COMMUTATOR SUBGROUPS OF FINITE p-GROUPS

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To Bernhard Hermann Neumann on his 60th birthday

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1. Introduction

Within five minutes of the start of my first meeting with Bernhard Neumann as his research student, late in 1954, he suggested the following problem to me. Let G be a group in which the cardinals of the classes of conjugate elements are boundedly finite with maximum n, say. Then the commutator subgroup G' is finite [6]. Is the order $|G'|$ of G' bounded in terms of n? I distinctly recall these words of Neumann: "That should provide us with a start, I think". He was right: more than just a start, the problem has been a continuing stimulus to a study of questions in fields as far apart as permutation group theory ([7], [11], [12], [14], and some unpublished work of Peter M. Neumann) and multiplicator theory ([13], [2]), as well as attracting interest in its own right.

That the answer is in the affirmative was not hard to establish. In my M.Sc. thesis [10] I gave a very bad estimate, and there are some improvements in [11], [5] and [14]. In the first of these articles I formulated the conjecture that

$$(1.1) \qquad |G'| \leqq n^{\frac{1}{2}(1+\lambda(n))}$$

in all cases, with $\lambda(n)$ denoting the number of (not necessarily distinct) prime divisors of n. The resolution of this problem seems to be a task of quite non-trivial difficulty. The best that is known up to now is an unpublished result of Peter M. Neumann, who has proved that

$$|G'| \leqq n^{q(n)},$$

where q is a named function quadratic in log. Peter Neumann's calculations for finite p-groups are to appear in this journal [8], and they represent an important step forward. Perhaps even more satisfying is the result embodied in the Ph. D. thesis [1] of Iain M. Bride, which states that (1.1) is correct for nilpotent groups of class 2. The satisfaction comes from the fact that the only known groups for which the bound in (1.1) is actually attained are themselves nilpotent of class 2 – apart from cases where n is prime (see [11]). Thus one believed that groups of class 2 are likely to be the worst behaved of all groups (counting a tendency to defeat a long-cherished conjecture as bad behaviour!), whereas Bride's grooming calls them to order. Experience suggests that the further a group is from nilpotency, the smaller is its derived group relative to the size of its largest conjugacy class; equally, the harder this becomes to establish as a general truth.

Recently I have seen how a very simple application of the elegant work of Gaschütz, Neubüser and Ti Yen [2] on the multiplicator proves that our conjecture is true for a finite p-group provided that its generating number d is small in comparison with the size p^{β} of its largest conjugacy class. Details are in §3; the main content is that *for each d, only finitely many values of β can possibly defeat* (1.1).

2. Multiplicators of finite p-groups

We shall follow the notation of [2] for certain numerical invariants of a finite p-group P. The multiplicator $M(P)$ has order $p^{m(P)}$ and the commutator subgroup has order $p^{k(P)}$. The size of the largest conjugacy class of P is $p^{b(P)}$, so that for finite p-groups, conjecture (1.1) becomes

$$k(P) \leqq \tfrac{1}{2}b(P)(b(P)+1). \tag{2.1}$$

Peter Neumann proves in [8] that

$$k(P) \leqq b(P)^2.$$

The number $b(P)$ is called the *breadth* of P (see [3], [8]) and has important connections with the nilpotency class of P ([3], [4]). Finally, $d(P)$ is the generating number of P.

Our considerations are based on the following three results:

2.2 (Schur [9]). *Let A be a central subgroup of the finite group G. Then $G' \cap A$ is isomorphic with a subgroup of $M(G/A)$.*

2.3 (Ibid.) *Let P be an abelian p-group of type*

$$(p^{\alpha_1}, p^{\alpha_2}, \cdots, p^{\alpha_r}) \text{ with } \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_r.$$

Then

$$m(A) = \alpha_{r-1}+2\alpha_{r-2}+\cdots+(r-1)\alpha_1.$$

2.4 (Gaschütz, Neubüser and Ti Yen [2]). *For any finite p-group P with centre Z,*

$$m(P) \leqq m(P/P')+k(P)(d(P/Z)-1).$$

From these last two results it is but a small step to:

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LEMMA 2.5. *Let P be a d-generator group of order p^s.* Then

$$m(P) \leqq \tfrac{1}{2}(d-1)(2s-d).$$

PROOF. Set $k(P) = k$ and suppose that P/P' is of type

$$(p^{\alpha_1}, p^{\alpha_2}, \cdots, p^{\alpha_d}) \text{ with } 0 < \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_d.$$

Then by 2.3 and 2.4,

$$m(P) \leqq (d-1)\alpha_1 + (d-2)\alpha_2 + \cdots + \alpha_{d-1} + k(d-1).$$

Since $\alpha_1+\alpha_2+\cdots+\alpha_d+k = s$ and each α_i is positive, it follows that

$$\begin{aligned} m(P) &\leqq (d-1)s - \alpha_2 - 2\alpha_3 - \cdots - (d-2)\alpha_{d-1} - (d-1)\alpha_d \\ &\leqq (d-1)s - (1+2+\cdots+d-1) \\ &= \tfrac{1}{2}(d-1)(2s-d), \end{aligned}$$

as required.

The bound given by the lemma is attained whenever $s = d(P)$, that is, when P is elementary; but I have no idea how far it is from the truth when $d(P)$ is less than s. Clearly, the bound can be attained only if P/P' is elementary abelian. One outcome of 2.5 is that the multiplicator of a two-generator group of order p^s has order strictly less than p^s. I have not met this result anywhere, and it seems worth noting.

{To digress for a moment, it is of course the case that the multiplicator of a two-generator p-group P may need many generators, even though it is smaller than P. For instance, let G be a two-generator p-group such that the last non-trivial term A of its lower central series needs many generators, l say. By 2.2, A is isomorphic with a subgroup of $M(G/A)$, and so this group needs at least l generators. A suitable G would be the free group of rank 2 of a variety of metabelian nilpotent groups of high nilpotency class and high p-power exponent.}

3. Commutator subgroups

Let P be a finite d-generator p-group with centre Z, and set $b(P) = \beta$ for short. Evidently $|P/Z| \leqq p^{\beta d}$, so that

$$(3.1) \qquad |P'Z/Z| \leqq p^{\beta d-2}.$$

A careful application of Lemma 2.5 now shows that

$$m(P/Z) \leqq \tfrac{1}{2}(d-1)\ (2\beta d-d),$$

so that, from 2.2,

$$(3.2) \qquad |P' \cap Z| \leqq p^{\frac{1}{2}(d-1)(2d\beta-d)}.$$

Putting (3.1) and (3.2) together, we get that

$$k(P) \leqq \beta d-2+\tfrac{1}{2}(d-1)(2d\beta-d).$$

This means that P satisfies (1.1) provided that

$$2(\beta d-2)+(d-1)(2d\beta-d) \leqq \beta(\beta+1),$$

that is, whenever

$$\beta^2+\beta(1-2d^2)+d^2-d+4 \geqq 0.$$

It is not hard to see that this inequality holds provided that $\beta \geqq 2d^2-1$; for $d = 2$, $\beta \geqq 6$ will do it.

In the two-generator case, I have more or less checked, using *ad hoc* arguments, that the conjecture is correct for $\beta = 2, 3$. I prefer not to reproduce these arguments here, for they are tedious and would intrude a note of non-sinplicity in an otherwise very easy discussion. The conjecture was confirmed for $\beta = 1$ in [11]. The cases $\beta = 4, 5$ prove to be annoyingly more complicated, and I have failed to make much headway. The case $\beta = 5$ would probably yield to a sustained combinational attack; for instance one knows that any 2-generator counterexample P of breadth 5 is such that $|P/Z| = p^{10}$, $k(P/Z) = 8$, $8 \leqq m(P/Z) \leqq 9$. But this is by the way.

To sum up:

THEOREM 3.3 *Let P be a finite d-generator p-group. Then*

$$k(P) \leqq \tfrac{1}{2}b(P)(b(P)+1)$$

provided that $b(P) \geqq 2d^2-1$; for $d = 2$ this inequality is satisfied whenever $b(P) \geqq 6$.

Added in proof. Bride's theorem is to appear in this Journal in his paper "Second nilpotent BFC groups".

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