its fundamental parallelogram.

For large k the inequality (3) is very little better than (2). The purpose of this paper is to prove an inequality which is as good as (3) for $k \geq 5$ but much stronger than (3) for all $k \geq 10$. It may happen that the new inequality is best possible for several values of k. Blundon [1] has shown that this is the case for $k = 5$ and $k = 6$, thereby proving that $d_5/5d_1 = \frac{8}{35}\sqrt{21} = 1.0475...$ and that $d_6/6d_1 = \frac{35}{48}\sqrt{2} = 1.3012....$ However, the results stated in the theorems that follow are far from complete, since simple considerations of density show that, as $k \to \infty$, $d_k/kd_1 \to 1/d_1 = 1.1026....$

Note: Every plane lattice has a reduced basis consisting of two points (P and Q, say) such that $|P| \leq |Q| \leq |Q-P|$. With suitable coordinates we can take $P = (a, 0)$ and $Q = (g, h)$ such that $a > 0$, $0 \leq g \leq \frac{1}{2}a$, $g^2 + h^2 \geq a^2$.

THEOREM I. Let d_k represent the density of closest k-fold lattice packing of equal circles in the plane. Let $c = [k\theta]$, where $\theta = \frac{1}{13}(6 - \sqrt{10}) = 0.21828....$ Let $f(x) = (1 - x^2)/(1 - 4x^2)^{1/2}$. Then

$$d_k/kd_1 \geq f(c/k), \quad \text{for } k \geq 5 , \tag{4}$$

and a reduced basis for the lattice providing this packing is given by the points

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$(a, 0)$ and $(0, h)$ for even k

$(a, 0)$ and $(\frac{1}{2}a, h)$ for odd k,

where $a^2 = 12/(k^2 - c^2)$ and $h^2 = (k^2 - 4c^2)/(k^2 - c^2)$.

THEOREM II. For every $\varepsilon > 0$, there exist arbitrarily large positive integers k such that

$$(5) \qquad d_k/kd_1 > f(\theta) - \varepsilon ,$$

where

$$(6) \qquad f(\theta) = \frac{41\sqrt{5} + 20\sqrt{2}}{845} (5 + 16\sqrt{10})^{1/2} = 1.0585\ldots.$$

2. <u>Construction of the lattice.</u> Let Λ_0 be a lattice with fundamental parallelogram $ABCD$ having $AB = BC = CD = DA = 2$ and $BD < 2$. Then circles of unit radius centred at the vertices of $ABCD$ cut the diagonal AC at points P, Q, R, S such that $AP = BQ = BR = DQ = DR = CS = 1$. Since QR/AC varies between 0 and $1/2$ and since $k \geq 5$, the length of AC may be so chosen that $k.QR/AC$ is an integer c. Let $AC/k = a$, so that $AC = ka$ and $QR = ca$. The diagram below illustrates the case $k = 14$, $c = 3$. Let $BD = 2h$. Then $1 - (\frac{1}{2}ca)^2 = h^2 = 4 - (\frac{1}{2}ka)^2$, whence

$$(7) \qquad a^2 = 12/(k^2 - c^2), \quad h^2 = (k^2 - 4c^2)/(k^2 - c^2).$$

The following lemma plays an important part in the proof of the theorems.

LEMMA. If $c < k\theta$, where $\theta = \frac{1}{13}(6 - \sqrt{10})$, then $PQ > \frac{1}{2}QR$.

<u>Proof.</u> We have $2AQ = (k - c)a$, $AP = 1$, $QR = ca$, $2PQ = (k - c)a - 2$. Now $c < k\theta$ implies that $13c^2 - 12ck + 2k^2 > 0$, which may be put in the form

