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A WEIGHTED HYPERPLANE MEAN ASSOCIATED WITH HARMONIC MAJORIZATION IN HALF-SPACES

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1. Introduction and main results

The purpose of this paper is to introduce a new kind of weighted hyperplane mean for subharmonic functions and to use this mean in giving results on the harmonic majorization of subharmonic functions of restricted growth in half-spaces.

An arbitrary point of the Euclidean space $\mathbf{R}^{n+1}$, where $n \geqq 1$, will be denoted by $M=(X,y)$ where $X=(x_1,\ldots,x_n)\in\mathbf{R}^n$ and $y\in\mathbf{R}$. We write

$$|X|=(x_1^2+\cdots+x_n^2)^{\frac{1}{2}}, \quad |M|=(|X|^2+y^2)^{\frac{1}{2}}$$

and, in the sense of Lebesgue, $dX=dx_1\ldots dx_n$. Throughout this paper a and b will be real numbers such that $0\leqq a<b$ and

$$D_a=\{M\in\mathbf{R}^{n+1}:y>a\}, \quad \Omega_{a,b}=\{M\in\mathbf{R}^{n+1}:a<y<b\}.$$

If f is a non-negative Lebesgue measurable function on $\mathbf{R}^n\times\{y\}$, where $y>a$, let

$$\Psi_a(f,y)=(y-a)^{-n-1}\int_{\mathbf{R}^n}\{1+|X|/(y-a)\}^{\frac{1}{2}(1-n)}e^{-\pi|X|/(y-a)}f(X,y)\,dX.$$

If f takes values of both signs, we write

$$\Psi_a(f,y)=\Psi_a(f^+,y)-\Psi_a(f^-,y),$$

provided at least one of the terms on the right-hand side is finite.

The weighted mean Ψ_a is related to the mean introduced by Brawn in his study of subharmonic functions in strips [4], and this paper depends upon his work. Our theorems, however, are more closely analogous to those of Kuran [9] on half-spherical means. Other hyperplane means which have been studied in relation to subharmonic functions in half-spaces are

$$\int_{\mathbf{R}^n} f(X,y)\,dX \tag{1}$$

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(see [1] and the papers cited there for a sample of the literature) and

$$\int_{\mathbf{R}^n} (1+|X|^2)^{-\frac{1}{2}(n+1)} f(X, y)\, dX \tag{2}$$

[9, 11, 12, 15]. An advantage of working with the mean Ψ_a is that $\Psi_a(f, \cdot)$ is finite on (a, ∞) for a large class of functions f, whereas the means (1) and (2) are finite for comparatively small classes of functions. However, in order to obtain interesting conclusions from hypotheses concerning the behaviour of $\Psi_a(s, \cdot)$ for a subharmonic function s in D_a, it is necessary to impose a general restriction on the growth of s^+. We shall say that a subharmonic function s in D_a belongs to the class $\mathscr{S}_a$ if for each $b>a$ and each positive number λ

$$\lim s^+(M)\, e^{-\lambda|M|} = 0 \tag{3}$$

as M tends to the Alexandroff point $\mathscr{A}$ (at infinity) from inside $\Omega_{a,b}$.

We denote the closure and boundary in $\mathbf{R}^{n+1}$ of a set E by $\bar{E}$ and ∂E.

Theorem 1. *Let s be a non-negative function in $\bar{D}_a$ such that $s \in \mathscr{S}_a$,*

$$s(N) = \limsup_{\substack{M \to N \\ M \in D_a}} s(M) < \infty \qquad (N \in \partial D_a), \tag{4}$$

and

$$\int_{\mathbf{R}^n} (1+|X|^2)^{-\frac{1}{2}(n+1)} s(X, a)\, dX < \infty. \tag{5}$$

Then $\Psi_a(s, y)$ is real-valued on (a, ∞) and tends to a limit $\psi_a(s)$ as $y \to \infty$ such that $0 \leqq \psi_a(s) \leqq \infty$.

This theorem is of the same type as [10], Theorem 2 and [12], Theorem 2, which deal with the limiting behaviour of half-spherical means and certain weighted hyperplane means, respectively.

Before giving our results on harmonic majorization, we need a brief discussion of Poisson integrals in strips and half-spaces. Let f and g be extended real-valued functions defined on $\mathbf{R}^n \times \{a\}$ and $\mathbf{R}^n \times \{b\}$, respectively, such that

$$\int_{\mathbf{R}^n} |f(X, a)|\, e^{-\pi|X|/(b-a)}\, dX < \infty \tag{6}$$

and

$$\int_{\mathbf{R}^n} |g(X, b)|\, e^{-\pi|X|/(b-a)}\, dX < \infty. \tag{7}$$

Then the Poisson integral in $\Omega_{a,b}$ of the function equal to f on $\mathbf{R}^n \times \{a\}$ and equal to 0 on $\mathbf{R}^n \times \{b\}$ exists and is harmonic in $\Omega_{a,b}$ (see [3], pp. 747, 748, 758). We denote this

Poisson integral by $I_{a,b,f}$. Similarly, the Poisson integral in $\Omega_{a,b}$ of the function equal to g on $\mathbf{R}^n \times \{b\}$ and equal to 0 on $\mathbf{R}^n \times \{a\}$ exists and is harmonic in $\Omega_{a,b}$. We denote this Poisson integral by $J_{a,b,g}$. If F is defined on $\partial\Omega_{a,b}$ and if $I_{a,b,F}$ and $J_{a,b,F}$ exist and are harmonic in $\Omega_{a,b}$, we write

$$H_{a,b,F} = I_{a,b,F} + J_{a,b,F}.$$

Further details of Poisson integrals in strips are given in Sections 2 and 3.

