Can. J. Math., Vol. XXVII, No. 3, 1975, pp. 528-544

CRITERIA FOR GROUPS WITH REPRESENTATIONS OF THE SECOND KIND AND FOR SIMPLE PHASE GROUPS

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1. Introduction. In this paper we consider matrix representations of compact groups over the field of the complex numbers. We shall deal mainly with finite groups.

The Kronecker product of two irreducible representations σ_1 and σ_2 of a group $\mathcal{G}$ is in general a reducible representation of $\mathcal{G}$. The explicit reduction of such a product to irreducible representations σ_3 can be performed by means of a unitary matrix, the elements of which are called Wigner coefficients or Clebsch-Gordan coefficients [**1**; **25**; **27**]. These coefficients are functions of σ_1, σ_2 and σ_3 and of m_1, m_2 and m_3, which number the rows and columns of the representation matrices of σ_1, σ_2 and σ_3, respectively. Wigner coefficients play an important role in theoretical physics.

The set of Wigner coefficients thus defined is not uniquely determined however, so that there remains a certain freedom in their choice. This freedom can sometimes be used to impose some simple symmetry relations upon the various Wigner coefficients. An important kind of group in theoretical physics are the simply reducible groups. These groups are defined as groups every element of which is conjugate to its inverse (i.e. all classes are ambivalent) and furthermore with the property that the Kronecker product of two irreducible representations contains no irreducible representation more than once (multiplicity free). As has been shown by Wigner [**27**], the Wigner coefficients of such a group can be chosen in such a way that their absolute values are invariant under every permutation of the σ_i and the corresponding m_i ($i = 1, 2, 3$). The Wigner coefficients change only by a multiplicative phase factor under such a permutation. This phase factor depends only on the σ_i. To determine its value one has to know whether the irreducible representations are of the first or second kind (cf. [**11**]), or stated in the language of physicists whether the irreducible representations are integer or half-integer (cf. [**27**; **28**]). (Because of the property of ambivalence of simply reducible groups, the irreducible representations σ_i cannot be of the third kind.) This means that in the case of a finite group one has to know whether the value of the Schur-Frobenius in-

Received November 6, 1973. This research was supported in part by the National Science Foundation and the Army Research Office (Durham).

https://doi.org/10.4153/CJM-1975-064-4 Published online by Cambridge University Press

variant [**11**; **12**; **28**]

$$(1) \quad c(\chi) = \frac{1}{g} \sum_{R} \chi(R^2)$$

is $+1$ or -1 for the various irreducible representations. In equation (1) χ is the character of an irreducible representation σ of the group $\mathcal{G}$ and g is the order of $\mathcal{G}$. The summation runs over all elements $R \in \mathcal{G}$.

If one drops the property of ambivalence no essential new difficulties arise in the definition and in the symmetry relations of Wigner coefficients. However, if the multiplicity free condition is dropped a multiplicity index enters the Wigner coefficients. Then it is not always possible to choose the Wigner coefficients such that they have symmetry properties as mentioned above. In fact difficulties arise with the Wigner coefficients in which all the three σ_i stand for the same irreducible representation. This problem was studied elaborately by Derome and Sharp [**9**]. Derome has shown in [**10**] that it is possible to choose Wigner coefficients of the form

$$(2) \quad \begin{pmatrix} \sigma & \sigma & \sigma \\ m_1 & m_2 & m_3 \end{pmatrix}_r$$

(where r is a multiplicity index) in such a way that their absolute values are invariant under every permutation of the m_i if and only if the following relation holds

$$(3) \quad \frac{1}{g} \sum_{R} \chi(R^3) = \frac{1}{g} \sum_{R} \{\chi(R)\}^3,$$

(χ is the character of the irreducible representation σ). Irreducible representations for which equation (3) holds are called simple phase representations. Groups for which equation (3) holds for all irreducible representations are called simple phase groups (S.P. groups) [**6**; **7**; **16**; **23**–**25**].

We want to point out that knowledge of the Schur-Frobenius invariant $c(\chi)$ is also of algebraic interest, because it is related to the Schur index (cf. [**11**; **15**; **22**]). It is unknown to us whether the non-simple phase property or in particular the value of $\sum_R (1/g)\chi(R^3)$ can in any way be related to an algebraic property.

In this paper we shall discuss some criteria according to which it is possible to decide whether a finite group is not a simple phase group. It will turn out that we can formulate a number of criteria which give a decision on the existence of representations of the second kind of a group $\mathcal{G}$ and that we can also formulate a number of criteria for non-simple phase groups. Many of the criteria for the existence of representations of the second kind are analogous to criteria for non-simple phase groups. That there exists a certain analogy can be understood by the following consideration. A group $\mathcal{G}$ has no representation of the second kind if

$$(4) \quad \frac{1}{g} \sum_{R} \chi(R^2) \neq -1$$

for all characters χ [**11**; **12**; **28**]. Now one has

$$(5)\quad \frac{1}{g}\sum_R \{\chi(R)\}^2 = \frac{1}{g}\sum_R \chi^{(2)}(R) + \frac{1}{g}\sum_R \chi^{(1^2)}(R)$$

and

$$(6)\quad \frac{1}{g}\sum_R \chi(R^2) = \frac{1}{g}\sum_R \chi^{(2)}(R) - \frac{1}{g}\sum_R \chi^{(1^2)}(R),$$

