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PROFINITE MODULES

BY
GERARD ELIE COHEN[1]

Introduction. An inverse limit of finite groups has been called in the literature a pro-finite group and we have extensive studies of profinite groups from the cohomological point of view by J. P. Serre. The general theory of non-abelian modules has not yet been developed and therefore we consider a generalization of profinite abelian groups. We study inverse systems of discrete finite length R-modules. Profinite modules are inverse limits of discrete finite length R-modules with the inverse limit topology.

Let R be a topological ring, C_R the category of all R-modules and R-homomorphisms. Let B_R be the category of profinite R-modules and continuous R-homomorphisms. Then B_R is a coreflective subcategory of C_R. Moreover it has exact inverse limits and we study the free and projective objects of B_R. B_R is not full unless the coreflection map is continuous $\forall B \in B_R$. B_R is an abelian subcategory of C_R, thus B_R is colocally finite.

I. The category of profinite R-modules: B_R. We consider an associative ring R with 1 and right-unitary R-modules unless otherwise stated.

1.1. Proposition. *Let R be a topological ring, A a simple R-module. The following are equivalent*:

(1) *A with the discrete topology is a topological R-module.*

(2) *There exists an open maximal right ideal M such that $A \simeq R/M$.*

(3) *$A \simeq R/M'$ implies that M' is open.*

Proof. (1)$\Rightarrow$(2): Let $a \in A$, $a \neq 0$, $M = \text{Ann}(a)$. Then $A \simeq R/M$. Let $f: A \to R/M$ be the isomorphism $(ar)f = r+M$. Also $g: A \times R \to A$ is continuous where $(a, r)g = ar$. $\text{Ker}(g) = \{(at, r): atr = 0\} = \bigcup_{t \in r} (\{at\} \times U_t)$ is open where U_t is open in R. $s \in M \Leftrightarrow as = 0 \Leftrightarrow (a, s) \in \text{Ker}(g) \Leftrightarrow s \in U_1$. Thus $M = U_1$ is open.

(2)$\Rightarrow$(3): Suppose $A \simeq R/M'$. There exists an open maximal right ideal M such that $A \simeq R/M$. Let $f: R/M' \to R/M$ be the isomorphism $(1+M')f = r+M$. Now $g: R \to R$ where $(x)g = rx$ is continuous. $(M)g^{-1} = \{x \in R: (x)g \in M\} = \{x \in R: x \in M'\} = M'$ is open.

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(3)$\Rightarrow$(1): The map $(x, y)\to(x-y)$ is obviously continuous. Also the map g: $A\times R\to A$ where $(a, r)g=ar$ is continuous since $(\{ar\})g^{-1}=\{(at, s):ats=ar\}=\bigcup_{t\in r}(\{at\}\times C_t)$ is open where $C_t=\{s\in R:ats=ar\}$: indeed if $C_t=\phi$, C_t is open; otherwise $\exists u\in C_t$ and $C_t=\text{Ann}(at)+u$; if $at=0$, $\text{Ann}(at)=R=C_t$ is open; and if $at\neq 0$, $\text{Ann}(at)$ is a maximal right ideal such that $A\cong R/\text{Ann}(at)$. By (3), $\text{Ann}(at)$ is open and thus C is open.

1.2. Definition. The simple R-modules satisfying the equivalent properties of 1.1 are called the *discrete simple R-modules.*

1.3. Definition. A discrete finite length R-module is an R-module A of finite length, (i.e., it has a composition series of length $l(A)<+\infty$) and A with the discrete topology is a topological R-module.

1.4. Lemma. *The class of discrete finite length R-modules is closed under taking submodules, factor modules, finite direct sums and homomorphic images.*

Proof. Left to reader.

1.4.1. Corollary. *Let D_R be the category whose objects are discrete finite length R-modules and whose morphisms are continuous R-homomorphisms. Then D_R is a full, abelian subcategory of C_R, the category of R-modules.*

Proof. Left to reader.

1.5. Lemma. *A is a discrete finite length R-module if and only if the composition factors are discrete simple.*

Proof. Left to reader.