A necessary and sufficient condition for the Poisson integral of f in D_a to exist and to be harmonic in D_a is

$$\int_{\mathbf{R}^n} |f(X,a)|(1+|X|^2)^{-\frac{1}{2}(n+1)}\, dX < \infty \tag{8}$$

(compare [7], Theorem 6). We denote this half-space Poisson integral by $I_{a,\infty,f}$. We shall also need, more generally, half-space Poisson integrals of measures. If μ is a signed measure on $\mathbf{R}^n$ such that

$$\int_{\mathbf{R}^n} (1+|X|^2)^{-\frac{1}{2}(n+1)}\, d|\mu|(X) < \infty, \tag{9}$$

then the half-space Poisson integral of μ in D_a is given by

$$I_{a,\infty,\mu}(M) = (2/s_{n+1}) \int_{\mathbf{R}^n} (y-a)\{|X-Z|^2 + (y-a)^2\}^{-\frac{1}{2}(n+1)}\, d\mu(Z)$$

and is harmonic in D_a. Here s_{n+1} is the surface area of the unit sphere in $\mathbf{R}^{n+1}$.

Theorem 2. *Let s be a function in $\bar{D}_a$ such that $s \in \mathcal{S}_a$, (4) holds, and*

$$\int_{\mathbf{R}^n} (1+|X|^2)^{-\frac{1}{2}(n+1)} s^+(X,a)\, dX < \infty. \tag{10}$$

Then $\Psi_a(s,\cdot)$ and $\Psi_a(s^+,\cdot)$ are real-valued on (a,∞) and $\Psi_a(s^+,y)$ tends to a limit $\psi_a(s^+)$ as $y \to \infty$ such that $0 \leqq \psi_a(s^+) \leqq \infty$.

For s to have a positive harmonic majorant in D_a it is necessary and sufficient that $\psi_a(s^+) < \infty$.

If $\psi_a(s^+) < \infty$, then

(i) *$\Psi_a(s,y)$ tends to a finite limit $\psi_a(s)$ as $y \to \infty$,*

(ii) $\int_{\mathbf{R}^n} (1+|X|^2)^{-\frac{1}{2}(n+1)} |s(X,a)|\, dX < \infty,$

(iii) $\lim_{b\to\infty} I_{a,b,s}(M) = I_{a,\infty,s}(M) \qquad (M \in D_a),$

(iv) $\lim_{b\to\infty} J_{a,b,s}(M) = (c_n)^{-1} \psi_a(s)(y-a) \qquad (M \in D_a),$

where

$$c_n = \int_{\mathbf{R}^n} (1+|Z|)^{\frac{1}{2}(1-n)} e^{-\pi|Z|}\, dZ,$$

(v) *the function $h_{s,a}$, defined in D_a by writing*

$$h_{s,a}(M) = I_{a,\infty,s}(M) + (c_n)^{-1}\psi_s(s)(y-a), \tag{11}$$

is a harmonic majorant of s in D_a.

Corollary. *If $s \in \mathscr{S}_a$ and*

$$\limsup_{\substack{M \to N \\ M \in D_a}} s(M) \leqq 0 \qquad (N \in \partial D_a)$$

and $\psi_a(s^+) = 0$, then $s \leqq 0$, in D_a.

Under the hypotheses of Theorem 2 it is possible that the function $h_{s,a}$, defined by (11), is a harmonic majorant of s in D_a but is not the least such majorant. However, the following theorem gives sufficient conditions for $h_{s,a}$ to be the least harmonic majorant of s in D_a.

Theorem 3. *Suppose that $a > 0$ and that $s \in \mathscr{S}_0$. Then s has a positive harmonic majorant in D_a if and only if* (10) *holds and $\psi_a(s^+) < \infty$. Further, if these conditions are satisfied, then the least harmonic majorant of s in D_a is the function $h_{s,a}$ given by* (11).

The example $s(M) = -\sqrt{(y-a)}$ shows that the conditions in Theorem 3 are not necessary for its final conclusion; this function is subharmonic in D_a but has no subharmonic extension to D_0.

Finally, we consider $\psi_a(s)$ as a function of a.

Theorem 4. *Let s be defined in $\bar{D}_a$. If $s \in \mathscr{S}_a$ and s satisfies* (4) *and* (10) *and if $\psi_a(s^+) < \infty$, then $\psi_{\cdot}(s)$ is constant on $[a, \infty)$.*

A similar result for half-spherical means is given in [**12**], Theorem 1.

2. Preliminaries on Poisson integrals in strips

We recapitulate some of Brawn's results.

Let $\Phi: [0, \infty) \times (0, 2) \to \mathbf{R}$ be defined by

$$\Phi(0, y) = (2\pi)^{-\frac{1}{2}n} 2^{1-\frac{1}{2}n} \{\Gamma(\tfrac{1}{2}n)\}^{-1} \int_0^\infty t^{n-1} \sinh\{(1-y)t\}(\sinh t)^{-1}\, dt$$

$$\Phi(r, y) = (2\pi)^{-\frac{1}{2}n} \int_0^\infty t^{\frac{1}{2}n} r^{1-\frac{1}{2}n} J_{\frac{1}{2}n-1}(rt) \sinh\{(1-y)t\}(\sinh t)^{-1}\, dt \qquad (r > 0),$$

where $J_{\frac{1}{2}n-1}$ denotes the Bessel function of the first kind of order $\frac{1}{2}n-1$ ([**14**], p. 40). Then Φ is positive and continuous on $[0, \infty) \times (0, 1)$. If f and g are functions satisfying

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(6) and (7), then $I_{a,b,f}$ and $J_{a,b,g}$ are given by

$$I_{a,b,f}(M)=(b-a)^{-n}\int_{\mathbf{R}^n}\Phi(|X-Z|/(b-a),(y-a)/(b-a))f(Z,a)\,dZ$$

and

$$J_{a,b,g}(M)=(b-a)^{-n}\int_{\mathbf{R}^n}\Phi(|X-Z|/(b-a),(b-y)/(b-a))g(Z,b)\,dZ.$$

Lemma A. *If f is a function on $\mathbf{R}^n\times\{a\}$ satisfying* (6) *then $I_{a,b,f}$ is harmonic in $\Omega_{a,b}$ and*

$$\lim_{M\to N} I_{a,b,f}(M)=0 \qquad (N\in\mathbf{R}^n\times\{b\}).$$

If, further, f is continuous at a point P of $\mathbf{R}^n\times\{a\}$, then

$$\lim_{M\to P} I_{a,b,f}(M)=f(P).$$

If f has compact support, then

$$\lim_{M\to\mathscr{A}} I_{a,b,f}(M)=0.$$

The same results, with $\mathbf{R}^n\times\{a\}$ and $\mathbf{R}^n\times\{b\}$ interchanged, hold for $J_{a,b,g}$, where g is a function on $\mathbf{R}^n\times\{b\}$ satisfying (7).

