SOME CONFIGURATIONS IN FINITE PROJECTIVE SPACES AND PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

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1. Introduction. Using the methods developed in **(2** and **3)**, in this paper we study some properties of the configuration of generators and points of a cone in an n-dimensional finite projective space. The configuration of secants and external points of a quadric in a finite plane of even characteristic is also studied. It is shown that these configurations lead to several series of partially balanced incomplete block (PBIB) designs. PBIB designs are defined in Bose and Shimamoto **(1)**. A PBIB design with m associate classes is an arrangement of v treatments in b blocks such that:

1. Each of the v treatments is replicated r times in b blocks each of size k and no treatment occurs more than once in any block.

2. There exists a relationship of association between every pair of the v treatments satisfying the following conditions:

2a. Any two treatments are either first associates, second associates, . . . , or mth associates.

2b. Each treatment has n_1 first associates, n_2 second associates, . . . , n_m mth associates.

2c. Given any two treatments which are ith associates, the number p_{jk}^i of treatments which are jth associates of the first and kth associates of the second is independent of the pair of treatments with which we start. Furthermore $p_{jk}^i = p_{kj}^i$, for $i, j, k = 1, 2, \ldots, m$.

In this paper we shall be interested in PBIB designs with three associate classes. The following simple lemma will be found useful for our purposes. Consider a relationship of association between v treatments such that any two treatments are first associates, second associates, or third associates. Let $p_{jk}^i(\theta, \phi)$ denote the number of treatments which are jth associates of θ and kth associates of ϕ, where (θ, ϕ) is a pair of ith associate treatments, $i, j, k = 1, 2, 3$. Also each treatment has n_1 first associates, n_2 second associates, and n_3 third associates.

LEMMA 1.1. *If the numbers $p_{11}^i(\theta, \phi)$, $p_{12}^i(\theta, \phi)$, and $p_{22}^i(\theta, \phi)$ are independent of the particular pair of ith associates (θ, ϕ) and $p_{12}^i(\theta, \phi) = p_{21}^i(\theta, \phi)$ for every pair of ith associates (θ, ϕ), $i = 1, 2, 3$, then the association scheme is a* PBIB *association scheme with three associate classes.*

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Proof. We need to show that the numbers $p_{13}{}^i(\theta, \phi)$, $p_{31}{}^i(\theta, \phi)$, $p_{32}{}^i(\theta, \phi)$, $p_{23}{}^i(\theta, \phi)$, and $p_{33}{}^i(\theta, \phi)$ are independent of the particular pair of ith associates (θ, ϕ) and $p_{13}{}^i(\theta, \phi) = p_{31}{}^i(\theta, \phi)$ and $p_{23}{}^i(\theta, \phi) = p_{32}{}^i(\theta, \phi)$, $i = 1, 2, 3$. Consider a pair of treatments (θ, ϕ) which are first associates. The n_1 first associates of θ are made up of ϕ, $p_{11}{}^1(\theta, \phi)$-treatments which are first associates of both θ and ϕ, $p_{12}{}^1(\theta, \phi)$-treatments which are first associates of θ and second associates of ϕ, and $p_{13}{}^1(\theta, \phi)$-treatments which are first associates of θ and third associates of ϕ. So we have the identity

$$(1.1) \qquad 1 + p_{11}{}^1(\theta, \phi) + p_{12}{}^1(\theta, \phi) + p_{13}{}^1(\theta, \phi) = n_1.$$

Applying similar considerations we can get the following identities:

$$(1.2) \qquad \sum_{k=1}^{3} p_{jk}{}^i(\theta, \phi) = \begin{cases} n_i - 1 & \text{for } i = j,\ i, j = 1, 2, 3, \\ n_j & \text{for } i \neq j. \end{cases}$$

Using the identities (1.3) we can express $p_{13}{}^i(\theta, \phi)$, $p_{31}{}^i(\theta, \phi)$, $p_{23}{}^i(\theta, \phi)$, $p_{32}{}^i(\theta, \phi)$, and $p_{33}{}^i(\theta, \phi)$ in terms of $p_{11}{}^i(\theta, \phi)$, $p_{12}{}^i(\theta, \phi)$, and $p_{22}{}^i(\theta, \phi)$. Hence the lemma follows.