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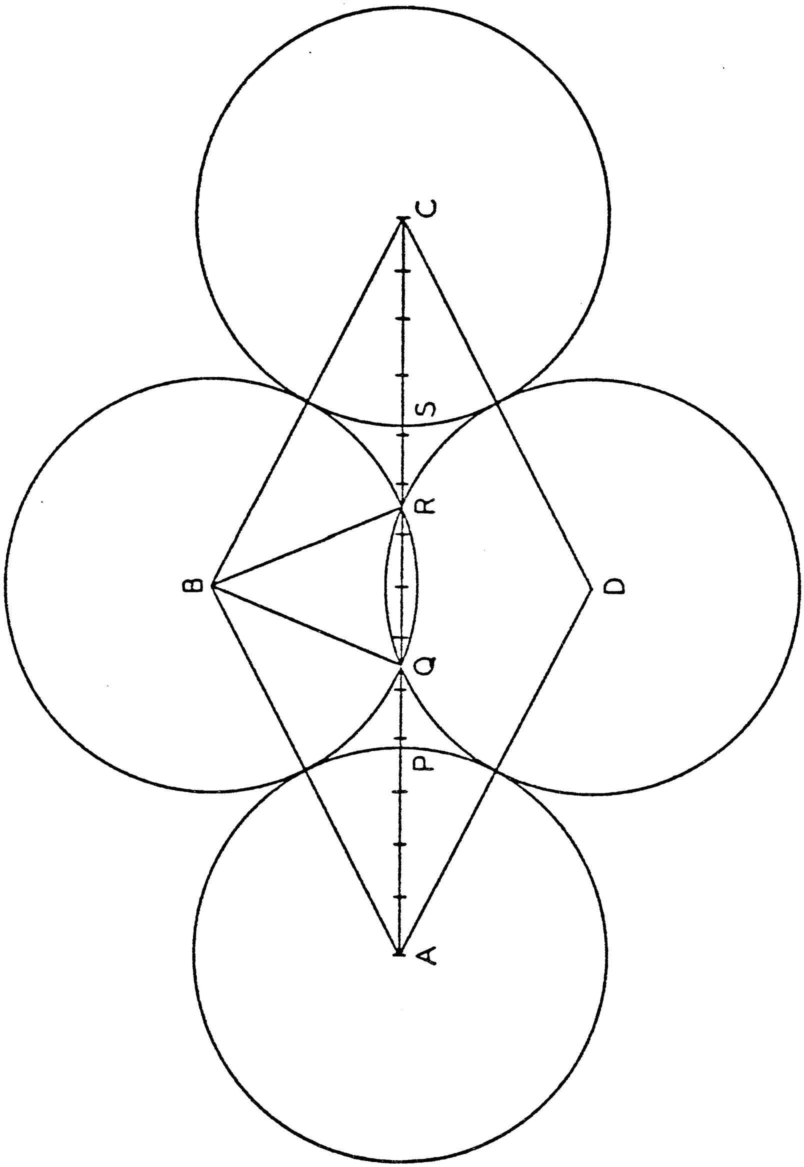

Figure 1

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$3(k - 2c)^2 > k^2 - c^2 = 12/a^2$. Since $k - 2c$ is clearly positive, this gives $k - 2c > 2/a$, that is, $(k - c)a - 2 > ca$. Thus $2PQ > QR$ and the lemma is proved.

3. Proof of Theorem I. The union of the k lattices $\Lambda_i = \Lambda_0 + \frac{i}{k}\vec{AC}$ $(i = 0, 1, 2, \ldots, k-1)$ is itself a lattice Λ. By the definition of Λ_0, it follows that Λ is generated by the points

$$(a, 0) \text{ and } (0, h) \qquad \text{for even } k$$

$$(a, 0) \text{ and } (\tfrac{1}{2}a, h) \qquad \text{for odd } k,$$

where a, h are given by (7) and the diagonals of $ABCD$ are taken as the coordinate axes. The relations $k \geq 5$ and $c/k < \theta = \frac{1}{13}(6 - \sqrt{10})$ give

$$4h^2/3a^2 = \frac{1}{9}k^2(1 - 4c^2/k^2) > \frac{25}{9}(1 - 4\theta^2) = \frac{25}{507}(16\sqrt{10} - 5) > 1,$$

so that $h/a > \sqrt{3}/2$. Hence the stated generating points form a reduced basis.

We prove next that Λ does in fact provide a k-fold packing of the plane. No point of the rhombus $ABCD$ can be covered by more than two circles of Λ_0, namely, those centred at B and D. The only other possibility is that the circle centred at D may be overlapped by a circle centred at $2B + \frac{i}{k}\vec{AC}$. A necessary condition for such an overlapping is that $3h < 2$. Since $h^2 = (k^2 - 4c^2)/(k^2 - c^2)$, this would give $c^2/k^2 > 5/32 > 1/9$ so that $\theta > c/k > \frac{1}{3}$, which contradicts the definition of θ.

It follows that the only points of $ABCD$ covered by exactly two circles of Λ_0 are those in the intersection of the interiors of the circles centred at B and D. No circle of Λ_0 contains any point of PQ or RS. The Lemma ensures

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that any point of ABCD covered twice by circles with centres in Λ_i is not covered by circles with centres in Λ_{i-c} or Λ_{i+c}. Therefore, no point of ABCD (and consequently by symmetry no point of the plane) can be covered by more than k circles with centres in Λ. Thus the lattice Λ provides a k-fold packing of the plane.

The determinant Δ of the lattice Λ is given by $\Delta = ah = 2\sqrt{(3k^2 - 12c^2)}/(k^2 - c^2)$. Now $d_k \geq \pi/\Delta$ and $d_1 = \pi/2\sqrt{3}$. Therefore $d_k/kd_1 \geq (k^2 - c^2)/k(k^2 - 4c^2)^{1/2}$ $= f(c/k)$, and the proof of the theorem is complete.