1.6. Definition. Let C_R be the category of R-modules, R is a topological ring. Then the subcategory B_R is defined as follows: its objects are inverse limits of discrete finite length modules with the inverse limit topology and its morphisms are continuous R-homomorphisms. We call B_R the *category of profinite modules.*

1.7. Example 1. Let Z be the ring of rational integers with the discrete topology. The discrete finite length Z-modules are finite abelian groups: being noetherian, they are finitely generated and being artinian, they cannot have infinite cycles in their decomposition. Thus B_Z is the category of profinite groups with the inverse limit topology.

1.8. Example 2. Consider Z, the ring of integers with the (p)-topology, (a basis for the neighborhood system of zero is given by the powers of the prime (hence maximal) ideal (p)). Thus $Z/(p)$ is a discrete simple R-module. If $q \neq p$ then (q) is not an open maximal ideal and thus $Z/(q)$ is not a discrete simple R-module although it is simple. $Z/(p)^k$ is a discrete finite length R-module. $\varprojlim Z/(p)^n$ is a profinite Z-module which is the uniform completion of Z when we give Z the (p)-topology.

1.9. EXAMPLE 3. Let R be a commutative local noetherian ring whose maximal ideal is M. We give R the M-topology. Let A be a finitely generated R-module. Then $B_k = A/AM^k$ is a discrete finite length R-module: B_k is the image of a finitely generated free module, $R \oplus \cdots \oplus R \to B_k$, whence the epimorphism

$$R/M^k \oplus \cdots \oplus R/M^k \to B_k;$$

one shows R/M^k (and hence B_k by 1.4) is a discrete finite length R-module. Also the $\{B_k\}$ forms an inverse system. Let $B = \varprojlim B_k$, $B \in B_R$. (B is the uniform completion of A if we give A the M-topology). In fact, $B = \varprojlim A/A_i$ where $\{A/A_i\}$ is the set of all the factor modules of A which are discrete finite length R-modules: it suffices to show that $\{A/AM^k = B_k\}$ is cofinal in $\{A/A_i\}$, i.e., $\forall i \in k \ \exists\ A_i \supseteq AM^k$. Consider the following chain

$$(A_i + AM^k)/A_i \supseteq (A_i + AM^{k+1})/A_i \supseteq \cdots$$

Since A/A_i is artinian, without loss of generality, we have $(A_i + AM^k)/A_i = (A_i + AM^{k+1})/A_i$, thus $((A_i + AM^k)/A_i)M = (A_i + AM^k)/A_i$. Also $(A_i + AM^k)/A_i$ is finitely generated since A/A_i is noetherian and Rad $R = M$. Thus $(A_i + AM^k)/A_i = 0$, $A_i + AM^k = A_i$, $AM^k \subseteq A_i$. (Thus if we give A the M-topology, the uniform completion of A is $\varprojlim A/A_i$.)

II. The coreflectivity of B_R. We refer the reader to [5, p. 128] for the definition of the terms: coreflection map, coreflective subcategory.

2.1. DEFINITION. A topological R-module is *linearly compact* if every family of closed cosets which has the finite intersection property has a nonvoid intersection.

2.2. LEMMA. *Every discrete finite length module is linearly compact and hence every object of B_R is linearly compact.*

Proof. (Cf. [6, p. 81, Propositions 5 and 4]).

2.3. LEMMA. *Let $A_1, \ldots, A_n$ be submodules of an R-module A such that A/A_i is a discrete finite length R-module. Then $A/\bigcap_{i=1}^n A_i$ is a discrete finite length module.*

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Proof. Consider the canonical monomorphism

$$A/\bigcap A_i \to A/A_1\oplus\cdots\oplus A/A_n$$

and it follows from 1.4.

2.4. LEMMA. *Let* $A=\varprojlim A_i \in B_R$, $p_i:A\to A_i$. *Let* $B_i=\operatorname{Im} p_i$. *Then A is topologically isomorphic to* $\varprojlim B_i$ *where the canonical projections* $q_i:A\to B_i$ *are onto.*

Proof. Left to reader.

2.4.1. REMARK. Thus $A=\varprojlim A/N_i$ where $N_i=\operatorname{Ker} q_i$.

2.5. *Definition of the coreflector* $G.\ C_R\to B_R$: For any $A\in C_R$ there corresponds a pair $(c_A, (A)G)$, $c_A:A\to(A)G$ such that the following universal property holds: given any R-homomorphism $f:A\to B$, $B\in B_R$, there exists a unique continuous R-homomorphism $g:(A)G\to B$ such that the following diagram commutes

$$\begin{array}{ccc} A & \longrightarrow & B \\ & \searrow \quad \nearrow & \\ & (A)G & \end{array}$$

$f=c_Ag$, we sometimes write $g=(f)G$.