Let $\mathbf{B}$ denote the class consisting of the sets $B_1, B_2, \ldots, B_b$ where B_j, $j = 1, 2, \ldots, b$, is a set of points in $PG(n, s)$, the finite projective space of n dimensions based on a Galois field with s elements. Let $\mathbf{V}$ denote another class consisting of sets $V_1, V_2, \ldots, V_v$, where V_i, $i = 1, 2, \ldots, v$, is a set of points in $PG(n, s)$. These two classes generate a design with the incidence matrix

$$\underset{(v \times b)}{N} = \underset{(v \times b)}{((n_{ij}))}$$

where $n_{ij} = 1$ (0) as $V_i \cap B_j \neq \emptyset (= \emptyset)$. The design generated by the classes $\mathbf{B}$ and $\mathbf{V}$ will be denoted by $D(\mathbf{B}, \mathbf{V})$.

2. Configuration of generators and points of a cone. Let Q_{n-1} be a non-degenerate quadric on an $(n-1)$-flat Σ_{n-1} in $PG(n, s)$ and Q_n be the cone with Q_{n-1} as the base and a point O outside Σ_{n-1} as the vertex. As in **(2)** we shall use $N(p, n-1-2t)$ to determine the number of p-flats contained in a non-degenerate quadric of the type of Q_{n-1} (hyperbolic or elliptic) in $PG(n-1-2t, s)$. We shall investigate some properties of the cone in the following lemmas.

LEMMA 2.1 *Let P be any point of Q_n other than O. Then the number of generators which pass through P but do not pass through O is $sN(0, n-3)$.*

Proof. Let PR be a generator of Q_n not passing through O. Then the three points O, P, and R are points of Q_n and mutually conjugate. By **(2**, Lemma 2.3**)**, the plane OPR is contained in Q_n. Let $P'R'$ be the intersection of OPR and Σ_{n-1}. Then $P'R'$ is a generator of Q_{n-1}, i.e. a line contained in Q_{n-1}. Hence any such generator PR is contained in a plane $OP'R'$, where P' is the intersection of OP and Σ_{n-1} and $P'R'$ is a generator of Q_{n-1}. The number of such planes $OP'R'$ is equal to the number of generators of Q_{n-1} passing through P' and hence by (**2**, Theorem 3.3) is $N(0, n-3)$. Every plane $OP'R'$ contains s generators passing through P but not passing through O. Hence the lemma follows.

LEMMA 2.2. *Let P_1 and P_2 be two points of Q_n other than O such that P_1P_2 is a generator not passing through O. Then the number of points P such that both PP_1 and PP_2 are generators not passing through O is $s^2 - s + s^3 N(0, n-5)$.*

Proof. By (**2**, Lemma 2.3), the plane OP_1P_2 is contained in Q_n. Hence the points of the plane other than those lying on the lines OP_1 and OP_2 possess the required property. So the plane OP_1P_2 contributes $(s^2 + s + 1) - (2s + 1)$ points P. Let P be a point not lying on the plane OP_1P_2 which has the required property. By (**2**, Lemma 2.3), the 3-flat OPP_1P_2 is contained in Q_n. Let the plane $P'_1P'P'_2$ be the intersection of the 3-flat OPP_1P_2 and Σ_{n-1} where P'_1 and P'_2 are respectively the intersections of OP_1 and OP_2 with Σ_{n-1}. We have shown that every such point P lies in a 3-flat $OP'_1P'_2P'$ where $P'_1P'_2P'$ is a plane contained in Q_{n-1} passing through $P'_1P'_2$. By (**2**, Theorem 3.3), the number of planes contained in Q_{n-1} passing through $P'_1P'_2$ is $N(0, n-5)$. Hence the number of 3-spaces of the type $OP'_1P'_2P'$ is also $N(0, n-5)$. Every such 3-space contributes s^3 points P with the property that both PP_1 and PP_2 are generators of Q_n not passing through O and that P does not lie on the plane OP_1P_2. This completes the proof.