The results for $I_{a,b,f}$ are contained in [3] (Theorem 1, Lemmas 1, 2) in the case where $a=0$ and $b=1$. For an indication of the modifications required to pass to the general case, see [3], p. 758. It is easy to see that the corresponding results hold for $J_{a,b,g}$.

Next, we give the results on harmonic majorization in strips that we shall need.

Lemma B. *If s is defined in $\bar{\Omega}_{a,b}$ and is subharmonic in $\Omega_{a,b}$ and satisfies*

$$\limsup_{\substack{M\to N\\ M\in\Omega_{a,b}}} s(M)=s(N)<\infty \qquad (N\in\partial\Omega_{a,b}),$$

$$\int_{\mathbf{R}^n}\{|s(Z,a)|+|s(Z,b)|\}\,e^{-\pi|Z|/(b-a)}\,dZ<\infty$$

and

$$\lim_{\substack{M\to\mathscr{A}\\ M\in\Omega_{a,b}}} s^+(M)\,e^{-\pi|X|/(b-a)}|X|^{\frac{1}{2}(n-1)}=0,$$

then $H_{a,b,s}$ is a harmonic majorant of s in $\Omega_{a,b}$.

Lemma C. *If $0\leqq\alpha<a<b<\beta$ and s is subharmonic in $\Omega_{\alpha,\beta}$ and has a positive harmonic majorant there, then the least harmonic majorant of s in $\Omega_{a,b}$ is $H_{a,b,s}$.*

In the case where $a=0$ and $b=1$ Lemma B is [3], Theorem 2, and in the case where $\alpha=0$ and $\beta=1$ Lemma C is [4], Theorem 2. The stated generalizations are easily obtained from the cited cases.

3. Further results on Poisson integrals in strips

We use A to denote a finite positive constant depending at most on n, not necessarily the same on any two occurrences.

Lemma 1. *If* $0\leqq a<c\leqq\frac{1}{2}(a+b)$ *and if* g *is a non-negative function on* $\mathbf{R}^n\times\{b\}$ *such that*

$$\int_{\mathbf{R}^n} g(X,b)\,e^{-\pi|X|/(b-a)}\,dX<\infty,$$

then

$$AJ_{a,b,g}(0,\ldots,0,c)\leqq(c-a)\Psi_a(g,b)\leqq AJ_{a,b,g}(0,\ldots,0,c).$$

We start by showing that

$$A\sin(\pi y)(1+r)^{\frac{1}{2}(1-n)}e^{-\pi r}\leqq\Phi(r,y)\leqq A\sin(\pi y)(1+r)^{\frac{1}{2}(1-n)}e^{-\pi r}\tag{12}$$

whenever $r\geqq 0$ and $\frac{1}{2}<y<1$. A similar but slightly less general result than (12) is given in [4], Lemma 1. Our proof of (12) for large r is modelled on the proof in [4]. We start from the equation

$$\Phi(r,y)=(2r)^{1-\frac{1}{2}n}\sum_{m=1}^{\infty}m^{\frac{1}{2}n}\sin(m\pi y)K_{\frac{1}{2}n-1}(m\pi r)\qquad(r>0,0<y<1),$$

where $K_{\frac{1}{2}n-1}$ denotes the Bessel function of the third kind of order $\frac{1}{2}n-1$ ([14], p. 78). For this equation, see [2], formula (22) and note that Φ is normalized in accordance with [4] and not [2]. Hence when $r\geqq 1$ and $0<y<1$

$$\begin{aligned}
&\left|\Phi(r,y)-(2r)^{1-\frac{1}{2}n}\sin(\pi y)K_{\frac{1}{2}n-1}(\pi r)\right|\\
&=(2r)^{1-\frac{1}{2}n}\left|\sum_{m=2}^{\infty}m^{\frac{1}{2}n}\sin(m\pi y)K_{\frac{1}{2}n-1}(m\pi r)\right|\\
&\leqq(2r)^{1-\frac{1}{2}n}\sin(\pi y)\sum_{m=2}^{\infty}m^{\frac{1}{2}n+1}K_{\frac{1}{2}n-1}(m\pi r)\\
&\leqq Ar^{\frac{1}{2}(1-n)}\sin(\pi y)\sum_{m=2}^{\infty}m^{\frac{1}{2}(n+1)}e^{-m\pi r}\\
&\leqq Ar^{\frac{1}{2}(1-n)}\sin(\pi y)\,e^{-2\pi r}\sum_{m=2}^{\infty}m^{\frac{1}{2}(n+1)}e^{-(m-2)}\\
&=Ar^{\frac{1}{2}(1-n)}\sin(\pi y)\,e^{-2\pi r}.
\end{aligned}\tag{13}$$

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The first of the above inequalities follows from the inequalities

$$K_{\frac{1}{2}n-1}(\xi)>0 \quad (\xi>0) \quad |\sin(m\pi y)| \leqq m\sin(\pi y) \qquad (0<y<1, m=1,2,\ldots),$$

and the second follows from the inequality

$$K_{\frac{1}{2}n-1}(\xi) \leqq A\xi^{-\frac{1}{2}}e^{-\xi} \qquad (\xi \geqq 1) \qquad ([\mathbf{14}], \text{ p. } 219).$$

Since as $r \to \infty$

$$K_{\frac{1}{2}n-1}(\pi r)=(2r)^{-\frac{1}{2}}e^{-\pi r}(1+o(1)) \qquad ([\mathbf{14}], \text{ p. } 219),$$

it follows from (13) that

$$\begin{aligned}\Phi(r,y)&=(2r)^{\frac{1}{2}(1-n)}\sin(\pi y)\,e^{-\pi r}(1+o(1))\\ &=2^{\frac{1}{2}(1-n)}\sin(\pi y)(1+r)^{\frac{1}{2}(1-n)}e^{-\pi r}(1+o(1)).\end{aligned}$$

Hence (12) holds when r is larger than some positive number $r_0=r_0(n)$ and $0<y<1$.

