Canad. Math. Bull. Vol. **32** (4), 1989

PROJECTIVE SOCLES

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ABSTRACT. Nicholson and Watters have recently investigated rings with projective socles and they have shown, among other things, that a ring R has a projective socle if and only if each matrix ring $M_n(R), n > 1$, has a projective socle. We generalize this result by showing that if S is an excellent extension of R, then the socle of R is projective if and only if the socle of S is projective. Examples of excellent extensions include, as well as matrix rings $M_n(R)$, skew group rings $R * G$ where G is a finite group and the order of G is invertible in R.

Nicholson and Watters [3] prove that R has a projective socle if and only if $M_n(R)$ has a projective socle by showing somewhat more. They prove that having a projective socle is a Morita invariant. In this paper we generalize the matrix ring result in a different direction by showing that if S is an excellent extension of R, then S has a projective socle if and only if R has a projective socle. The proof is elementary and avoids the Morita context machinery used in [3].

All rings considered in this paper are associative and have multiplicative identities, all modules are unital right modules. Let R and S be rings with the same identity, $R \subseteq S$. The ring S is an *excellent extension* of R if

(i) there is a finite set $\{1 = s_1, \ldots, s_n\} \subseteq S$ such that S is a free right and left R-module with basis $\{s_1, \ldots, s_n\}$ and $s_iR = Rs_i$ for all $i = 1, \ldots, n$, and

(ii) S is R-projective; that is, if N is an S-submodule of the S-module M and N is a direct summand of M as an R-module, then N is a direct summand as an S-module (note that the use of 'projective' in 'R-projective' differs from the usual homological use of 'projective'). See [4] for further information about excellent extensions.

The right socle of a ring A will be denoted by $\sum(A)$, it is the sum of all the minimal right ideals of A.

THEOREM. *If S is an excellent extension of R, then $\sum(S)$ is a projective S-module if and only if $\sum(R)$ is a projective R-module.*

PROOF. First assume that $\sum(S)$ is projective and that $x \in R$ is such that xR is a minimal right ideal. For each $i = 1, \ldots, n$, xRs_i is a simple R-module and so $xS = \bigoplus_{i=1}^n xRs_i$ is a completely reducible R-module. Thus every S-submodule of xS is a direct summand because S is R-projective. It follows that xS is a completely

Received by the editors June 20, 1988 and, in revised form, November 10, 1988.
AMS Subject Classification (1980): Primary 16A50, 16A56.

reducible S-module and so, since $\sum(S)$ is projective, xS is a direct summand of a free S-module F. Now, since $s_1 = 1$, xR is a direct summand of the R-module xS and since S is a free R-module, so is F. Thus xR is a projective R-module and so $\sum(R)$ is projective.

Now assume that $\sum(R)$ is projective and that $t \in S$ is such that tS is a minimal right ideal. It follows from [2, Theorem 4] that tS is completely reducible as an R-module so, since $\sum(R)$ is projective, there is a free R-module F and an R-submodule M of F such that $F = tS \oplus M$ (we have used the fact that simple submodules of free modules are isomorphic to minimal right ideals). Thus $F \otimes_R S = (tS \oplus M) \otimes_R S \cong (tS \otimes_R S) \oplus (M \otimes_R S)$. Since F is a direct sum of copies of R and $R \otimes_R S \cong S$, $F \otimes_R S$ is a free S-module. Moreover, viewing tS as an R-module we have

$$tS \otimes_R S = \bigoplus_{i=1}^{n} tS \otimes_R s_i$$

and so, since $s_1 = 1$, $tS \cong tS \otimes_R 1$ is a direct summand, as an R-module, of $F \otimes_R S$. Because S is R-projective, tS is also a direct summand of $F \otimes_R S$ as an S-module. Thus tS is a projective S-module and it follows that $\sum(S)$ is projective.

Let A be a ring graded by a finite group G. The smash product $A\#G^*$ is a free right and left A-module with basis $\{p_g : g \in G\}$ and multiplication determined by $(ap_g)(bp_h) = ab_{gh^{-1}}p_h$ where $g, h \in G, a, b \in A$ and $b_{gh^{-1}}$ is the gh^{-1} component of b.

Corollary. *If A is graded by a finite group G and $|G|^{-1} \in A$, then $\sum(A)$ is a projective A-module if and only if $\sum(A\#G^*)$ is a projective $A\#G^*$-module.*

Proof. From [1, Theorem 3.5] we know that G acts as automorphisms on $A\#G^*$ and that the skew group ring $(A\#G^*) * G$ is isomorphic to the matrix ring $M_n(A)$ where $n = |G|$. Since both skew group rings and matrix rings are excellent extensions, the corollary follows from two applications of the theorem.

Acknowledgments. The author appreciates the support of NSERC (grant #8789) and the hospitality of the University of Ottawa where this paper was written while he was there on sabbatical leave.

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