LEMMA 2.3. *Let P_1 and P_2 be two points of Q_n such that P_1P_2 is a generator passing through O. Then the number of points P such that both PP_1 and PP_2 are generators not passing through O is $s^2N(0, n-3)$.*

Proof. Let P'_1 be the intersection of the line OP_1P_2 and Σ_{n-1}. Let P be a point such that both PP_1 and PP_2 are generators not passing through O. By (**2**, Lemma 2.3), the plane PP_1P_2 is contained in Q_n. Let P'_1P' be the intersection of the plane PP_1P_2 and Σ_{n-1}. P'_1P' is contained in Q_{n-1}. Hence we have shown that every such point P lies in a plane OP'_1P' where P'_1P' is a generator of Q_{n-1} passing through P'_1. The number of planes OP'_1P' is equal to the number of generators P'_1P' of Q_{n-1} passing through P'_1 and hence is equal to $N(0, n-3)$ by (**2**, Theorem 3.3). Every plane OP'_1P' contributes s^2 points P such that both PP_1 and PP_2 are generators not passing through O. Hence the lemma follows.

LEMMA 2.4. *Let P_1 and P_2 be two points of Q_n such that P_1P_2 is not a generator. Then the number of points P such that both PP_1 and PP_2 are generators not passing through O is $sN(0, n-3)$.*

Proof. Let P'_1 and P'_2 be respectively the intersection of OP_1 and OP_2 with Σ_{n-1}. Let P be a point such that both PP_1 and PP_2 are generators not passing through O. Let $P' = OP \cap \Sigma_{n-1}$. It can be shown that $P'P'_1$ and $P'P'_2$ are a pair of intersecting generators of Q_{n-1} and also $P'_1P'_2$ is not a generator of Q_{n-1}. Hence every point P lies on a line OP' where P' has the property that $P'P'_1$ and $P'P'_2$ are generators of Q_{n-1}. By **(3**, Section 3, Lemma 4**)**, the number of such points is $N(0, n-3)$. Every line OP' contributes s points. This completes the proof.

LEMMA 2.5. *Let P_1 and P_2 be two points of Q_{n-1} such that $P_1 P_2$ is a generator not passing through O. Then the number of points P other than P_1 and P_2 such that PP_1 is a generator not passing through O and PP_2 is a generator passing through O is $s-1$.*

Proof. Since $P_1 P_2$ is a generator not passing through O, the plane $OP_1 P_2$ is contained in Q_n. Let P be a point with the required property. PP_2 is a generator passing through O. Hence P must be a point of OP_2. Since the plane $OP_1 P_2$ is contained in Q_n, every point P of OP_2 is such that PP_1 is a generator of Q_n. Hence the required points P are the points of the line OP_2 other than O and P_2.

The following simple lemma is stated without proof.

LEMMA 2.6. *Let P_1 and P_2 be two points of Q_n such that $P_1 P_2$ is a generator passing through O. Then the number of points P such that PP_1 is a generator not passing through O and PP_2 is a generator passing through O is* 0.

LEMMA 2.7. *Let P_1 and P_2 be two points of Q_n such that $P_1 P_2$ is not a generator of Q_n. Then the number of points P such that PP_1 is a generator not passing through O and PP_2 is a generator passing through O is* 0.

Proof. Any point P with the stated property must lie on the line OP_2. Suppose there is a point P on the line OP_2 such that PP_1 is a generator. Then the plane OPP_1 is contained in Q_n by **(2**, Lemma 2.3**)**. $P_1 P_2$ is a line of OPP_1 and hence is contained in Q_n. But this contradicts the hypothesis of the lemma.

The following three simple lemmas are stated without proof for the sake of reference.

LEMMA 2.8. *Let P_1 and P_2 be two points of Q_n such that $P_1 P_2$ is a generator not passing through O. Then the number of points P other than O such that both PP_1 and PP_2 are generators passing through O is* 0.