Now define a function h_1 in $\Omega_{0,2}$ by writing

$$h_1(X,y)=\Phi(|X|,y).$$

Then h_1 is harmonic in $\Omega_{0,2}$ and vanishes on $\mathbf{R}^n \times \{1\}$. (It is the Poisson kernel of $\Omega_{0,1}$ with pole at the origin, see [2]). Hence $|\partial h_1/\partial y| \leqq A$ in the set $\{(X,y): |X| \leqq r_0, \frac{1}{2} \leqq y \leqq 1\}$ and therefore if $0 \leqq r \leqq r_0$ and $\frac{1}{2} \leqq y < 1$, then by the mean value theorem, there exists $y' \in (y,1)$ such that

$$\begin{aligned}|\Phi(r,y)|&=|h_1(r,0,\ldots,0,y)-h_1(r,0,\ldots,0,1)|\\ &=(1-y)\left|\frac{\partial h}{\partial y}(r,0,\ldots,0,y')\right|\\ &\leqq A(1-y) \leqq A\sin(\pi y),\end{aligned}$$

and it now follows that the right-hand inequality in (12) holds whenever $r \geqq 0$ and $\frac{1}{2} \leqq y < 1$.

Next define h_2 in $\mathbf{R}^{n+1}$ by writing

$$h_2(X,y)=\cos(\pi x_1/4r_0)\ldots\cos(\pi x_n/4r_0)\sinh(\pi\sqrt{n}(1-y)/4r_0).$$

It is easy to check that h_2 is harmonic in $\mathbf{R}^{n+1}$. Further, if

$$\omega=\{(X,y): |x_i|<2r_0 (i=1,\ldots,n), \qquad \tfrac{1}{2}<y<1\},$$

then $h_1 > A$ and $h_2 \leqq \sinh(\pi\sqrt{n}/8r_0)$ on $\partial\omega \cap \partial D_{\frac{1}{2}}$ and $h_1 \geqq 0 = h_2$ on $\partial\omega \cap D_{\frac{1}{2}}$. Hence $h_1 \geqq Ah_2$ on $\partial\omega$, and it follows from the minimum principle that $h_1 \geqq Ah_2$ in $\bar{\omega}$. Hence if $0 \leqq r \leqq r_0$ and $\frac{1}{2} \leqq y \leqq 1$, then

$$\begin{aligned}\Phi(r, y) = h_1(r, 0, \ldots, 0, y) &\geqq Ah_2(r, 0, \ldots, 0, y)\\ &\geqq 2^{-\frac{1}{2}} \sinh(\pi\sqrt{n}(1-y)/4r_0)\\ &\geqq 2^{-5/2}(r_0)^{-1}\pi\sqrt{n}(1-y) \geqq A\sin(\pi y).\end{aligned}$$

It now follows that the left-hand inequality in (12) holds whenever $r \geqq 0$ and $\frac{1}{2} \leqq y \leqq 1$, and the proof of (12) is complete.

If a, b and c are as in the lemma, then $\frac{1}{2} \leqq (b-c)/(b-a) < 1$. Hence, by (12), for each $Z \in \mathbf{R}^n$

$$\Phi(|Z|/(b-a), (b-c)/(b-a)) \tag{14}$$

lies between positive multiples of

$$\sin\{\pi(b-c)/(b-a)\}\{1+|Z|/(b-a)\}^{\frac{1}{2}(1-n)} e^{-\pi|Z|/(b-a)}$$

(the implied constants depending only on n). Since, for such a, b and c,

$$2(c-a)/(b-a) < \sin\{\pi(b-c)/(b-a)\} < \pi(c-a)/(b-a),$$

it follows that (14) lies between positive multiples of

$$(c-a)(b-a)^{-1}\{1+|Z|/(b-a)\}^{\frac{1}{2}(1-n)} e^{-\pi|Z|/(b-a)}$$

(the implied constants again depending only on n). Hence the lemma follows.

We need some results on the Perron–Wiener–Brelot (PWB) solution of the Dirichlet problem (see, for example, [8] for a general account). If Ω is an unbounded domain in $\mathbf{R}^{n+1}$, we denote its compactified boundary $\partial\Omega \cup \{\mathscr{A}\}$ by $\partial^*\Omega$. A function F, defined at least on $\partial^*\Omega$, such that the PWB solution of the Dirichlet problem in Ω with boundary data F exists and is harmonic in Ω is called resolutive, and we denote the PWB solution by $H(\Omega, F)$.