where $\chi^{(2)}$ is the character of the symmetrized part of the Kronecker product of the representation σ with itself and $\chi^{(1^2)}$ is the character of the anti-symmetrized part (cf. [**24**; **25**]). Therefore equation (4) is equivalent with

$$(7)\quad \frac{1}{g}\sum_R \chi^{(1^2)}(R) = 0$$

or

$$(8)\quad \frac{1}{g}\sum_R \chi(R^2) = \frac{1}{g}\sum_R \{\chi(R)\}^2$$

for all characters χ of $\mathscr{G}$. Condition (8) is very similar to condition (3), which has to hold for all characters χ of $\mathscr{G}$ if the group is to be simple phase. From this similarity one can expect that criteria for non-simple phase groups and criteria for groups having no representations of the second kind will also be similar in some cases.

The criteria which we shall derive in the next sections will all be formulated for finite groups, although some of them can be generalized to compact groups. The classification of representations of connected compact simple Lie groups into representations of first, second and third kind has been discussed at various places in the literature (see [**2**; **3**; **7**; **19–21**]). Results on the simple phase property of representations of such groups are given by Butler and King [**7**]. In the same paper the authors give also results on the symmetric and alternating groups on n symbols.

2. Criteria for groups with representations of the second kind. In this section we shall derive some criteria for the occurrence of representations of the second kind in finite groups.

THEOREM 1. *Let $\mathscr{G}$ be a finite group. Let $\zeta^{(2)}(1)$ be the number of solutions of the equation $X^2 = 1$, where X must be an element of $\mathscr{G}$ and where 1 is the unit element of $\mathscr{G}$. Let furthermore $\sum'_\chi \chi(1)$ denote the sum of the degrees of the irreducible representations with real characters. The group $\mathscr{G}$ has at least one representation of the second kind if and only if the following inequality holds*

$$(9)\quad \zeta^{(2)}(1) < \sum_\chi{}' \chi(1).$$

https://doi.org/10.4153/CJM-1975-064-4 Published online by Cambridge University Press

Proof. From equations (5) and (6) we have

$$(10)\quad \frac{1}{g}\sum_R \chi(R^2) = \frac{1}{g}\sum_R \{\chi(R)\}^2 - \frac{2}{g}\sum_R \chi^{(1^2)}(R)$$
$$= s_2(\chi) - \frac{2}{g}\sum_R \chi^{(1^2)}(R),$$

where $s_2(\chi)$ denotes the number of times that the trivial representation $1_{\mathscr{G}}$ is contained in the Kronecker square $\sigma \otimes \sigma$ (χ is the character of the irreducible representation σ). Multiplying by $\chi(1)$ and summing over χ provides us with

$$(11)\quad \frac{1}{g}\sum_\chi \sum_R \chi(1)\chi(R^2) = \sum_\chi \chi(1)s_2(\chi) - \frac{2}{g}\sum_\chi \sum_R \chi(1)\chi^{(1^2)}(R).$$

The left hand side of equation (11) equals $\zeta^{(2)}(1)$ (cf. [**11**; **12**; **24**]). The first term in the right hand side of equation (11) equals $\sum'_\chi \chi(1)$, because $s_2(\chi) = 1$ if χ is a real character whereas $s_2(\chi) = 0$ if χ is a non-real character. The second term in the right hand side of equation (11) equals twice the sum of the degrees of the representations of the second kind of $\mathscr{G}$, which is a non-negative integer.

We remark that Theorem 1 is a statement about whether a finite group $\mathscr{G}$ has one or more irreducible representations which satisfy

$$\frac{1}{g}\sum_R \chi(R^2) \neq \frac{1}{g}\sum_R \{\chi(R)\}^2.$$

If one can decide with Theorem 1 that this inequality must hold for some χ, then it is still unknown in general for which character this inequality holds. For Theorem 1 one has to know which characters are real and the degrees of these characters. However, if one wants to find out for which characters χ the inequality holds, one has to calculate the Schur-Frobenius invariant $c(\chi) = \sum_R (1/g)\chi(R^2)$ for the characters χ, which may be much more difficult. (In incidental cases it may be possible to determine whether a certain representation is of the second kind by other means than a calculation of $c(\chi)$. Cf. the example of the group $U_3(3)$ in section 4.) In this sense the property for which Theorem 1 gives a necessary and sufficient condition is a "global" property of the group $\mathscr{G}$. In concrete cases it still often happens that Theorem 1 cannot be applied, because one does not always know sufficient details of the character table of the group considered. Therefore we shall derive some other criteria in which the conditions are formulated directly in terms of the group structure.

Before we come to this we want to point out that if the degree $\chi(1)$ of an irreducible representation σ is odd then this representation is not of the second kind (cf. [**11**; **12**]).

THEOREM 2. *Let $\mathscr{G}$ be a finite group and let $\zeta^{(2)}(S)$ be the number of solutions in the group $\mathscr{G}$ of the equation $X^2 = S$, where S is a fixed element of $\mathscr{G}$. The group $\mathscr{G}$ has at least one representation of the second kind if there exists an element R with*

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the property that

(12) $\zeta^{(2)}(R) > \zeta^{(2)}(1)$.