LEMMA 2.9. *Let P_1 and P_2 be two points of Q_n such that $P_1 P_2$ is a generator passing through O. Then the number of points P other than O, P_1, and P_2 such that both PP_1 and PP_2 are generators passing through O is $s-2$.*

LEMMA 2.10. *Let P_1 and P_2 be two points of Q_n such that $P_1 P_2$ is not a generator. Then the number of points P other than O, P_1, and P_2 such that both PP_1 and PP_2 are generators passing through O is* 0.

THEOREM 1. *Let* **B** *be the class of generators of Q_n not passing through O and* **V** *be the class of points of Q_n other than O. The design $D(\mathbf{B}, \mathbf{V})$ is a* PBIB *design with* 3 *associate classes and the following parameters:*

$$v = sN(0, n-1), \quad b = s^2N(1, n-1), \quad k = s+1, \quad r = sN(0, n-3),$$

$$\lambda_1 = 1, \quad \lambda_2 = \lambda_3 = 0, \quad n_1 = s^2N(0, n-3), \quad n_2 = s-1,$$

$$p_{11}^{\ 1} = s^2 - s + s^3N(0, n-5), \quad p_{12}^{\ 1} = (s-1), \quad p_{22}^{\ 1} = 0,$$

$$p_{11}^{\ 2} = s^2N(0, n-3), \quad p_{12}^{\ 2} = 0, \quad p_{22}^{\ 2} = s-2, \quad p_{11}^{\ 3} = sN(0, n-3),$$

$$p_{12}^{\ 3} = p_{22}^{\ 3} = 0.$$

The other parameters of the design can be obtained from the equalities (1.2) *between the parameters of a* PBIB *design.*

Proof. The points of Q_n other than O and the generators of Q_n not passing through O are respectively regarded as treatments and blocks. Two points P_1 and P_2 are first associates if $P_1 P_2$ is a generator not passing through O, second associates if $P_1 P_2$ is a generator passing through O, and third associates if $P_1 P_2$ is not a generator of Q_n. Using (**2**, Theorem 3.3) and Lemma 2.1, the following results are obtained easily:

$v = sN(0, n-1)$ = number of points of Q_n other than O,
$b = s^2N(1, n-1)$ = number of generators of Q_n not passing through O,
$k = s+1$ = number of points on a generator,
$r = sN(0, n-3)$ = number of generators passing through a point P $(\neq O)$ which do not pass through O.

The expressions for n_1 and n_2 are obvious. The other parameters of the design follow from Lemmas 2.2 to 2.10 and the theorem follows from Lemma 1.1.

Taking $n = 2t+1$, $t \geqslant 2$, Q_{n-1} a non-degenerate quadric, and using (**2**, Theorem 3.2), we get the following series of PBIB designs with three associate classes:

$$v = \frac{s(s^{2t}-1)}{s-1}, \quad b = \frac{s^2(s^{2t-2}-1)(s^{2t}-1)}{(s^2-1)(s-1)},$$

$$k = s+1, \quad r = \frac{s(s^{2t-2}-1)}{s-1}, \quad \lambda_1 = 1, \quad \lambda_2 = \lambda_3 = 0,$$

$$n_1 = s^2\frac{s^{2t-2}-1}{s-1},$$

$$n_2 = s-1, \quad p_{11}^{\ 1} = s^2 - s + \frac{s^3(s^{2t-4}-1)}{s-1}, \quad p_{12}^{\ 1} = (s-1), \quad p_{22}^{\ 1} = 0,$$

$$p_{11}^{\ 2} = s^2\frac{s^{2t-2}-1}{s-1}, \quad p_{12}^{\ 2} = 0, \quad p_{22}^{\ 2} = s-2, \quad p_{11}^{\ 3} = s\frac{s^{2t-2}-1}{s-1},$$

$$p_{12}^{\ 3} = p_{22}^{\ 3} = 0.$$

Taking $n = 2t, t \geqslant 3$, Q_{n-1} a non-degenerate elliptic quadric in PG$(2t - 1, s)$, and using the results of (**2**, Theorem 3.2), we get the following series of PBIB designs with three associate classes:

$$v = s\frac{s^{2t-1} + s^{t-1} - s^t - 1}{s - 1},$$

$$b = s^2\frac{(s^{2t-1} + s^{t-1} - s^t - 1)(s^{2t-3} + s^{t-2} - s^{t-1} - 1)}{(s - 1)(s^2 - 1)},$$

$$k = s + 1, \qquad r = s\frac{s^{2t-3} + s^{t-2} - s^{t-1} - 1}{s - 1}, \qquad \lambda_1 = 1, \qquad \lambda_2 = \lambda_3 = 0,$$

$$n_1 = s^2\frac{s^{2t-3} + s^{t-2} - s^{t-1} - 1}{s - 1}, \qquad n_2 = s - 1,$$

$$p_{11}^{\ 1} = s^2 - s + s^3\frac{s^{2t-5} + s^{t-3} - s^{t-2} - 1}{s - 1}, \qquad p_{12}^{\ 1} = s - 1, \qquad p_{22}^{\ 1} = 0,$$

$$p_{11}^{\ 2} = s^2\frac{s^{2t-3} + s^{t-2} - s^{t-1} - 1}{s - 1}, \qquad p_{12}^{\ 2} = 0, \qquad p_{22}^{\ 2} = s - 2,$$

$$p_{11}^{\ 3} = s\frac{s^{2t-3} + s^{t-2} - s^{t-1} - 1}{s - 1}, \qquad p_{12}^{\ 3} = p_{22}^{\ 3} = 0.$$

Taking $n = 2t$, $t \geqslant 2$, Q_{n-1} a non-degenerate hyperbolic quadric in PG$(n - 1, s)$, and using results of (**2**, Theorem 3.3), we get the following series of PBIB designs:

$$v = s\frac{s^{2t-1} - s^{t-1} + s^t - 1}{s - 1},$$

$$b = s^2\frac{(s^{2t-1} - s^{t-1} + s^t - 1)(s^{2t-3} - s^{t-2} + s^{t-1} - 1)}{(s - 1)(s^2 - 1)},$$

$$k = s + 1, \qquad r = s\frac{s^{2t-3} - s^{t-2} + s^{t-1} - 1}{s - 1}, \qquad \lambda_1 = 1, \qquad \lambda_2 = \lambda_3 = 0,$$

$$n_1 = s^2\frac{s^{2t-3} - s^{t-2} + s^{t-1} - 1}{s - 1}, \qquad n_2 = s - 1,$$

$$p_{11}^{\ 1} = s^2 - s + s^3\frac{s^{2t-5} - s^{t-3} + s^{t-2} - 1}{s - 1}, \qquad p_{12}^{\ 1} = s - 1,$$

$$p_{22}^{\ 1} = 0, \qquad p_{11}^{\ 2} = s^2\frac{s^{2t-3} - s^{t-2} + s^{t-1} - 1}{s - 1}, \qquad p_{12}^{\ 2} = 0, \qquad p_{22}^{\ 2} = s - 2,$$

$$p_{11}^{\ 3} = \frac{s(s^{2t-3} - s^{t-2} + s^{t-1} - 1)}{s - 1}, \qquad p_{12}^{\ 3} = p_{22}^{\ 3} = 0.$$

The three series given contain many designs useful for statistical experiments.

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3. Configuration of secants and external points of a quadric in a plane of even characteristic. Let Q_2 be a non-degenerate quadric in PG$(2, s)$, $s = 2^m$. Any line of PG$(2, s)$ which intersects the quadric in two points is called a secant. We shall prove some properties of the configuration of secants and external points in the following lemmas.

LEMMA 3.1. *Let P be an external point of Q_2 other than the nucleus of polarity of Q_2 in* PG$(2, s)$, $s = 2^m$ $(m > 1)$. *The number of secants passing through P is $s/2$.*

Proof. Let $\tau(P)$ denote the polar of P with respect to Q_2. $\tau(P)$ is a line; it can intersect Q_2 in k points, $k = 0, 1, 2$. Hence $\tau(P)$ contains another external point P_1. The points P and P_1 are external points of Q_2 and mutually conjugate with respect to Q_2. By **(2**, Lemma 4.1**)**, the line PP_1 intersects Q_2 in a single point. Let R_1 be any point of Q_2 not lying on $\tau(P)$. By **(2**, Lemma 4.1**)**, the line PR_1 contains another point of Q_2. Hence PR_1 is a secant of Q_2. Hence for every point R of the quadric not lying on $\tau(P)$, PR is a secant. The number of points of Q_2 not lying on $\tau(P)$ is s and every secant contains 2 points. Hence the lemma follows.