Lemma 2. *Let f and g be functions on $\mathbf{R}^n \times \{a\}$ and $\mathbf{R}^n \times \{b\}$ respectively.*

(i) *Define F_1 on $\partial^*\Omega_{a,b}$ by writing*

$$F_1(M) = f(M)(M \in \mathbf{R}^n \times \{a\}), \quad F_1(M) = g(M)\,(M \in \mathbf{R}^n \times \{b\}), \qquad F_1(\mathscr{A}) = 0.$$

If f and g satisfy (6) *and* (7), *then F_1 is resolutive and $H(\Omega_{a,b}, F_1) = I_{a,b,f} + J_{a,b,g}$ in $\Omega_{a,b}$.*

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(ii) *Define F_2 on $\partial^* D_a$ by writing*

$$F_2(M) = f(M) \qquad (M \in \partial D_a), \quad F_2(\mathscr{A}) = 0.$$

Then F_2 is resolutive if and only if (8) *holds, and in this case $H(D_a, F_2) = I_{a,\infty,f}$.*

We prove only (i), the proof of (ii) being similar. If f and g are real-valued and continuous in their domains of definition and have compact supports, then $I_{a,b,f} + J_{a,b,g}$ is harmonic in $\Omega_{a,b}$ and by Lemma A,

$$\lim_{M \to N} \{I_{a,b,f}(M) + J_{a,b,g}(M)\} = F_1(N) \qquad (N \in \partial^* \Omega_{a,b}).$$

It follows that $I_{a,b,f} + J_{a,b,g}$ is the classical solution and hence the PWB solution of the Dirichlet problem in $\Omega_{a,b}$ with boundary data F_1. It follows from this special case that the harmonic measure on $\partial^* \Omega_{a,b}$ relative to a point (X, y) of $\Omega_{a,b}$ is given on $\mathbf{R}^n \times \{a\}$ by

$$(b-a)^{-n} \Phi(|X-Z|/(b-a), (y-a)/(b-a))\, dZ$$

and on $\mathbf{R}^n \times \{b\}$ by

$$(b-a)^{-n} \Phi(|X-Z|/(b-a), (b-y)/(b-a))\, dZ,$$

whence the general result follows.

4. Means of half-space Poisson integrals and potentials

Lemma 3. *Let μ be a signed measure on $\mathbf{R}^n$ such that* (9) *holds. Then $\Psi_a(I_{a,\infty,\mu}, y)$ is finite on (a, ∞) and tends to 0 as $y \to \infty$.*

We may suppose, without loss of generality, that $a = 0$. Then

$$\begin{aligned} \tfrac{1}{2} s_{n+1} &|\Psi_0(I_{0,\infty,\mu}, y)| \\ &= y^{-n-1} \left| \int_{\mathbf{R}^n} (1 + |X|/y)^{\frac{1}{2}(1-n)} e^{-\pi |X|/y} \int_{\mathbf{R}^n} y(y^2 + |X-Z|^2)^{-\frac{1}{2}(n+1)}\, d\mu(Z)\, dX \right| \\ &\leqq y^{-n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} (1 + |X|/y)^{\frac{1}{2}(1-n)} (y^2 + |X-Z|^2)^{-\frac{1}{2}(n+1)} e^{-\pi |X|/y}\, dX\, d|\mu|(Z) \\ &\leqq y^{-n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} (y^2 + |X-Z|^2)^{-\frac{1}{2}(n+1)} e^{-\pi |X|/y}\, dX\, d|\mu|(Z). \end{aligned} \tag{15}$$

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Now, for each $Z \in \mathbf{R}^n$, putting $X = yX'$ and $Z = yZ'$, we have

$$\int_{\mathbf{R}^n} (y^2 + |X - Z|^2)^{-\frac{1}{2}(n+1)} e^{-\pi|X|/y} \, dX$$

$$= y^{-1} \int_{\mathbf{R}^n} (1 + |X' - Z'|^2)^{-\frac{1}{2}(n+1)} e^{-\pi|X'|} \, dX'$$

$$\leqq Ay^{-1} \int_{\mathbf{R}^n} \{(1 + |X' - Z'|^2)(1 + |X'|^2)\}^{-\frac{1}{2}(n+1)} \, dX' \tag{16}$$

$$= Ay^{-1}(4 + |Z'|^2)^{-\frac{1}{2}(n+1)}$$

$$= Ay^n(y^2 + |Z|^2)^{-\frac{1}{2}(n+1)}. \tag{17}$$

To prove the last written equality, note that the integral in (16) is a constant positive multiple of the value at $(Z', 1)$ of the Poisson integral in D_0 of the function $(1 + |X'|^2)^{-\frac{1}{2}(n+1)}$ and use the reproductive property of the Poisson kernel. From (15) and (17) we obtain

$$|\Psi_0(I_{0,\infty,\mu}, y)| \leqq A \int_{\mathbf{R}^n} (y^2 + |Z|^2)^{-\frac{1}{2}(n+1)} \, d|\mu|(Z). \tag{18}$$

Since μ satisfies (9), the right-hand side of (18) is finite for each positive y and tends to 0 as $y \to \infty$, by Lebesgue's dominated convergence theorem.

Recall that a superharmonic function in a domain Ω is called a potential if its greatest harmonic minorant in Ω is identially zero.

Lemma 4. *If u is a potential in D_a, then $\Psi_a(u, y)$ is finite on (a, ∞) and tends to 0 as $y \to \infty$.*

Again it suffices to work with $a = 0$. In **[12]**, Theorem 3 it was shown that if u is a potential in D_0, then the function

$$K(u, y) = \int_{\mathbf{R}^n} \{|X|^2 + (y+1)^2\}^{-\frac{1}{2}(n+1)} u(X, y) \, dX$$

is real-valued for $y > 0$ and tends to 0 as $y \to \infty$. We use this result to prove Lemma 4. Suppose that $y_0 > 0$ and that $(X, y) \in D_{y_0}$. Then

$$|X|^2 + (y+1)^2 \leqq C(y + |X|)^2,$$

where C depends only on y_0. Hence

$$y^{-n-1}(1 + |X|/y)^{\frac{1}{2}(1-n)} e^{-\pi|X|/y} \{|X|^2 + (y+1)^2\}^{\frac{1}{2}(n+1)}$$

$$\leqq C^{\frac{1}{2}(n+1)}(1 + |X|/y)^{\frac{1}{2}(n+3)} e^{-\pi|X|/y}$$

which is bounded on D_{y_0}. It now follows that $\Psi_0(u, \cdot)$ is dominated by a constant multiple of $K(u, \cdot)$ and, in view of the properties of $K(u, \cdot)$, this proves the lemma.