Proof. Assume that condition (12) holds and that the group $\mathscr{G}$ has no representation of the second kind. From this it follows that in the equation

(13) $\zeta^{(2)}(R) = \sum_{\chi} c(\chi)\chi(R)$

(cf. [**11**; **12**]), the numbers $c(\chi)$ are 1 or 0. Hence,

$$\zeta^{(2)}(R) = \left| \sum_{\chi} c(\chi)\chi(R) \right| \leqq \sum_{\chi} c(\chi)|\chi(R)|$$
$$\leqq \sum_{\chi} c(\chi)\chi(1) = \zeta^{(2)}(1),$$

which contradicts condition (12).

Theorem 2 is a sufficient criterion for a finite group $\mathscr{G}$ to have representations of the second kind. In section 5 it will be shown that the criterion is not necessary by giving a counter-example. We shall now indicate a kind of group for which the criterion is necessary as well.

THEOREM 3. *Let $\mathscr{G}$ be a finite group and let $\zeta^{(2)}(S)$ be the number of solutions in the group $\mathscr{G}$ of the equation $X^2 = S$, where S is a fixed element of $\mathscr{G}$. Let furthermore the representations of the first and third kind generate a subring in the ring of characters of $\mathscr{G}$. If $\mathscr{G}$ has at least one representation of the second kind then there exists an element R with the property*

$$\zeta^{(2)}(R) > \zeta^{(2)}(1).$$

Proof. The ring of characters of $\mathscr{G}$ (over the rational integers) is defined in [**15**, S. 586]. From the conditions of the theorem it follows by applying the Corollary on p. 299 of [**4**] that $\mathscr{G}$ has an invariant subgroup $\mathscr{H}$ not equal to the trivial subgroup consisting of the unit element only, with the property that all elements of $\mathscr{H}$ are represented by the unit matrix in the representations of the first and third kind, whereas this is not the case for representations of the second kind. Hence there exists an element $R \in \mathscr{H}$, $R \neq 1$ for which

(14) $\chi(R) = \chi(1)$,

if χ is of the first or third kind, and

(15) $\chi(R) \neq \chi(1)$,

if χ is of the second kind. Because the characters appearing in equation (15) are real we can write instead of equation (15)

(16) $\chi(R) < \chi(1)$,

if χ is of the second kind. (Remember that one always has $|\chi(R)| \leqq \chi(1)$.)

For the number $\zeta^{(2)}(R)$ we now have

$$(17)\quad \zeta^{(2)}(R) = \sum_{\chi} c(\chi)\chi(R) = \sum_{\chi}{}'' \chi(R) - \sum_{\chi}{}''' \chi(R),$$

where $\sum''_{\chi}$ and $\sum'''_{\chi}$ denote summations over all representations of the first kind and the second kind respectively. From equations (14), (16) and (17) it follows that for an element $R \in \mathscr{H}$, $R \neq 1$ we have

$$\zeta^{(2)}(R) > \sum_{\chi}{}'' \chi(1) - \sum_{\chi}{}''' \chi(1) = \zeta^{(2)}(1).$$

COROLLARY 1. *Let $\mathscr{G}$ be a finite simply reducible group. The group $\mathscr{G}$ has at least one representation of the second kind if and only if $\mathscr{G}$ has an element R with the property that*

$$\zeta^{(2)}(R) > \zeta^{(2)}(1).$$

Proof. From the definition of simply reducible groups we know that $\mathscr{G}$ has no representation of the third kind. Furthermore it has been proved by Wigner in [**28**, Lemma 1] that the representations of the first kind generate a subring of the ring of characters of $\mathscr{G}$. The corollary now follows from Theorems 2 and 3.

Wigner (cf. [**28**, Theorem 2]) has given a necessary and sufficient condition for a group to be simply reducible. He proved that a finite group is simply reducible if and only if

$$(18)\quad \sum_{R} \{\zeta^{(2)}(R)\}^3 = \sum_{R} \left(\frac{g}{g_R}\right)^2.$$

Here g_R stands for the number of elements of the class to which R belongs. (This condition was also discussed by Mackey in [**18**].) Hence we can formulate:

COROLLARY 2. *Let $\mathscr{G}$ be a finite group for which one has*

$$\sum_{R} \{\zeta^{(2)}(R)\}^3 = \sum_{R} \left(\frac{g}{g_R}\right)^2.$$

This group has representations of the second kind if and only if $\mathscr{G}$ has an element R with the property that

$$\zeta^{(2)}(R) > \zeta^{(2)}(1).$$

There is still another kind of group to which Theorem 3 can be applied, vis. those groups which have only one non-linear character. For these groups we have the following corollary.

COROLLARY 3. *Let $\mathscr{G}$ be a finite group and let $\mathscr{G}$ have only one non-linear* *character. The irreducible representation corresponding to this non-linear character is of the second kind if and only if $\mathscr{G}$ has an element R with the property that*

$$\zeta^{(2)}(R) > \zeta^{(2)}(1).$$

Proof. Representations of degree 1 are either of the first kind or of the third kind. Because the linear characters always form a subring of the ring of characters we can apply Theorem 3 immediately.

From Theorem 2 we can derive another criterion, which is weaker.