LEMMA 3.2. *Let S denote the nucleus of polarity of Q_2. Let P be any point of* PG$(2, s)$ *not lying on Q_2. Then $\tau(P)$ contains S.*

Proof. In the proof of Lemma 3.1 it is shown that $\tau(P)$ contains one point R of Q_2. P and R are mutually conjugate. So $\tau(R)$, the tangent line at R, is the same as $\tau(P)$ and therefore $\tau(P)$ contains S.

LEMMA 3.3. *Let P_1 be an external point other than the nucleus of polarity S. The number of external points P other than P_1 such that PP_1 is a secant is $s(s-2)/2$.*

Proof. The required points are those which lie on a secant passing through P_1 but do not lie on Q_2. By Lemma 3.1 the number of secants passing through P_1 is $s/2$ and every secant contains two points of Q_2. Hence the lemma follows.

LEMMA 3.4. *Let P_1 be an external point of Q_2 other than S. The number of external points P other than P_1 and S such that PP_1 intersects the quadric in a single point is $s - 2$.*

Proof. The required points are those external points other than S and P which lie on $\tau(P_1)$. $\tau(P_1)$ contains P, S, and one point R of Q_2. The total number of points of $\tau(P_1)$ is $s + 1$. Hence the lemma follows.

LEMMA 3.5. *Let P_1 and P_2 be two external points of Q_2 other than S such that P_1P_2 is a secant. The number of external points P other than S, P_1, and P_2 such that both PP_1 and PP_2 are secants is $(\frac{1}{2}s - 2)^2 + s - 3$.*

Proof. By **(2**, Lemma 4.1**)**, P_1 and P_2 are not mutually conjugate. Let R_1 and R_2 be respectively the points at which $\tau(P_1)$ and $\tau(P_2)$ intersect the quadric.

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$P_1 R_2$ must be a secant. Suppose $P_1 R_2$ is a tangent line intersecting Q_2 in a single point R_2. Then P_1 and R_2 are mutually conjugate and R_2 lies in $\tau(P_1)$, the polar of P_1. So $\tau(P_1) = \tau(R_1)$ intersects Q_2 at two points, which is a contradiction. It follows that $P_1 R_2$ is a secant. Similarly $P_2 R_1$ is a secant.

Let $t = s/2$. Let the t secants passing through P_1 and P_2 respectively be $P_1 M_{11}$, $P_1 P_2 M_{12}, \ldots, P_1 M_{1t}$ and $P_2 M_{21}$, $P_1 P_2 M_{22}, \ldots, P_2 M_{2t}$, where $M_{11} = R_2$, $M_{21} = R_1$, $M_{12} = M_{22}$. All the points P with the required property must lie on the secants $P_1 M_{11}, P_1 P_2 M_{12}, \ldots, P_1 M_{1t}$. Consider the common secant $P_1 P_2 M_{12}$ which contains two points of Q_2 and the points P_1 and P_2. Hence the common secant contains $s - 3$ points P satisfying the required conditions. Next we consider $P_1 M_{11}$. The external points lying on $P_1 M_{11}$ which are points of intersection of $P_1 M_{11}$ and a secant passing through P_2 will have the required property. Let the two points of $P_1 M_{11}$ which lie on Q_2 be M_{11} and M'_{11}. $M_{11} = R_2$. The line $P_2 R_2$ is a tangent and $P_2 M'_{11}$ is a secant. $P_1 P_2 M_{22}$ intersects $P_1 M_{11}$ at P_1. Hence $t - 2$ of the secants passing through P intersect $P_1 M_{11}$ at external points other than P_1, P_2, and S. So the secant $P_1 M_{11}$ contributes $t - 2$ points P with the required property. Next we consider the secant $P_1 M_{13}$ which contains two points M_{13} and M'_{13} of Q_2. Both the lines $P_2 M_{13}$ and $P_2 M'_{13}$ are secants. Also $P_1 P_2 M_{22}$ intersects $P_1 M_{13}$ at P_1. So $t - 3$ of the secants passing through P_2 intersect $P_1 M_{13}$ at external points other than P_1, P_2, and S, and $P_1 M_{13}$ contributes $t - 3$ points P possessing the required property. Counting together all these points, we get the total number of points P as given in the lemma.