Lemma 5. *If*

$$v(M)=y-a \qquad (M\in D_a),$$

then $\Psi_a(v,\cdot)\equiv c_n$ *on* (a,∞).

This is the result of a simple calculation which we omit.

5. Proof of Theorem 1

The following lemmas will be useful in the proofs of Theorems 1 and 2.

Lemma 6. *If* f *is a function on* ∂D_a *which satisfies* (8), *then for each* $M\in D_a$

$$\lim_{b\to\infty} I_{a,b,f}(M)=I_{a,\infty,f}(M). \tag{19}$$

If, further, $f\geqq 0$ *on* ∂D_a, *then for each* $b\in(a,\infty)$, *we have* $I_{a,b,f}\leqq I_{a,\infty,f}$ *in* $\Omega_{a,b}$.

Lemma 7. *If* s *satisfies the hypotheses of Theorem 1, then for each* $M=(X,y)\in D_a$, *we have that* $H_{a,b,s}(M)$ *is increasing (in the wide sense) as a function of* b *on* (y,∞).

The proof of Lemma 6 depends on the following result.

Lemma D. *Let* Ω_0 *and* Ω *be unbounded domains in* $\mathbf{R}^{n+1}$ *such that* $\Omega\subset\Omega_0$. *Let* F *be a function on* $\Omega_0\cup\partial^*\Omega_0$ *such that* F *is resolutive on* $\partial^*\Omega_0$ *and* $F=H(\Omega_0,F)$ *in* Ω_0. *Then* F *is resolutive on* $\partial^*\Omega$ *and* $F=H(\Omega,F)$ *in* Ω.

See [5], p. 98, for the corresponding result in bounded domains.

To prove Lemma 6, define F in $D_a\cup\partial^*D_a$ by putting

$$F(M)=I_{a,\infty,f}(M)\quad (M\in D_a),\quad F(M)=f(M)\quad (M\in\partial D_a),\quad F(\mathscr{A})=0.$$

Then, by Lemma 2(ii), F is resolutive on ∂^*D_a and $F=H(D_a,F)$ in D_a. Hence, by Lemma D, F is resolutive on $\partial^*\Omega_{a,b}$ and $F=H(\Omega_{a,b},F)$ in $\Omega_{a,b}$. By Lemma 2(i), we also have in $\Omega_{a,b}$

$$H(\Omega_{a,b},F)=H_{a,b,F}=I_{a,b,f}+J_{a,b,F}.$$

Hence

$$I_{a,\infty,f}=I_{a,b,f}+J_{a,b,F}$$

in $\Omega_{a,b}$. If $f\geqq 0$ on ∂D_a, then $F\geqq 0$ in D_a and $J_{a,b,F}\geqq 0$ in $\Omega_{a,b}$, so the inequality stated in the lemma now follows. To prove (19), it now suffices to show that $J_{a,b,F}(M)\to 0$ as $b\to\infty$ for each $M\in D_a$. Since F is a half-space Poisson integral in D_a, we have by

Lemma 3, $\Psi_a(F, b) \to 0$ as $b \to \infty$. From Lemma 1 it now follows, in the case where $f \geqq 0$ on ∂D_a that

$$\lim_{b \to \infty} J_{a,b,F}(0, \ldots, 0, y) = 0$$

for each $y > a$. In the case where f takes values of both signs, the same conclusion follows by working with f^+ and f^-. Since we may translate the origin parallel to the $x_1, \ldots, x_n$-axes, we find that $J_{a,b,F}(M) \to 0$ as $b \to \infty$ for each $M \in D_a$, as required.

To prove Lemma 7, suppose that $a < b < b'$ and define w in $\bar{D}_a$ to be equal to $H_{a,b,s}$ in $\Omega_{a,b}$ and equal to s elsewhere in $\bar{D}_a$. Then $w \geqq s$ in $\Omega_{a,b}$ ([**3**], Theorem 2, interpreted for $\Omega_{a,b}$) and w is subharmonic in D_a ([**4**], p. 280). It is easy to check that w satisfies the conditions of [**3**], Theorem 2, interpreted for $\Omega_{a,b'}$. Hence $H_{a,b',s} \geqq w = H_{a,b,s}$ in $\Omega_{a,b}$.

Lemma E. *If s is subharmonic in D_a and s has a positive harmonic majorant in D_a, then s is expressible in the form*

$$s(M) = I_{a,\infty,\mu}(M) + k(y - a) - u(M) \qquad (M \in D_a), \tag{20}$$

where μ is a signed measure on $\mathbf{R}^n$ satisfying (9), *k is a real number and u is a potential in D_a.*

This result is essentially known. It can be deduced from [**12**], Theorem 5(ii) and the Riesz decomposition theorem in the form given, for example, in [**12**], Theorem C.

We can now complete the proof of Theorem 1. Since $s \in \mathscr{S}_a$, it is clear that $\Psi_a(s, \cdot)$ is finite on (a, ∞). Since, by Lemma 7, $H_{a,b,s}$ is an increasing function of b in D_a, and since, by Lemma 6, $I_{a,b,s} \to I_{a,\infty,s}$ in D_a as $b \to \infty$, it follows that either $J_{a,b,s} \to \infty$ in D_a or $J_{a,b,s}$ tends to a harmonic limit in D_a as $b \to \infty$. In the former case, it follows from Lemma 1 that $\Psi_a(s, b) \to \infty$ as $b \to \infty$. In the latter case, s has a harmonic majorant in D_a, since, by Lemma B, $H_{a,b,s} \geqq s$ in $\Omega_{a,b}$ and since $\lim_{b \to \infty} H_{a,b,s}$ is harmonic in D_a. Hence, in this case, by Lemma E, s has the representation (20) in D_a, so that

$$\Psi_a(s, y) = \Psi_a(I_{a,\infty,\mu}, y) + k\Psi_a(y - a, y) - \Psi_a(u, y)$$

$$\to 0 + c_m k - 0 \qquad (y \to \infty),$$

by Lemmas 3, 4 and 5.