4. Proof of Theorem II. Let $f(x) = (1-x^2)/(1 - 4x^2)^{1/2}$, with $0 < x < 1/2$, so that f is continuous at every point. Then $f'(x)/f(x) = 2x(1 + 2x^2)/(1 - x^2)(1 - 4x^2) > 0$, so that $f(x)$ increases with x. Since θ is irrational, a rational fraction can always be found less than θ and arbitrarily close to θ. Take the denominator of this fraction as k and the numerator as c. Then it is clearly possible to find integers c and k such that $f(c/k)$ is less than $f(\theta)$ and arbitrarily close to $f(\theta)$. Hence for every positive ε, there exist integers c, k such that $0 < f(\theta) - f(c/k) < \varepsilon$. Thus, by Theorem I, $d_k/kd_1 \geq f(c/k) > f(\theta) - \varepsilon$. (6) follows by straightforward computation. This completes the proof of Theorem II.

5. Remarks. A graph (for $k \leq 25$) of lower bounds for d_k/kd_1 as given by the preceding theorems appears below. Suitable values for c, k in Theorem II can be found by expressing θ as a continued fraction and selecting those convergents less that θ. From $\theta = \frac{1}{13}(6 - \sqrt{10}) = [4, \dot{1}, 1, \dot{2}]$ we obtain the following table.

c	1	5	12	43	191	456	1633	...
k	5	23	55	197	875	2089	7481	...

Note that the restriction on k in (4) is made for convenience. Actually (4) holds for all $k \geq 1$, since, for $1 \leq k \leq 4$,

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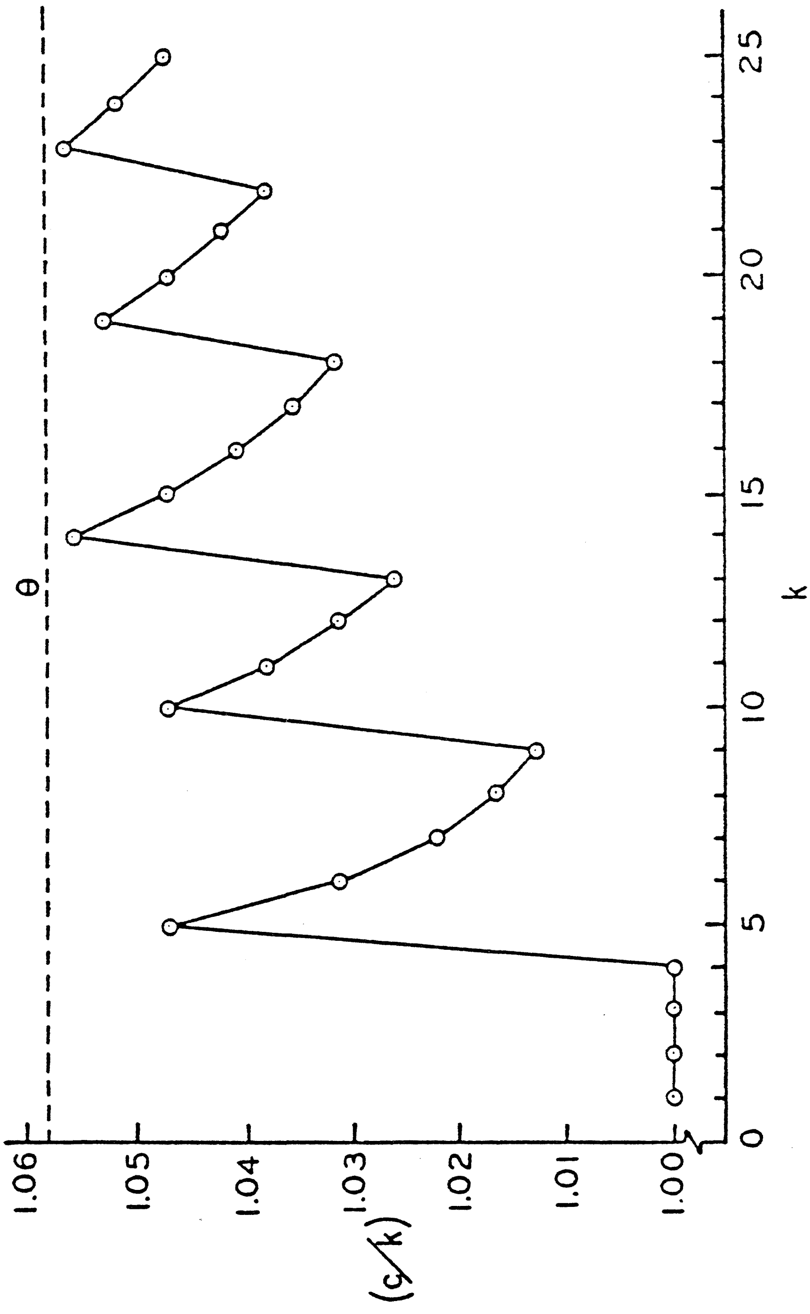

Figure 2

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we have $c = 0$ so that $f(0) = 1$. The inequality for these values then reduces to (2).

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