LEMMA 3.6. *Let P_1 and P_2 be two external points other than S such that $P_1 P_2$ is a tangent line intersecting Q_2 in a single point. Then the number of external points P, other than P_1, P_2, and S such that both PP_1 and PP_2 are secants is* $\frac{1}{2}s(\frac{1}{2}s - 2)$.

Proof. Let R_0 be the point at which $P_1 P_2$ intersects Q_2. If R is any other point of Q_2, $P_1 R$ and $P_2 R$ are secants by (**2**, Lemma 4.1). Let the $t = s/2$ secants passing through P_1 and P_2 respectively be $P_1 M_{11}, P_1 M_{12}, \ldots, P_1 M_{1t}$ and $P_2 M_{21}$, $P_2 M_{21}, \ldots, P_2 M_{2t}$. Any point P with the required property must lie on one of the secants $P_1 M_{11}$, $P_1 M_{12}, \ldots, P_1 M_{1t}$. Let us consider $P_1 M_{11}$. Let M_{11} and M'_{11} be the points of $P_1 M_{11}$ which lie on Q_2. Both the lines $P_2 M_{11}$ and $P_2 M'_{11}$ are secants. So $t - 2$ of the secants passing through P_2 intersect $P_1 M_{11}$ at external points other than P_1, P_2, and S. Hence the secant $P_1 M_{11}$ contributes $t - 2$ points P with the required property. The same is true for all the secants passing through P_1. Hence the lemma follows.

LEMMA 3.7. *Let P_1 and P_2 be two external points of Q_2 other than S such that $P_1 P_2$ is an external line which does not intersect Q_2. Then the number of external points P other than P_1, P_2, and S such that both PP_1 and PP_2 are secants is* $(\frac{1}{2}s - 1)^2$.

Proof. Let R_1 and R_2 respectively be the points at which $\tau(P_1)$ and $\tau(P_2)$,

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polars of P_1 and P_2 respectively, intersect Q_2. $P_1 R_2$ and $P_2 R_1$ are secants. Let the secants passing through P_1 and P_2 be respectively $P_1 M_{11}$, $P_1 M_{12}$, $\ldots$, $P_1 M_{1t}$ and $P_2 M_{21}$, $P_2 M_{22}$, $\ldots$, $P_2 M_{2t}$ where $t = \frac{1}{2}s$, $M_{11} = R_2$, and $M_{21} = R_1$. By an argument similar to that used in Lemmas 3.4 and 3.5, it can be shown that $P_1 M_{11}$ contributes $t - 1$ points P possessing the required property and the remaining secants passing through P_1 contribute $t - 2$ points each. Hence the lemma follows.

LEMMA 3.8. *Let P_1 and P_2 be two external points of Q_2 other than S such that $P_1 P_2$ is a secant. Then the number of external points P other than P_1, P_2, and S such that PP_1 is a tangent line of Q_2 and PP_2 is a secant is* $\frac{1}{2}s - 2$.

Proof. Let $P_1 R_1$ be the polar of P_1 with respect to Q_2, where R_1 is a point of Q_2. The required points P must lie on the line $P_1 R_1$. Since P_1 and P_2 are mutually non-conjugate, $P_2 R_1$ must be a secant. Hence of the $\frac{1}{2}s$ secants passing through P_2, $\frac{1}{2}s - 2$ secants intersect $P_1 R_1$ at an external point other than P_1, P_2, and S. Hence the lemma follows.

The following lemma is obvious.