6. Proof of Theorem 2

Clearly, if s satisfies the hypotheses of Theorem 2, then s^+ satisfies the hypotheses of Theorem 1, so that $\Psi_a(s^+, y)$ is finite on (a, ∞) and tends to a limit $\psi_a(s^+)$ as $y \to \infty$ such that $0 \leqq \psi_a(s^+) \leqq \infty$.

For each positive integer m, define s_m in $\bar{\Omega}_{a,b}$ to be $\max\{-m, s\}$. Then each s_m satisfies the hypotheses of Lemma B, so that H_{a,b,s_m} is a harmonic majorant of s_m in $\Omega_{a,b}$. By monotone convergence, $H_{a,b,s_m} \to H_{a,b,s}$ in $\Omega_{a,b}$ as $m \to \infty$. Hence $H_{a,b,s}$ is a harmonic majorant of s in $\Omega_{a,b}$. In particular, this implies that

$$J_{a,b,s}(0,\ldots,0,\tfrac{1}{2}(a+b)) > -\infty,$$

so that

$$J_{a,b,s^-}(0,\ldots,0,\tfrac{1}{2}(a+b)) < \infty.$$

Hence, by Lemma 1, $\Psi_a(s^-, b) < \infty$, and since $\Psi_a(s^+, b) < \infty$, it now follows that $\Psi_a(s,\cdot)$ is finite on (a,∞).

Now suppose that s has a positive harmonic majorant in D_a. Then

$$\int_{\mathbf{R}^n} (|X|^2 + y^2)^{-\frac{1}{2}(n+1)} s^+(X,y)\, dX$$

is bounded on $(a+1,\infty)$ ([**9**], Theorem 4) and since $\Psi_a(s^+, y)$ is dominated by a positive multiple of this integral for $y > a+1$ (cf. proof of Lemma 4 above), we have $\Psi_a(s^+) < \infty$.

Conversely, suppose that $\psi_a(s^+) < \infty$. By Lemmas B and 7, H_{a,b,s^+} is a harmonic majorant of s^+ in $\Omega_{a,b}$ and increases with b. Hence it follows easily that as $b \to \infty$, either $H_{a,b,s^+} \to \infty$ in D_a or H_{a,b,s^+} tends to a limit function which is a harmonic majorant of s^+ in D_a. To show that s has a positive harmonic majorant in D_a, it now suffices to prove that for some $M \in D_a$

$$\lim_{b\to\infty} H_{a,b,s^+}(M) < \infty. \tag{21}$$

By Lemma 1, if $b \geqq a+2$, then

$$J_{a,b,s^+}(0,\ldots,0,a+1) \leqq A\Psi_a(s^+, b),$$

so that

$$\limsup_{b\to\infty} J_{a,b,s^+}(0,\ldots,0,a+1) < \infty,$$

and by Lemma 6,

$$I_{a,b,s^+}(0,\ldots,0,a+1) \leqq I_{a,\infty,s^+}(0,\ldots,0,a+1) < \infty.$$

Hence (21) holds with $M = (0,\ldots,0,a+1)$.

For the remainder of this section we suppose that $\psi_a(s^+) < \infty$ and we show that (i)–(v) hold.

Since s has a positive harmonic majorant in D_a, by Lemma E, we can write s in the form (20), so that, by Lemmas 3, 4 and 5

$$\lim_{y\to\infty} \psi_a(s,y) = c_n k. \tag{22}$$

To prove (ii), note that

$$\int_{\mathbf{R}^n} \{(y+1-a)^2+|X|^2\}^{-\frac{1}{2}(n+1)}s^-(X,y)\,dX$$

is bounded for $y\in(a,\infty)$, by [**12**], Theorem 5(i), interpreted for D_a. Since s^- is lower semi-continuous in $\bar{D}_a$, on letting $y\to a^+$, we obtain, by Fatou's lemma,

$$\int_{\mathbf{R}^n} (1+|X|^2)^{-\frac{1}{2}(n+1)}s^-(X,a)\,dX<\infty,$$

and this, with (10), gives the result.

Conclusion (iii) now follows from Lemma 6.

To prove (iv), we use again the representation (20) of s. Writing $H=I_{a,\infty,\mu}$, we have $J_{a,b,H}\to 0$ in D_a as $b\to\infty$ (cf. proof of Lemma 6). Also, by Lemmas 1 and 4, if $y>a$, then

$$\begin{aligned}0&\leqq \lim_{b\to\infty} J_{a,b,u}(0,\dots,0,y)\\&\leqq A(y-a)\lim_{b\to\infty}\Psi_a(u,b)=0.\end{aligned}$$

Since we may translate the origin parallel to the $x_1,\dots,x_n$-axes, we find that $J_{a,b,u}\to 0$ in D_a as $b\to\infty$. It now follows that

$$\lim_{b\to\infty} J_{a,b,s}(M)=k\lim_{b\to\infty} J_{a,b,y-a}(M)\qquad (M\in D_a).$$

From Lemma 2(i) it is easy to see that $J_{a,b,y-a}(M)=y-a$ when $M\in\Omega_{a,b}$. Hence

$$\lim_{b\to\infty} J_{a,b,s}(M)=k(y-a)\qquad (M\in D_a),$$

and since $\psi_a(s)=c_nk$ (see (22)), the result follows.

The conclusion (v) now follows from (iii) and (iv), since, by Lemma B, $H_{a,b,s}$ is a harmonic majorant of s in $\Omega_{a,b}$.

To prove the corollary, first extend s to $\bar{D}_a$ by writing

$$s(N)=\limsup_{\substack{M\to N\\ M\in D_a}} s(M)\qquad (N\in\partial D_a).$$

Thus extended, s satisfies the hypotheses of Theorem 2, and therefore the function $h_{s,a}$, given by (11), is a harmonic majorant of s in D_a. Since $s\leqq 0$ on ∂D_a, we have $I_{a,\infty,s}\leqq 0$ in D_a. Since, also, $\psi_a(s)\leqq\psi_a(s^+)=0$, it follows that $h_{s,a}\leqq 0$ in D_a. Hence $s\leqq 0$ in D_a.

