THE DENSITY OF REDUCIBLE INTEGERS

S. D. CHOWLA AND JOHN TODD

Introduction. The concept of a reducible integer was introduced recently [3] : if $P(m)$ denotes the greatest prime factor of m then n is said to be reducible if $P(1 + n^2) < 2n$. The reason for the term is that reducibility is a condition necessary and sufficient for the existence of a relation of the form

$$\arctan n = \sum_{i=1}^{r} f_i \arctan n_i$$

where the f_i are integers and the n_i positive integers less than n. J. C. P. Miller pointed out to us the regularity of the distribution of the reducible integers (less than 600). In collaboration with Dr. J. W. Wrench, using his tables of factors of $1 + n^2$, we carried the count still further, and observed the same regularity. The following conjecture suggested itself:

C. *"Reducible integers have a density about* 0.3."

We have not been able to make very much headway with this but have succeeded in establishing the following:

THEOREM A. *The density of the set of integers n for which $P(n) < 2n^{\frac{1}{2}}$ is* $1 - \log 2 = .3069. \ldots$

This note contains a proof of this theorem, and a table summarizing the numerical evidence in support of C.

1. Numerical evidence. We give here a summary of the numerical evidence relating to the conjecture C together with corresponding results related to Theorem A. The table below gives, in each range $(1 + 100\, n, 100(n + 1))$, for $n = 0(1)49$, on the right, the number of reducible integers in that range, and on the left, the number of integers in that range which satisfy $P(n) < 2n^{\frac{1}{2}}$.

Totals in the various chiliads and a grand total for the complete range (1-5000) are given in the last line of the table.

	0	1000	2000	3000	4000	
1-100	(29, 57)	(31, 43)	(29, 43)	(33, 41)	(29, 42)	
101-200	(29, 50)	(25, 43)	(30, 42)	(28, 43)	(28, 40)	
201-300	(28, 47)	(33, 44)	(23, 42)	(23, 43)	(27, 41)	
301-400	(26, 45)	(28, 41)	(32, 41)	(32, 43)	(31, 40)	
401-500	(30, 45)	(31, 44)	(28, 44)	(29, 38)	(27, 42)	
501-600	(30, 44)	(23, 44)	(32, 39)	(32, 41)	(38, 39)	
601-700	(30, 44)	(27, 40)	(26, 43)	(25, 40)	(30, 41)	
701-800	(29, 44)	(34, 43)	(32, 41)	(30, 43)	(35, 39)	
801-900	(27, 44)	(28, 45)	(27, 42)	(29, 40)	(30, 43)	
901-1000	(23, 42)	(31, 39)	(29, 41)	(19, 41)	(38, 41)	
	(281, 462)	(291, 426)	(288, 418)	(280, 413)	(313, 408)	(1453, 2127)

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2. Proof of Theorem A. It is more convenient to show that the density of the integers n for which $P(n) \geq 2n^{\frac{1}{2}}$, is log 2. That is, we shall show that

$$Q(x) = \sum_{\substack{n \leq x \\ P(n) \geq 2n^{\frac{1}{2}}}} 1 \sim x \log 2\ ;$$

to do this we establish the two following results:

A_1.
$$Q_1(x) = \sum_{\substack{n \leq x \\ P(n) \geq 2x^{\frac{1}{2}}}} 1 \sim x \log 2;$$

A_2.
$$Q_2(x) = Q(x) - Q_1(x) = \sum_{\substack{n \leq x \\ 2n^{\frac{1}{2}} \leq P(n) \leq 2x^{\frac{1}{2}}}} 1 = o(x).$$

2.1. Proof of A_1. This is carried out by a modification of a method used recently [1] to evaluate lim $x^{-1}R_a(x)$ where $R_a(x)$ is the number of integers $n \leq x$ for which $P(n) \geq x^a$.

For any p the number of integers $n \leq x$ which are multiples of p is $[x/p]$. In $Q_1(x)$ we consider only primes $p = P(n) \geq 2x^{\frac{1}{2}}$: for such primes the residual factor $(n/p) \leq \frac{1}{2} x^{\frac{1}{2}} < p$ and so every multiple of p which does not exceed x has p for its greatest prime factor. Hence

$$\begin{aligned} Q_1(x) &= \sum_{2x^{\frac{1}{2}} \leq p \leq x} [x/p] \\ &= \sum_{2x^{\frac{1}{2}} \leq p \leq x} \{(x/p) + O(1)\} \\ &= x \sum_{2x^{\frac{1}{2}} \leq p \leq x} p^{-1} + O(x/\log x), \end{aligned}$$

since $\sum_{2x^{\frac{1}{2}} \leq p \leq x} 1 \leq \sum_{p \leq x} 1 = O(x/\log x)$.