2.5.1. REMARK. This is the same as saying that the inclusion functor $F:B_R\to C_R$ (which forgets the topology of objects of B_R) has a left-adjoint $G:C_R\to B_R$, i.e., $C_R[A, (B)F]\cong B_R[(A)G, B]$.

2.6. *Construction of the coreflection G.* Let $A\in C_R$. We define $(A)G=\varprojlim A/A_i$ where (A/A_i)'s are all the factor modules of A which are discrete finite length R-modules: $\{A/A_i\}$ forms an inverse system (2.3), $(A)G\in B_R$. Let $p_i:\varprojlim A/A_i\to A/A_i$, $c_A:A\to(A)G$ is defined by $(a)c_Ap_i=a+A_i$. Let $f:A\to B$ be a given R-homomorphism, $B\in B_R$. $B\cong\varprojlim B/B_j$ (2.4.1); let $q_j:B\to B/B_j$, $(a)fq_j=(a)f+B_j$. Define g as follows: $(\ldots, a_i+A_i, \ldots)gq_j=(a_k)f+B_j$ where $A_k=(B_j)f^{-1}$. Now A/A_k is a discrete finite length R-module since it is isomorphic to a submodule of B/B_j, where the explicit map is given by $a+A_k\mapsto(a)f+B_j$. One shows g is a continuous R-homomorphism, makes the diagram commutative and is unique. (The following two facts are used: first, if $q_{ij}:B/B_j\to B/B_i$, $(B_j)q_{ij}\subseteq B_i$, thus $A_e=(B_j)f^{-1}=(B_j)q_j^{-1}f^{-1}\subseteq(B_i)q_{ij}^{-1}(fq_j)^{-1}=(B_i)f^{-1}=A_k$. Thus $A_e\subseteq A_k$ and $q_{ke}:A/A_e\to A/A_k$; also, $(A)c_A$ is dense in $(A)G$, thus $g:(A)G\to B$ is the unique extension of the continuous mapping $(A)c_A\to B$ defined by the commutativity of the diagram by [1, p. 85, Corollary 1 to Proposition 2]).

2.7. PROPOSITION. *Every object $B \in B_R$ is linearly topologized.*

Proof. Let U be any open neighborhood of 0, $U \subseteq B$. U is the union of basic open sets. Thus $0 \in$ some basic open set V, $V = (\{0\} \times \cdots \times \{0\} \times B/B_{n+1} \times \cdots) \cap B$. V is a submodule.

2.8. PROPOSITION. *Let U be an open submodule of $B \in B_R$. Then B/U is a discrete finite length R-module.*

Proof. Left to reader.

2.9. LEMMA. *Let $C \cong \varprojlim C/C_i \in B_R$, $q_i: C \to C/C_i$. Let D be a linearly compact R-module. If $f: D \to C$ is an R-homomorphism such that $p_i = fq_i: D \to C/C_i$ is continuous and onto $\forall i$, then f is onto.*

Proof. Let $y \in C$. We have to find $x \in D \ni (x)f = y$: let $(y)q_i = y_i = c_i + C_i$. Consider $V_i = (y_i)p_i^{-1}$. The V_i's are closed cosets of D, moreover they have the finite intersection property: consider $V_1, \ldots, V_n$, since the index set is directed $\exists k \ni i \leq k$, $i = 1, \ldots, n$; V_k is a nonempty closed coset of D, thus $\exists t \in V_k \ni (t)p_k = y_k = c_k + C_k$. Let q_{ik}: $C/C_k \to C/C_i$, $(t)p_i = (t)fq_i = (t)fq_kq_{ik} = (t)p_kq_{ik} = (c_k + C_k)q_{ik} = c_i + C_i = y_i$; thus $t \in V_i \forall i = 1, \ldots, n$; since D is linearly compact, the intersection of all V_i's contains an element x.