THEOREM 4. *Let $\mathscr{G}$ be a finite group and let $\zeta^{(n)}(1)$ be the number of solutions in the group $\mathscr{G}$ of the equation $X^n = 1$, where 1 is the unit element of $\mathscr{G}$. The group $\mathscr{G}$ has at least one representation of the second kind if the following inequality holds*

$$(19) \quad \zeta^{(4)}(1) > \{\zeta^{(2)}(1)\}^2.$$

Proof. The equation $X^4 = 1$ has $\zeta^{(4)}(1)$ solutions in $\mathscr{G}$. The squares of these solutions all satisfy the equation $X^2 = 1$. However, these squares cannot all be different from each other, for $\zeta^{(4)}(1) > \{\zeta^{(2)}(1)\}^2 \geqq \zeta^{(2)}(1)$. There cannot be more than $\zeta^{(2)}(1)$ different squares. Because $\zeta^{(4)}(1) > \{\zeta^{(2)}(1)\}^2$, among the squares of the solutions of $X^4 = 1$ there must be an element R which is more often than $\zeta^{(2)}(1)$ times the square of a solution of $X^4 = 1$. Hence for this element we have $\zeta^{(2)}(R) > \zeta^{(2)}(1)$ and we can apply Theorem 2.

In practical cases one often knows the orders of the elements of a group. Therefore the following corollary, which is equivalent to the above theorem might be useful.

COROLLARY. *Let $\mathscr{G}$ be a finite group and let a_n be the number of elements of order n. The group $\mathscr{G}$ has at least one representation of the second kind if the following inequality holds*

$$(20) \quad a_4 > a_2(a_2 + 1).$$

Now we come to some theorems on representations of the second kind which are specific in the sense that they can be applied only to irreducible representations of degree 2.

THEOREM 5. *Let $\mathscr{G}$ be a finite group. Let $\mathscr{G}$ have an irreducible representation of degree 2. This representation is of the second kind if and only if the representation provided by the determinants of the representation of degree 2 is the trivial representation.*

Proof. Let $D(R)$ be the representation matrix of the element R in the irreducible representation of degree 2. Let λ_1 and λ_2 be the characteristic values of the matrix $D(R)$. Then we have

$$\chi(R^2) = \lambda_1^2 + \lambda_2^2 = (\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2 = \{\chi(R)\}^2 - 2 \det D(R).$$

Now suppose that $\det D(R) = 1$, for all $R \in \mathscr{G}$, then it follows

$$\frac{1}{g}\sum_R \chi(R^2) = \frac{1}{g}\sum_R \{\chi(R)\}^2 - 2.$$

Because $\sum_R (1/g)\chi(R^2) = \pm 1$ or 0 and $\sum_R (1/g)\{\chi(R)\}^2 = 0$ or 1 we must have here

$$\frac{1}{g}\sum_R \chi(R^2) = -1,$$

which means that the irreducible representation of degree 2 is of the second kind.

Conversely if the given representation is of the second kind $\det D(R)$ must be the trivial representation.

COROLLARY. *An irreducible representation of degree 2 of a perfect group is of the second kind.*

Proof. A perfect group is a group the commutator subgroup of which is equal to the group itself (cf. [**17**, p. 105]). Therefore such a group has only one linear character and hence Theorem 5 can be applied.

Another consequence of Theorem 5 is:

THEOREM 6. *Let $\mathscr{G}$ be a finite group. An irreducible representation of degree 2 with real character χ is a representation of the second kind if and only if for all elements R of $\mathscr{G}$ with $\chi(R) = 0$ one has $\chi(R^2) = -2$.*

Proof. Because the character χ is real, the two characteristic values of a matrix $D(R)$ of the corresponding representation are either $e^{i\phi}$ and $e^{-i\phi}$ or 1 and -1. In the former case $\det D(R) = 1$ and in the latter case $\det D(R) = -1$. Further we observe that as a consequence of a theorem of Burnside the group $\mathscr{G}$ always has at least one element S with $\chi(S) = 0$ (see [**4**, p. 319] and [**11**, p. 36]).

(i) Suppose that $\chi(R^2) = -2$ if $\chi(R) = 0$. If $\chi(R) \neq 0$ then the characteristic values of the matrix $D(R)$ are $e^{i\phi}$ and $e^{-i\phi}$ with $\phi \neq \pi/2 + k\cdot\pi$ ($k = 0, \pm 1, \pm 2, \ldots$). In this case we have that $\det D(R) = 1$. If $\chi(R) = 0$ then the characteristic values of $D(R)$ are 1 and -1 or i and $-i$. In the first case we should have $\chi(R^2) = 2$, which contradicts our assumption. Hence if $\chi(R) = 0$ the characteristic values of $D(R)$ are i and $-i$ and therefore $\det D(R) = 1$. From Theorem 5 it now follows that the irreducible representation of degree 2 with character χ is of the second kind.

(ii) Suppose the irreducible representation of degree 2 is of the second kind. From Theorem 5 we know that $\det D(R) = 1$, for all $R \in \mathscr{G}$. If $\chi(R) = 0$ then the characteristic values of $D(R)$ are 1 and -1 or i and $-i$. In the first case we should have $\det D(R) = -1$, which is excluded. Hence if $\chi(R) = 0$ the eigenvalues of $D(R)$ are i and $-i$ and $\chi(R^2) = -2$.

COROLLARY. *An irreducible representation of a finite group $\mathscr{G}$ of degree 2 with*

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real character χ is a representation of the first kind if $\mathcal{G}$ does not possess an element R with $\chi(R) = -2$.