It is, however, well known [2, pp. 100-102] that

B.
$$\sum_{p \leq x} p^{-1} = \log \log x - l + O(1/\log x)$$

where l is a certain constant. Hence

$$\begin{aligned} x^{-1}Q_1(x) &= \log \log x - \log \log 2x^{\frac{1}{2}} + o(1) \\ &= \log \{(\log x)/(\tfrac{1}{2} \log x + \log 2)\} + o(1) \\ &= \log 2 + o(1), \end{aligned}$$

which establishes A_1.

2.2. Proof of A_2. This is carried out in the following manner. First, it will be sufficient to restrict the values of n considered to the range

$$x/(\log x)^2 \leq n \leq x,$$

for this implies a change in the sum of $O(x/(\log x)^2) = o(x)$. Secondly, we do not decrease the sum if we replace $2n^{\frac{1}{2}}$, the variable limit in the lower inequality, by its smallest value $2x^{\frac{1}{2}}/\log x$. Thirdly, we do not decrease the sum by now allowing n to cover the full range $1 \leq n \leq x$. Thus it will be sufficient to show that

$$Q_3(x) = \sum_{\substack{n \leq x \\ (2x^{\frac{1}{2}}/\log x) \leq p(n) \leq 2x^{\frac{1}{2}}}} 1 = o(x).$$

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In order that an integer should contribute to Q_3 it is necessary that it should have a prime factor p in the range $(2x^{\frac{1}{2}}/\log x,\ 2x^{\frac{1}{2}})$. For p fixed the number of such n is $[x/p]$. Hence

$$Q_3 \leq \sum_{(2x^{\frac{1}{2}}/\log x) \leq p \leq 2x^{\frac{1}{2}}} [x/p] \, .$$

(It is possible for an integer $n \leq x$ to have two factors in the range and so we must allow for inequality, which was not so in the case of Q_1.)

We now proceed as before:

$$\begin{aligned} Q_3(x) &\leq \sum_{(2x^{\frac{1}{2}}/\log x) < p \leq 2x^{\frac{1}{2}}} [x/p] = x \sum_{(2x^{\frac{1}{2}}/\log x) \leq p \leq 2x^{\frac{1}{2}}} p^{-1} + O(x^{\frac{1}{2}}/\log x) \\ &= x \left\{\log\log 2x^{\frac{1}{2}} - \log\log (2x^{\frac{1}{2}}/\log x)\right\} + O(x/\log x), \end{aligned}$$

using B. Since

$$\begin{aligned} &\log\log 2x^{\frac{1}{2}} - \log\log (2x^{\frac{1}{2}}/\log x) \\ &\quad = \log \left\{(\tfrac{1}{2}\log x + \log 2)/(\tfrac{1}{2}\log x + \log 2 - \log\log x)\right\} \\ &\quad = \log \left[\left\{1 + (\log 4)/(\log x)\right\} \left\{1 + (\log 4 - 2\log\log x)/\log x\right\}^{-1}\right] \\ &\quad = \log \left\{1 + O(1/\log x))(1 + O(\log\log x/\log x)\right\} \\ &\quad = O(\log\log x/\log x) = o(1), \end{aligned}$$

the proof of A_2 is complete.

3. Possible generalizations. It is clear that $2n^{\frac{1}{2}}$ in Theorem A can be replaced by $An^{\frac{1}{2}}$ for any $A \geq 1$ without affecting the conclusion.

Similar arguments show that the density of the integers n for which $P(n) > An^{\alpha}$ $(\frac{1}{2} < \alpha < 1, A > 1)$ is exactly $\log \alpha$.

The case when $\alpha < \frac{1}{2}$ requires more careful study along the lines indicated in [1] and it can be shown that the device used here (replacing a summation over $1 \leq n \leq x$ by one over $(x/(\log x)^2 \leq n \leq x)$ will enable the density to be evaluated explicitly in this case, too.

It is clear that an estimate for the error term

$$x^{-1}Q(x) - \log 2$$

is

$$O(\log\log x/\log x),$$

and this explains the slowness of the convergence apparent in the table.

References

[1] S. Chowla and T. Vijayaraghavan, *J. Indian Math. Soc.* (New Series), vol. 11 (1947), 31-37.

[2] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen* (Leipzig, 1909).

[3] John Todd, "A Problem of J. C. P. Miller on Arctangent Relations," *Amer. Math. Monthly* (1949).

Institute for Advanced Study
King's College, London

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