2.10. THEOREM. *Let $B \in B_R$. Let c_B, the coreflection map, be continuous. Then B is topologically isomorphic to $((B)F)G$. In fact the coreflection map is a topological isomorphism.*

Proof. Consider the following diagram
$$\begin{array}{ccc} B & \rightarrow & B \\ \downarrow & \nearrow & \\ (B)FG & & \end{array}$$
where $c_B: B \to (B)FG$, $g: (B)FG \to B$, $c_Bg = 1_B$. Thus c_B is mono; $(B)FG = \varprojlim B/B_k$ where $\{B/B_k\}$ is the set of all the factor modules of B which are discrete finite length; in the commutative diagram
$$\begin{array}{ccc} (B)FG & \rightarrow & B/B_k \\ \nwarrow & & \nearrow \\ & B & \end{array}$$
where $q_k: (B)FG \to B/B_k$, $p_k: B \to B/B_k$, $c_Bq_k = p_k$. Since q_k's and c_B are continuous, p_k's are continuous, p_k's are also onto, B is linearly compact (2.2), thus c_B is onto (2.9); c_B is an R-module isomorphism, $\exists c^{-1} \in$ $c_Bc^{-1} = 1_B$, $c^{-1}c_B = 1_{(B)FG} \cdot g(c_Bc^{-1}) = g = (c^{-1}(c_Bg) = c^{-1}$, g is continuous (2.6), thus $c^{-1} = g$ is continuous and c_B is open.

2.10.1. COROLLARY. *If c_B is continuous, then any R-homomorphism $f: B \to C$, where B, $C \in B_R$, is a continuous R-homomorphism.*

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Proof. $c_B: B \to (B)FG$ is continuous, $g: (B)FG \to C$ is continuous, (2.6), $\therefore f = c_B g$ is continuous.

2.10.2. Corollary. *B_R is full if and only if c_B is continuous $\forall B \in B_R$.*

Proof. Left to reader.

2.11. Proposition. *B_R is not necessarily a full subcategory.*

Proof. Consider $\prod Z_2$, $Z_2 \in B_{Z_2}$ where Z_2 is the 2-element field with the discrete topology. Let M be a maximal submodule of $\prod Z_2$, thus $\prod Z_2/M \cong Z_2$. Now M is the kernel of a map $f: \prod Z_2 \to Z_2$. Now M is dense in $\prod Z_2$, if f is continuous, M is closed and $M = \bar{M} = \prod Z_2$.

III. Subjects and quotient objects of B_R.

3.1. Proposition. *Let $A \in B_R$. Let B be a submodule of A with the relative topology. B is closed if and only if $B \in B_R$.*

Proof. If $B \in B_R$, B is linearly compact (2.2), A is linearly topologized (2.8) thus B is closed [6, p. 82, Proposition 7]; conversely, if B is closed, B is linearly compact. Now $A = \underset{\longleftarrow}{\text{Lim}}\, A_i$, $q_i: A \to A_i$, let $(B)q_i = B_i$, the $\{B_i\}$ forms an inverse system of discrete finite length R-modules: consider the following diagram

$$\begin{array}{ccc} B & \to & \underset{\longleftarrow}{\text{Lim}}\, B_i \\ & \searrow \quad \swarrow & \\ & B_i & \end{array}$$

where $p_i: B \to B_i$ is continuous $\forall i$, since p_i is the restriction of q_i. Let $m_i: \underset{\longleftarrow}{\text{Lim}}\, B_i \to B_i$. By properties of inverse limits, we have a unique R-homomorphism $g: B \to \underset{\longleftarrow}{\text{Lim}}\, B_i$. One shows g is a topological isomorphism and thus $B \in B_R$.

3.2. Proposition. *Let C be a linearly compact (and hence closed) submodule of B, $B \in B_R$. Then $B/C \cong \underset{\longleftarrow}{\text{Lim}}\, B_i/C_i$ where $B = \underset{\longleftarrow}{\text{Lim}}\, B_i$, $p_i: B \to B_i$, $(C)p_i = C_i$ and where B/C has the quotient topology.*

Proof. By (3.1), $C = \underset{\longleftarrow}{\text{Lim}}\, C_i$. Consider $p_i m_i: B \to B_i \to B_i/C_i$ where B_i/C_i has the quotient topology which coincides here with the discrete topology: $\text{Ker}(p_i m_i) \supseteq \underset{\longleftarrow}{\text{Lim}}\, C_i$. Thus $p_i m_i$ induces $v_i: B/C \to B_i/C_i$. One shows v_i's are continuous, $\{B_i/C_i\}$ is an inverse system of discrete finite length R-modules and that $g: (B/C \to \underset{\longleftarrow}{\text{Lim}}\, B_i/C_i$ induced by the v_i's is a topological isomorphism.