In [**12**, p. 199] Frobenius and Schur gave without proof a theorem on the existence of representations of the second kind in a special type of group. We shall state here this theorem in a slightly different form and give a proof. The proof is such that it enables us to give a similar proof for an analogous theorem for non-simple phase groups.

THEOREM 7 [Frobenius and Schur]. *Let $\mathcal{G}$ be a finite group with one and only one element of order 2. If $\mathcal{G}$ is not the direct product of a cyclic group of order 2^n and a group of odd order, then $\mathcal{G}$ has at least one representation of the second kind.*†

Proof. We shall prove the theorem by showing that a group of order $g = 2^n m$ (m odd) with only one element Z of order 2 and which has no representation of the second kind is necessarily of the type $\mathcal{G} = C_{2^n} \otimes \overline{\mathcal{G}}$, where C_{2^n} is the cyclic group of order 2^n and $\overline{\mathcal{G}}$ is a group of odd order m. In the proof we shall make use of the principle of mathematical induction.

We note that $n \geqq 1$ for otherwise $\mathcal{G}$ could not have an element of order 2. Because Z is the only element of order 2 there can be no elements conjugate to Z. Hence Z lies in the centre of $\mathcal{G}$. Because we suppose that $\mathcal{G}$ has no representation of the second kind it follows from Theorem 2 that $\zeta^{(2)}(R) \leqq \zeta^{(2)}(1) = 2$, for all $R \in \mathcal{G}$. However, if $X_1^2 = R$ then also $(ZX_1)^2 = R$ from which it follows that $\zeta^{(2)}(R) = 0$ or 2. From the absence of representations of the second kind and from $\zeta^{(2)}(1) = \sum_\chi c(\chi)\chi(1) = 2$ we now have that besides the trivial representation $1_{\mathcal{G}}$ there is only one other irreducible representation with real character and this representation must be of degree 1. We shall denote the character of this representation by χ'. Because $\chi'(R)$ is real there exist $g/2$ elements of $\mathcal{G}$ with $\chi'(R) = 1$, whereas for the other $g/2$ elements one has $\chi'(R) = -1$ (remember that $\sum_R \chi'(R) = 0$). The $g/2$ elements of $\mathcal{G}$ with $\chi'(R) = 1$ form an invariant subgroup $\mathcal{G}_1$ of order $g_1 = 2^{n-1}m$ (see, e.g., [**4**, p. 92]). With the relation $\zeta^{(2)}(R) = \sum_\chi c(\chi)\chi(R)$ it is easy to see that the elements of $\mathcal{G}_1$ are just the squares of the elements of $\mathcal{G}$.

(i) Suppose $n = 1$: In this case $g_1 = m$ and therefore g_1 is odd. The element Z cannot lie in $\mathcal{G}_1$, from which it follows that the subgroups $C_2 = \{E, Z\}$ and $\overline{\mathcal{G}} = \mathcal{G}_1$ have only the unit element in common. We proved above that $\overline{\mathcal{G}}$ is an invariant subgroup. The subgroup C_2 is also an invariant subgroup, because Z is in the centre of $\mathcal{G}$. Furthermore it is not difficult to see that $C_2 \cup \overline{\mathcal{G}} = \mathcal{G}$. Hence, we can conclude

(21) $\mathcal{G} = C_2 \otimes \overline{\mathcal{G}}$

(cf. [**13**, Theorem 2.5.1]).

†Dr. L. C. A. van Leeuwen (Mathematical Institute, Rijksuniversiteit, Groningen) has given a proof (which is rather lengthy) that under the conditions of Theorem 7 there exists an element $R \in \mathcal{G}$ such that $\zeta^{(2)}(R) > \zeta^{(2)}(1)$. Therefore Theorem 7 is a special case of Theorem 2 (private communication). See also Nieuw Arch. Wisk. *22* (1974), p. 92.

(ii) Suppose now $n \geqq 2$ and let the theorem be proved for $n - 1$: In this case the order of $\mathcal{G}_1$ is even and hence $\mathcal{G}_1$ must have an element of order 2, i.e. $Z \in \mathcal{G}_1$. We take an arbitrary element $R \in \mathcal{G}_1$. We saw that R has two square roots in $\mathcal{G}$, which we shall call X_1 and ZX_1. If $X_1 \in \mathcal{G}_1$, then also $ZX_1 \in \mathcal{G}_1$. Therefore the elements of $\mathcal{G}_1$ have either 0 or 2 square roots in $\mathcal{G}_1$, which gives

$$(22) \quad \zeta_1^{(2)}(R) = \sum_{\chi_1} c_1(\chi_1)\chi_1(R) = 0 \text{ or } 2.$$

The indices 1 denote that all symbols are to be taken in relation to $\mathcal{G}_1$. Because $\zeta_1^{(2)}(R)$ is a linear combination of characters in which the trivial character occurs once it must satisfy $\sum_R^{(1)} (1/g_1)\zeta_1^{(2)}(R) = 1$, where $\sum^{(1)}$ means that the summation runs only over the elements of $\mathcal{G}_1$. Hence, $\zeta_1^{(2)}(R) = 2$ for $g_1/2$ elements of $\mathcal{G}_1$, whereas for the remaining $g_1/2$ elements of $\mathcal{G}_1$ we have $\zeta_1^{(2)}(R) = 0$. But this gives that

$$(23) \quad \frac{1}{g_1} \sum_R^{(1)} |\zeta_1^{(2)}(R)|^2 = \sum_{\chi_1} |c_1(\chi_1)|^2 = 2,$$

from which it follows that $\mathcal{G}_1$ has only two irreducible representations with real character. Now $\zeta_1^{(2)}(1) = 2$ and thus both of these representations are of degree one (one of them being the trivial representation). These representations of degree one have to be of the first kind, whereas all other irreducible representations of $\mathcal{G}_1$ are of the third kind.