7. Proof of Theorem 3

If (10) holds and if $\psi_a(s^+)<\infty$, then it follows from Theorem 2 that s has a positive harmonic majorant in D_a.

Conversely, if s has such a majorant, then s^+ has a harmonic majorant in D_a and (10) holds, by [9], Theorem 3 and $\psi_a(s^+)<\infty$ by Theorem 2.

To prove the last assertion in the theorem, suppose that $0<\alpha<a<b<\beta<\gamma$ and define h in $\mathbf{R}^{n+1}$ by

$$h(X, y)=\cosh(x_1\pi/\gamma\sqrt{n})\ldots\cosh(x_n\pi/\gamma\sqrt{n})\sin(y\pi/\gamma).$$

It is easy to check that h is harmonic in $\mathbf{R}^{n+1}$. Also, $h(M)\geqq e^{C|M|}$ when $M\in\Omega_{\alpha,\beta}$, where C is a positive constant depending only on α, β, γ and n. Since $s\in\mathscr{S}_0$, it is clear that s is majorized in $\Omega_{\alpha,\beta}$ by some multiple of h. Hence, by Lemma C, the least harmonic majorant of s in $\Omega_{a,b}$ is $H_{a,b,s}$. If (10) holds and $\psi(s^+)<\infty$, then s has a harmonic majorant in D_a and it is now clear that the least such majorant is $\lim_{b\to\infty}H_{a,b,s}$. By Theorem 2 (iii), (iv), this limit is given by (11).

8. Proof of Theorem 4

If the hypotheses of Theorem 4 are satisfied, then, by Theorem 2, s has a positive harmonic majorant in D_a. Hence, by Lemma E, s has the representation (20) in D_a, and by Lemmas 3, 4 and 5, $\psi_a(s)=c_nk$. If we write $H=I_{a,\infty,\mu}$ and if $a'>a$, then in $D_{a'}$ we have $H=I_{a',\infty,H}$, as is well known. Hence, by Lemma 3, $\psi_{a'}(H)=0$. Also $\psi_{a'}(y-a)=\psi_{a'}(y-a')+\psi_{a'}(a'-a)=c_n$, by Lemma 5 and the special case of Lemma 3 in which the Poisson integral is a constant function. Hence to show that $\psi_{a'}(s)=c_nk=\psi_a(s)$, it remains to prove that $\psi_{a'}(u)=0$. Since u is positive and superharmonic in $D_{a'}$, we can apply Lemma E to $-u$ to obtain the representation

$$u(M)=I_{a',\infty,v}(M)+l(y-a')+w(M) \qquad (M\in D_{a'}),$$

where v is a non-negative measure on $\mathbf{R}^n$, l is a non-negative constant and w is a potential in $D_{a'}$. From Lemmas 3, 4 and 5, we have $\psi_{a'}(u)=c_nl$. Since $u(M)\geqq l(y-a')$ in $D_{a'}$, it follows that $\psi_a(u)\geqq l\psi_a(y-a)+l\psi_a(a-a')=c_nl$, by Lemmas 5 and 3 (trivial case). By Lemma 4, $\psi_a(u)=0$. Hence $l=0$, and therefore $\psi_{a'}(u)=0$, as required.

9. Examples

We give two examples to show how our theorems break down if the condition on the growth of s is relaxed. For simplicity, we work only with $n=1$ and $a=0$. A point of $\mathbf{R}^2$ is denoted by (x, y). Let ε be a positive number and define h_ε in $\mathbf{R}^2$ by

$$h_\varepsilon(x, y)=e^{\varepsilon x}\sin(\varepsilon y).$$

Then h_ε is harmonic in $\mathbf{R}^2$. Define functions s_1 and s_2 in $\bar{D}_0$ by writing $s_1=|h_\varepsilon|$ and

$$s_2(x, y)=h_\varepsilon(x, y) \quad (0\leqq y<\pi/\varepsilon), s_2(x, y)=0 \qquad (y\geqq\pi/\varepsilon).$$

Then s_1 and s_2 are subharmonic in D_0 and vanish on ∂D_0. Also,

$$\lim_{M \to \mathscr{A}} s_j(M)\, e^{-\lambda|M|} = 0 \qquad (j=1,2)$$

for any $\lambda > \varepsilon$. (Recall that if $s \in \mathscr{S}_0$, then (3) holds for all positive λ.) Straightforward calculations give

$$\Psi_0(s_1, y) = \Psi_0(s_2, y) = \pi y^{-1} \sin(\varepsilon y)(\pi^2 - \varepsilon^2 y^2)^{-1} \qquad (0 < y < \pi/\varepsilon),$$

$$\Psi_0(s_1, y) = \infty (y > \pi/\varepsilon, y \neq \pi/\varepsilon, 2\pi/\varepsilon, \ldots), \quad \Psi_0(s_1, y) = 0 \qquad (y = \pi/\varepsilon, 2\pi/\varepsilon, \ldots),$$

$$\Psi_0(s_2, y) = 0 \qquad (y \geqq \pi/\varepsilon).$$

Hence $\Psi_0(s_1, y)$ takes both finite and infinite values on $(0, \infty)$ and has no limit as $y \to \infty$. Thus the conclusions of Theorem 1 fail for s_1. On the other hand, $\Psi_0(s_2, y)$ is real-valued on $(0, \infty)$ and possesses a finite limit as $y \to \infty$, but s_2 does not possess a harmonic majorant in any half-space D_a with $0 \leqq a < \pi/\varepsilon$. (If s_2 had a harmonic majorant in D_a with $0 < a < \pi/\varepsilon$, then we would have

$$\int_{-\infty}^{\infty} (1+x^2)^{-1} s(x, a)\, dx < \infty$$

([9], Theorem 3, which is false). Thus Theorem 2 fails with $s = s_2$.

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