3.3. Proposition. *Let A, $B \in B_R$. Form $A \times B = A \oplus B \in C_R$. Then $A \times B \in B_R$ when we give $A \times B$ the product topology.* (*In fact it is the sum and the product of A and B in B_R*).

Proof. Left to reader.

3.4. Proposition. *Every morphism in B_R has a kernel and a cokernel.*

Proof. Let $f: A \to B \in B_R$. Let $K = \text{Ker}(f) = (0)f^{-1}$, then K is a closed submodule of A, $K \in B_R$ (3.2), one shows that $i: K \to A$ the canonical monomorphism is the kernel of $f: A \to B$. Also $(A)f \in B_R$ using [6, p. 81, Proposition 2], (2.8) [6, p. 82, Proposition 7], (3.2), $\therefore$ $B/(A)f \in B_R$, (3.3); one shows Coker $(f) \cong B/(A)f$.

3.5. Proposition. *Let $f: A \to B \in B_R$ be a monomorphism, then $f: A \to B$ is a monomorphism in C_R and hence* 1-1.

Proof. Let $a, b: D \to A$ be R-homomorphisms such that $af = bf$. Now $c_D(a)G = a$, $c_D(b)G = b$, $\therefore$ $c_D(a)Gf = c_D(b)Gf$, thus $(a)Gf$ and (b) Gf agree on the dense subset $(D)c_D$ of $(D)G$, $\therefore$ $(a)Gf = (b)Gf$ on $(D)G$, thus $(a)G = (b)G$, and $a = c(a)G = c(b)G = b$.

3.6. Proposition. *Let $f: A \to B \in B_R$ be an epimorphism, then f is onto.*

Proof. Consider 0, $x: B \to B/(A)F$, now $f0 = fx$, $\therefore$ $0 = x$, $B = (A)f$.

3.7. Proposition. *$F: B_R \to C_R$ is exact and $G: C_R \to B_R$ is right exact.*

Proof. F is exact (3.5, 3.6, 3.1, 3.2, 3.4). Now consider $0 \to A \to B \to C \to 0$ an exact sequence in C_R where $f: A \to B$, $g: B \to C$; we show $(A)G \to (B)G \to (C)G \to 0$ is exact in B_R. First $(g)G$ is onto: let $y \in (C)G$, we have to find $x \in (B)G$ such that $(x)(f)G = y$; let $p_i: (C)G \to C/C_i$, $(y)p_i = y_i$, now $B/B_j \cong C/C_i$ where $B_j = (C_i)g^{-1}$ since $g: B \to C$ is onto, $q_j: (B)G \to B/B_j$ is onto since $(b + B_j) \in (B)G$, $\therefore$ $(B)G \to C/C_i$ is continuous and onto: moreover $(B)G$ is linearly compact, $\therefore$ $(g)G$ is onto (2.9). Now $\text{Im}((f)G) \subseteq \text{Ker}((g)G)$ since $fg = 0$; conversely, let $(y)(g)G = (0 + C_k)$, $(y)q_j = y_j = (b_j + B_j)$, consider $r_i: (A)G \to A/A_i$ and the monomorphism $t_j: A/A_i \to B/B_j$ derived from f where $A_i = B_j f^{-1}$, let $s_j = r_i t_j$, let $V_j = (y_j)s_j^{-1}$, let $B_m = B_j g g^{-1} = B_j + N$ where $N = \text{Ker}(g) = \text{Im}(f)$; B/B_m is a discrete finite length R-module since $B/B_j \to B/B_m$ is onto (1.4), $\therefore$ $B/B_m = (B)g/(B_m)g = C/(B_m)g$ and since $(b_m)g + (B_m)g = 0 + (B_m)g$ $\therefore$ $b_m \in B_m$; since $B_j \subseteq B_m$, $b_j + B_m = b_m + B_m = 0 + B_m$, $\therefore$ $b_j \in B_m$, $\therefore$ $b_j = s + (a)f$, where $s \in B_j$ and $(a)f \in N$, $b_j + B_j = (a)f + B_j$, $\therefore$ $(a + A_i) \in V_j$; thus $\{V_j\}$ are nonempty closed cosets, they have the finite intersection property as in (2.9), and there exists $x \in (A)G$ such that $(x)(f)G = y$.