We see that $\mathcal{G}_1$ just as $\mathcal{G}$ is a group with only one element of order 2, not having representations of the second kind. From our assumption it follows that

$$(24) \quad \mathcal{G}_1 = C_{2^{n-1}} \otimes \mathcal{G}.$$

Hence there must be an element $A_1 \in \mathcal{G}_1$ with

$$(25) \quad A_1^{2^{n-1}} = 1.$$

Because $A_1 \in \mathcal{G}_1$ we can find an element $A \in \mathcal{G}$ with $A^2 = A_1$ which gives

$$(26) \quad A^{2^n} = 1.$$

We shall show now that $C_{2^n} = \{A\}$ and $\mathcal{G}$ are invariant subgroups of $\mathcal{G}$.

For an arbitrary element $B \in \mathcal{G}$ we have

$$(27) \quad B^m = 1.$$

Let furthermore

$$(28) \quad BAB^{-1} = A',$$

which gives $BA^2B^{-1} = A'^2$. We also know that $BA^2B^{-1} = A^2$ ($A^2 \in C_{2^{n-1}}$ and $B \in \mathcal{G}$) and so we have either $A' = A$ or $A' = ZA$. This last case leads to a contradiction, for then $B^mAB^{-m} = Z^mA = ZA$ (m is odd), whereas from

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equation (27) it follows that $B^m A B^{-m} = A$. Instead of equation (28) we can write

(29) $BAB^{-1} = A,$

which means that every element of C_{2^n} commutes with every element of $\overline{\mathscr{G}}$. Completely analogous to part (i) of the proof we can conclude that

(30) $\mathscr{G} = C_{2^n} \otimes \overline{\mathscr{G}}.$

The theorem has been proved now by the principle of mathematical induction.

3. Criteria for non-simple phase groups. In this section we shall derive some criteria for the occurrence of finite non-simple phase groups. Most of the criteria are similar to the criteria for groups with representations of the second kind.

THEOREM 8. *Let $\mathscr{G}$ be a finite group. Let $\zeta^{(3)}(1)$ be the number of solutions of the equation $X^3 = 1$, where X must be an element of $\mathscr{G}$ and where 1 is the unit element of $\mathscr{G}$. Let furthermore $s_3(\chi)$ be the number of times that the trivial representation is contained in the Kronecker* 3rd-*power of the irreducible representation with character χ. The group $\mathscr{G}$ is a non-simple phase group if and only if the following inequality holds*

(31) $\zeta^{(3)}(1) < \sum_{\chi} \chi(1) s_3(\chi).$

Proof. In [**24**; **25**] it is shown that

(32) $$s_3(\chi) = \frac{1}{g} \sum_R \{\chi(R)\}^3$$
$$= \frac{1}{g} \sum_R \chi^{(3)}(R) + \frac{2}{g} \sum_R \chi^{(2,1)}(R) + \frac{1}{g} \sum_R \chi^{(1^3)}(R),$$

and

(33) $$\frac{1}{g} \sum_R \chi(R^3) = \frac{1}{g} \sum_R \chi^{(3)}(R) - \frac{1}{g} \sum_R \chi^{(2,1)}(R) + \frac{1}{g} \sum_R \chi^{(1^3)}(R).$$

Here, $\chi^{(3)}(R)$ is the character of the totally symmetric part of the Kronecker third power, $\chi^{(1^3)}(R)$ of the totally antisymmetric part and $\chi^{(2,1)}(R)$ of the part which is partially symmetric and partially antisymmetric (for a precise description see [**24**; **25**]).

From equations (32) and (33) we have

(34) $$\frac{1}{g} \sum_R \chi(R^3) = s_3(\chi) - \frac{3}{g} \sum_R \chi^{(2,1)}(R),$$

hence

$$(35)\quad \frac{1}{g}\sum_{\chi}\sum_{R}\chi(1)\chi(R^3) = \sum_{\chi}\chi(1)s_3(\chi) - \frac{3}{g}\sum_{\chi}\sum_{R}\chi(1)\chi^{(2,1)}(R),$$

from which equation (31) follows.

COROLLARY. *A finite group $\mathscr{G}$ the order of which is not divisible by 3 is a non-simple phase group if and only if it has an irreducible representation, which is not the trivial representation, with $s_3(\chi) \neq 0$.*

THEOREM 9. *Let $\mathscr{G}$ be a finite group and let $\zeta^{(3)}(S)$ be the number of solutions in the group $\mathscr{G}$ of the equation $X^3 = S$, where S is a fixed element of $\mathscr{G}$. The group $\mathscr{G}$ is a non-simple phase group if there exists an element R with the property that*

$$(36)\quad \zeta^{(3)}(R) > \zeta^{(3)}(1).$$

Proof. The proof is analogous to the proof of Theorem 2. One has to apply the formula

$$(37)\quad \zeta^{(3)}(R) = \sum_{\chi} s_3(\chi)\chi(R),$$

which holds for simple phase groups and can be derived easily from equation (35).