3.8. Remark 1. (3.7) is also the consequence of the fact that G is left adjoint to F (2.5.1) and thus right exact. It preserves all colimits [4].

3.9. Remark 2. If B_R is full then B_R is abelian for then every monomorphism is the kernel of a morphism and every epimorphism is the cokernel of a morphism. i.e., by (2.10.2) if c_B is continuous $\forall B \in B_R$, B_R is abelian.

IV. Exact inverse limits and cogenerators in B_R.

4.1. Lemma. *If U_i is closed in B_i, then $\prod U_i$ is closed in $\prod B_i$.*

Proof. $\prod U_i = \bigcap S_i$ where $S_i = B_1 \times \cdots \times B_{i-1} \times U_i \times B_{i+1} \times \cdots$ is closed $\forall_i$.

4.2. Lemma. *Let $\{B_i\}$ be a family of discrete finite length modules. Then $\prod B_i \in B_R$.*

Proof. Left to reader.

4.3. Theorem. *B_R is closed under inverse limits.*

Proof. Let $\{B_i\}$ be an inverse system of profinite modules, $B_i = \varprojlim B(i, j_i)$. Now $\prod B_i = \prod \varprojlim_{j_i} B(i, j_i) \subseteq \prod_i \prod_{j_i} B(i, j_i) = P$, $P \in B_R$ by 4.2. Also since B_i is a closed submodule of $\prod B(i, j_{j_i})$, $\therefore$ $\prod B_i$ is a closed submodule of P by 4.1.

$$\therefore \prod B_i \in B_R \text{ by } 3.1$$

$\therefore$ $\varprojlim B_i$, being a closed submodule of $\prod B_i$, belongs to B_R (3.1).

4.4. Theorem. $\varinjlim$ *is an exact functor: $T_R \to B_R$ where B_R is the category having for objects inverse systems of objects of B_R and for morphisms inverse systems of morphisms of B_R.*

Proof. Since $\varprojlim$ is left exact on C_R, it is left exact on B_R. Given $B_i \to C_i \to 0$ exact in $B_R \forall_i$, $v_i : B_i \to C_i$, let $v : B \to C \in B_R$ be the morphism induced by the v_i's. We have to show that v is onto. Let $K = \ker(v_i)$, $t_i : \varprojlim B_i/K_i \to B_i/K_i$, $(\varprojlim B_i/K_i)t_i = E_i/K_i$, where $E_i \subseteq B_i$. One shows $\varprojlim B_i/K_i \cong \varprojlim E_i/K_i$ (2.4), $\{E_i\} \in T_R$, $\varprojlim E_i \subseteq \varprojlim B_i$. It is thus sufficient to show that the restricted morphism $u : \varprojlim E_i \to \varprojlim E_i/K_i$ is onto, let $q_i : \varprojlim E_i \to E_i$, $p_i : \prod E_i \to E_i$, $m_i : E_i \to E_i/K_i$, $p_i m_i$ is onto, one shows $q_i m_i$ is onto and thus u is onto (2.9).

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4.5. Proposition. *B_R has a family of cogenerators $\{U_i\}$.*

Proof. Let A/A_j be a discrete finite length R-module. Let $A=X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n=A_j$ be a composition series with discrete simple composition factors (1.5). $X_i/X_{i+1} \cong R/M_{i+1}$, where M_{i+1} is a maximal open right ideal of R (1.1). Thus $X_i/X_{i+1} \cong \bar{a}_{i+1}R$ where $\bar{a}_{i+1}=a_{i+1}+X_{i+1}$, $a_{i+1} \notin X_{i+1}$. Let $x \in A$,

$$x+X_1 = a_1r_1+X_1, \quad x = a_1r_1+x_1, \quad x_1 \in X_1; \quad x_1+X_2 = a_2r_2+X_2,$$

$$x_1 = a_2r_2+x_2, \quad x_2 \in X_2; \ldots ; \quad x = a_1r_1+a_2r_2+\cdots+a_nr_n+s_n,$$

$s_n \in A_j$; $x+A_j=a_1r_1+\cdots+a_nr_n+A_j$. Thus the mapping $f:R^n \to A/A_j$ defined by $(r_1, \ldots, r_n)f=a_1r_1+\cdots+a_nr_n+A_j$ is onto. Let $\operatorname{Ker} f=N_j$ and $R^n/N_j \cong A/A_j$, when we give R^n/N_j the discrete topology, it is a finite length discrete R-module. Let $U(n,j)=R^n/N_j$ where N_j is any right ideal of R^n such that R^n/N_j is a discrete finite length R-module and n a positive integer. The $\{U(n,j)\}$ forms a set of cogenerators of B_R since $\overleftarrow{\operatorname{Lim}}\, A/A_j$ is a closed submodule of $\prod (A/A_j)$ which is topologically isomorphic to $\prod U(n,j)$.

V. Free and projective objects of B_R.

5.1. Definition. Let $F:A \to Ens$ be a functor, where *Ens* is the category of sets. If F has a left adjoint $G:Ens \to A$ then an object $A \in A$ is *free* if $A=(S)G$ for $S \in Ens$.

5.2. Example. Let $F':C_R \to Ens$ be the "forgetful" functor that assigns to each module its underlying set. Then F' has a left adjoint G' where $(S)G'=\oplus_{sGs}R_s$ where $R_s=R\ \forall s \in S$.

5.3. Proposition. *Let $G:C_R \to B_R$ be defined as in* (2.6). *Then the free objects of B_R are of the form $(\oplus R_s)G=\sum (R_s)G$ where $\sum$ denotes direct sums in B_R.*

Proof. Since $B_R((A)G, B)=C_R(A, (B)F)$ and $C_R((C)G', D)=Ens(C, (D)F')$, $\therefore$ $Ens(S, (B)FF')=C_R((S)G', (B)F)=B_R((S)G'G, B)$. Now $(\oplus R_s)G=\sum (R_s)G$ since G is a coreflector.

5.4. Proposition. *Let P be a projective object of C_R, then $(P)G$ is a projective object in B_R.*

Proof. Let $A \to B \to 0$ be exact in B_R, $f:A \to B$. Now $A \to B \to 0$ is exact in C_R, (3.6). Let $g:(P)G \to B$ in B_R be given, $c_P:P \to (P)G$ be the coreflection map, thus $c_Pg:P \to B$, thus there exists $h:P \to A$ such that $c_Pg=hf$; also there exists $k=(h)G:(P)G \to A \in B_R$ such that $c_Pk=h$; thus $c_Pg=c_Pkf$, thus g and kf agree on the dense subset $(P)c_P$ of $(P)G$; thus $g=kf$ [1, p. 85, Corollary 1 to Proposition 2].

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5.5. PROPOSITION. *B_R has enough projectives.*

Proof. Let $A \in B_R$. Since C_R has enough projectives, there exists $P \in C_R$, P projective such that $P \to A \to 0$ is exact in C_R; one shows that the corresponding $(P)G \to A$ is also onto. $\therefore$ $(P)G \to A \to 0$ is exact in B_R, $(P)G$ projective (5.4).

5.6. PROPOSITION *Every free object of B_R is projective.*

Proof. Let $D \in B_R$ be free, $D = (\oplus R)G$; now R is projective in C_R, $\therefore$ $\oplus R$ is projective in C_R, thus $(\oplus R)G = D$ is projective in B_R (5.4.).

5.7.1. DEFINITION. (5) Let $\mathscr{A}$ be any category, $c: A \to B \in \mathscr{A}$; if there exists c': $B \to A$ such that $c'c = 1_B$, then B is called a *coretract* of A.

5.7.2. PROPOSITION. *In B_R every projective object is a coretract of a free object.*

Proof. Let P be a projective object of B_R; there exists $\oplus R$ such that $\oplus R \to P \to 0$ is exact in C_R, $\therefore$ $(\oplus R)G \to P \to 0$ is exact in B_R where $f: (\oplus R)G \to P$. Now $1_P: P \to P$, P projective, $\therefore$ there exists $g: P \to (\oplus R)G$ such that $gf = 1_P$.