THEOREM 10. *Let $\mathscr{G}$ be a finite group and let $\zeta^{(n)}(1)$ be the number of solutions in the group $\mathscr{G}$ of the equation $X^n = 1$, where 1 is the unit element of $\mathscr{G}$. The group $\mathscr{G}$ is a non-simple phase group if the following inequality holds*

$$(38)\quad \zeta^{(9)}(1) > \{\zeta^{(3)}(1)\}^2.$$

The proof of Theorem 10 is analogous to the proof of Theorem 4.

COROLLARY. *Let $\mathscr{G}$ be a finite group and let a_n be the number of elements of order n. The group $\mathscr{G}$ is a non-simple phase group if the following inequality holds*

$$(39)\quad a_9 > a_3(a_3 + 1).$$

THEOREM 11. *Let $\mathscr{G}$ be a finite group with only two elements of order 3. Let these elements of order 3 lie in the centre of $\mathscr{G}$. If $\mathscr{G}$ is not the direct product of a cyclic group of order 3^n and a group the order of which is not divisible by 3, $\mathscr{G}$ is a non-simple phase group.*

Proof. We shall prove the theorem by showing that a simple phase group $\mathscr{G}$ of order $g = 3^n m$ (m not divisible by 3) with only two elements of order 3, both of which lie in the centre of $\mathscr{G}$, is necessarily of the type

$$(40)\quad \mathscr{G} = C_{3^n} \otimes \overline{\mathscr{G}}.$$

Because the proof is quite similar to the proof of Theorem 7 we shall only indicate the characteristic points: Let the elements of order 3 be Z and Z^2. If $\mathscr{G}$

https://doi.org/10.4153/CJM-1975-064-4 Published online by Cambridge University Press

is a simple phase group we have

$$\zeta^{(3)}(1) = \sum_{\chi} \chi(1)s_3(\chi) \qquad \text{(cf. equation (37))}.$$

Now $\zeta^{(3)}(1) = 3$ and therefore one of the following possibilities must occur:

(a) $\mathscr{G}$ has 3 representations of degree 1 with $s_3(\chi) = 1$, whereas for all other irreducible representations $s_3(\chi) = 0$;

(b) $\mathscr{G}$ has a representation of degree 2 with $s_3(\chi) = 1$, whereas for all other irreducible representations unequal to the trivial representation we have $s_3(\chi) = 0$.

Analogous to the proof of Theorem 7 it can be seen that $\zeta^{(3)}(R) = 0$ or 3 for all $R \in \mathscr{G}$ and more precisely $\zeta^{(3)}(R) = 0$ for $2g/3$ elements R and $\zeta^{(3)}(R) = 3$ for the remaining elements, because $\sum_R (1/g)\chi^{(3)}(R) = 1$. This means that in case (b) we would have $\chi(R) = -1$ for $2g/3$ elements and $\chi(R) = 2$ for the remaining elements of the irreducible representation of degree 2. However, this would contradict the orthonormality relation for a character of an irreducible representation, because this would give

$$(41) \quad \frac{1}{g}\sum_{R} |\chi(R)|^2 = 2.$$

Only case (a) remains. Of the three representations of degree 1 having $s_3(\chi) = 1$ one is the trivial representation. The characters of the two other representations of degree 1 will be denoted by χ' and χ''. Because in particular $s_3(\chi') = 1$ the character $\chi'(R)$ can only assume the values 1, ω and ω^2, where $\omega = \exp(2\pi i/3)$. Furthermore $\chi'(R) = \chi''(R)^*$, where * denotes complex conjugation. From the orthogonality relation $\sum_R (1/g)\chi'(R) = 0$ and the relation $1 + \omega + \omega^2 = 0$ it follows that $\chi'(R) = \chi''(R) = 1$ for $g/3$ elements of $\mathscr{G}$. These $g/3$ elements form an invariant subgroup $\mathscr{G}_1$ of the order $g_1 = 3^{n-1}m$. The subgroup $\mathscr{G}_1$ consists of the third powers of the elements of $\mathscr{G}$.

As in the proof of Theorem 7, we can show now by the principle of mathematical induction that

$$(42) \quad \mathscr{G} = C_{3^n} \otimes \overline{\mathscr{G}}.$$

Corollary. *Let $\mathscr{G}$ be a finite group of odd order. Let $\mathscr{G}$ have only two elements of order* 3. *If $\mathscr{G}$ is not the direct product of a cyclic group of order 3^n and a group, the order of which is not divisible by* 3 *then $\mathscr{G}$ is a non-simple phase group.*

Proof. As the order of $\mathscr{G}$ is odd, $\mathscr{G}$ cannot have classes with two elements (the number of elements in a class divides the order of the group). Therefore the two elements of order 3 form each a class of one element and hence lie in the centre of $\mathscr{G}$.

We have not been able to give theorems for non-simple phase groups which are analogous to the remaining theorems of the previous section. However, we shall now state another theorem for non-simple phase groups.

THEOREM 12. *Let $\mathscr{G}$ be a finite group the order of which is not divisible by* 3. *If there is a class other than the class consisting of the unit element, which admits the substitution $R \rightarrow R^{-2}$ then the group $\mathscr{G}$ is a non-simple phase group.*

This theorem was proved in [**25**].

We remark that the corresponding theorem, saying that a group of odd order has at least one representation of the second kind if there is a class other than the unit class which admits the substitution $R \rightarrow R^{-1}$, does not make sense. In a group of odd order the unit class is always the only class which admits this substitution.

We remarked in the previous section that the degree of a representation of the second kind is always even. We do not know an analogous property for representations to be simple phase. Concerning the degree of non-simple phase representation we can only say that this degree has to be at least equal to 3. This follows readily from equations (32) and (33). From these equations and equation (3) it is clear that an irreducible representation with character χ is non-simple phase if and only if the non-negative integer

$$(43) \quad \frac{1}{g} \sum_{R} \chi^{(2,1)}(R) > 0$$

(the left hand side of the inequality (43) is equal to the number of times that the trivial representation $1_{\mathscr{G}}$ is contained in that part of the Kronecker 3rd-power which is denoted by (2, 1) cf. [**24**; **25**]). From equation (32) it is seen that $s_3(\chi) \geqq 2$ if equation (43) holds. However, one can easily verify that for an irreducible representation of degree 2 one has $s_3(\chi) \leqq 1$.

4. Examples. In this section we shall give simple examples of groups which satisfy the criteria derived in sections 2 and 3.

We shall start by giving examples relevant to Theorems 2, 4 and 7, in which sufficient criteria for the existence of representations of the second kind are formulated. We shall choose the examples such that they show that the criteria are not equivalent to each other. (Note that Theorems 4 and 7 follow from Theorem 2.)

Table 1	Theorem 2	Theorem 4	Theorem 7
$Q \cong \langle 2, 2, 2 \rangle$	+	+	+
$\langle -2, 2, 3 \rangle$	+	−	+
$Q \otimes C_4$	+	+	−
$\langle 2, 2 \vert 4; 2 \rangle$	+	−	−
$U_3(3)$	−	−	−

For the definition of the groups $\langle 2, 2, 2 \rangle$, $\langle -2, 2, 3 \rangle$ and $\langle 2, 2|4; 2 \rangle$ see [**8**, Table 1] and for the group $U_3(3)$ see [**14**, p. 441]. The group $Q \otimes C_4$ is the direct product of the group $Q = \langle 2, 2, 2 \rangle$ and the cyclic group of order 4. All these groups have representations of the second kind. A plus sign in the table indicates that the existence of representations of the second kind is covered by the theorem which labels the column in which this sign appears, whereas a minus sign indicates that it is not covered by the corresponding theorem. The example of $U_3(3)$ shows that the criterion of Theorem 2 is not a necessary criterion. From the character table of this group (cf. [**14**]) and Theorem 1 it follows that $U_3(3)$ must have a representation of type 2, because $\zeta^{(2)}(1) = 64$, whereas $\sum'_\chi \chi(1) = 76$, hence the inequality (9) holds. Because the difference $\sum'_\chi \chi(1) - \zeta^{(2)}(1) = 12$ equals twice the sum of the degrees of the representations of the second kind, we see from the character table that the irreducible representation of degree 6 has to be of the second kind. This example illustrates a remark made at the beginning of section 2.

The dihedral group D_4 of order 8 (cf. [**8**]) which does not have a representation of the second kind and the quaternion group provide examples for Theorem 3 and its corollaries and Theorems 5 and 6 and the corollary of Theorem 6. An example to the corollary of Theorem 5 is the binary icosahedral group $\langle 2, 3, 5 \rangle$ (cf. [**8**, p. 69]).

In Table 2 we present examples relevant to Theorems 9, 10 and 11, in which sufficient criteria for non-simple phase groups are expressed. Once again the examples are chosen in such a way that they show the non-equivalence of the various criteria.

Table 2	Theorem 9	Theorem 10	Theorem 11
G_{63}	+	+	+
G_{189}	+	−	+
$G_{63} \otimes C_3$	+	+	−
$G_{63} \otimes C_3 \otimes C_3$	+	−	−
G_{27}	−	−	−

The group G_{63} is a group of order 63 and is defined by the relations

$$(44) \quad S^7 = T^9 = 1, \qquad T^{-1}ST = S^2$$

(cf. [**4**, p. 319; **22**, S. 179; **24**]). The group G_{189} is a group of order 189 and is defined by

$$(45) \quad S^7 = T^{27} = 1, \qquad T^{-1}ST = S^2.$$

Furthermore G_{27} is a group of order 27. A set of defining relations of this group is

$$(46) \quad S^9 = T^3 = 1, \qquad T^{-1}ST = S^4$$

(cf. [**4**, p. 145]). Finally C_3 is the cyclic group of order 3.

All the groups in Table 2 are non-simple phase groups. The plus and minus signs have a similar meaning as in Table 1.

From the conditions of the Theorems 9, 10 and 11 it follows that they can be applied only to groups the orders of which are divisible by 3. If 3 is prime to the order of a group one can attempt to apply Theorem 12. An example of a non-simple phase group which is covered by Theorem 12 is a group of order 20, which can be defined by

$$(47)\quad S^5 = T^4 = 1, \qquad T^{-1}ST = S^3$$

(cf. [**8**; **25**]).

Some more examples can be found in [**23**].

Acknowledgement. One of us (A.J.v.Z.) would like to thank the Netherlands America Commission for Educational Exchange for financial support.

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https://doi.org/10.4153/CJM-1975-064-4 Published online by Cambridge University Press