5.8. PROPOSITION. *$(R)G$ is a generator of B_R.*

Proof. Let $i: C \to B \in B_R$ be a proper monomorphism, thus i is 1-1, and C is a closed submodule of B, $C \neq B$. Thus there exists $b \in B$ such that $b \notin C$. Let $f: R \to B \in C_R$ be defined by $(1_R)f = b$; thus $c_R(f)G = f$ where c_R is the coreflection map; $(R)G = \varprojlim R/N_i$, $(1_R + N_i)(f)G = (1_R)c_R(f)G = (1_R)f = b$; thus the morphism $(f)G$ cannot factor through C: for if there exists $g: (R)G \to C$, $gi = (f)G$, then $(1_R + N_i)gi = (1_R + N_i)(f)G = b$, but since $b \notin C$, $(1_R + N_i)gi \neq b$.

VI. B_R, an abelian subcategory (colocally finite).

6.1. LEMMA. *Let $f: A \to B \in B_R$ be a continuous R-isomorphism, then f is a topological isomorphism.*

Proof. We have to show that f is open, i.e., $\forall$ open submodule A' of A, $(A')f$ contains an open submodule B' of B. Consider the basis of the neighborhood system of 0 given by the open submodules $\{B_i\}$ of B (2.7), and the corresponding family $\{(A' + B_i f^{-1})/A'\}$. Since A/A' is a discrete finite length module (2.8), we have a minimal element $(A' + f^{-1}B_0)/A'$. Since $\varprojlim$ is exact (4.4), $\therefore$ by the dual of the equivalent conditions of [2, p. 337, Proposition 6], $(\bigcap B_i f^{-1}) + A' = \cap (A' + B_i f^{-1})$. Now $\bigcap B_i f^{-1} = 0$ since $\bigcap B_i = 0$ by properties of inverse limit topology; also $\bigcap (A' + B_i f^{-1}) = A' + B_0 f^{-1}$ since it is the minimal element. $\therefore$ $A' = A' + B_0 f^{-1}$, $B_0 f^{-1} \subseteq A'$ and $B_0 \subseteq (A')f$ [2, pp. 392–393].

6.2. LEMMA. *$\forall$ monomorphism $f: A \to B \in B_R$ is the kernel of some morphism in B_R.*

Proof. f is 1-1 (3.5). Consider the canonical epimorphism $g: B \to B/(A)f \in B_R$ (3.2), then $f = \ker g: \forall x: C \to B \in B_R \ni xg = 0$, $(C)x \subseteq (A)f$. Now $\bar{f}: A \to (A)f$ where $(x)f = (x)\bar{f}\, \forall x \in A$ is a continuous R-isomorphism, $\therefore$ it is open (6.1) $\therefore \bar{f}^{-1} \in B_R$ and $u = x\bar{f}^{-1}$ is a unique mapping $\ni uf = x$.

6.3. LEMMA. *$\forall$ epimorphism $f: A \to B \in B_R$ is the cokernel of some morphism in B_R.*

Proof. f is onto (3.6). Let $i: K \to A$ be $\ker f$, then $f = \operatorname{coker} i: A/K \in B_R$ and is topologically isomorphic to B (6.1) $\therefore$ $\forall x: A \to C \in B_R \ni ix = 0$, $(K)x = 0$, $\therefore$ x factors through $B \simeq A/K$ in B_R.

6.4. THEOREM. *B_R is an abelian subcategory of C_R and is colocally finite.*

Proof. B_R is abelian (6.2), (6.3), (3.3), (3.4). $F: B_R \to C_R$ is exact (3.7) $\therefore$ B_R is an abelian subcategory. Since B_R is abelian, has exact inverse limits (4.6) and has cogenerators of finite length (4.5) $\therefore$ B_R is colocally finite [2, p. 356].

REFERENCES

1. N. Bourbaki, *Topologie générale*, Chapitres 1 et 2, Hermann, Paris, 1965.
2. P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France, **90** (1962), 323–448.
3. A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. **9** (1957), 119–221.
4. J. Lambek, *Completion of categories*, Springer lecture notes in Mathematics no. 24, 1966.
5. B. Mitchell, *Theory of categories*, Academic Press, New York, 1965.
6. D. Zelinsky, *Linearly compact modules and rings*, Amer. J. Math. **75** (1953), 79–90.

SIR GEORGE WILLIAMS UNIVERSITY,
MONTREAL, QUEBEC