LEMMA 3.9. *Let P_1 and P_2 be two external points other than S such that $P_1 P_2$ is a tangent line. Then the number of external points P other than P_1, P_2, and S such that PP_1 is a tangent line and PP_2 is a secant is* 0.

LEMMA 3.10. *Let P_1 and P_2 be two external points of Q_2 other than S such that $P_1 P_2$ is an external line. Then the number of external points P other than S, P_1, and P_2 such that PP_1 is a tangent line and PP_2 is a secant is* $\frac{1}{2}s - 1$.

Proof. Let $P_1 R_1$ be the polar of P_1 with respect to Q_2 where R_1 is a point of Q_2. $P_2 R_1$ must be a secant of Q_2. Hence of the $\frac{1}{2}s$ secants of P_2, $\frac{1}{2}s - 1$ secants, namely the secants other than $P_2 R_1$, intersect $P_1 R_1$ in an external point. Hence the lemma follows.

LEMMA 3.11. *Let P_1 and P_2 be two external points other than S such that $P_1 P_2$ is a secant. The number of external points P other than P_1, P_2, and S such that both PP_1 and PP_2 are tangents is* 0.

Proof. Let $P_1 R_1$ and $P_2 R_2$ be the polars of P_1 and P_2 respectively, where R_1 and R_2 are points of Q_2. Every point P satisfying the required condition lies on both $P_1 R_1$ and $P_2 R_2$. The only point which lies on both $P_1 R_1$ and $P_2 R_2$ is S. Hence the lemma follows.

LEMMA 3.12. *Let P_1 and P_2 be two external points of Q_2 other than S such that $P_1 P_2$ is a tangent line. Then the number of external points P other than P_1, P_2, and S such that both PP_1 and PP_2 are tangent lines is* $s - 3$.

The proof is obvious.

LEMMA 3.13. *Let P_1 and P_2 be two external points of Q_2 such that $P_1 P_2$ is an* *external line. Then the number of external points P other than P_1, P_2, and S such that both PP_1 and PP_2 are tangent lines is* 0.

The proof is similar to that of Lemma 3.11.

THEOREM 2. *Let* **B** *be the class of lines of* PG(2, s), $s = 2^m$, *which are secants of* Q_2, *and* **V** *be the class of external points of* Q_2 *other than S, the nucleus of polarity of* Q_2. *Then* $D(\mathbf{B}, \mathbf{V})$ *is a* PBIB *design with three associate classes with the following parameters:*

$$v = s^2 - 1, \quad r = \tfrac{1}{2}s, \quad b = (s+1).\tfrac{1}{2}s, \quad k = s - 1, \quad \lambda_1 = 1,$$

$$\lambda_2 = \lambda_3 = 0, \quad n_1 = \tfrac{1}{2}s(s-2), \quad n_2 = s - 2,$$

$$p_{11}{}^1 = (s-3) + (\tfrac{1}{2}s - 2)^2, \quad p_{12}{}^1 = \tfrac{1}{2}s - 2, \quad p_{22}{}^1 = 0,$$

$$p_{11}{}^2 = \tfrac{1}{2}s(\tfrac{1}{2}s - 2), \quad p_{12}{}^2 = 0, \quad p_{22}{}^2 = s - 3,$$

$$p_{11}{}^3 = (\tfrac{1}{2}s - 1)^2, \quad p_{12}{}^3 = \tfrac{1}{2}s - 1, \quad p_{22}{}^3 = 0.$$

Other parameters of the design can be obtained from the equalities (1.3).

Proof. The external points other than S are considered to be treatments. The secants with points of Q_2 excluded are the blocks. Two external points P_1 and P_2 are first associates if the line P_1P_2 is a secant, second associates if the line P_1P_2 is a tangent line at a point of Q_2, and third associates if P_1P_2 is an external line not intersecting Q_2. We can easily see that:

1. The number v of points of PG(2, s) other than S which do not lie on Q_2 is equal to $s^2 - 1$.

2. The number r of secants passing through a given external point other than S is equal to $\frac{1}{2}s$, by Lemma 3.1.

3. The number k of external points lying on a secant is equal to $s - 1$.

Lemmas 1.1, 3.1 to 3.13 imply Theorem 2. The series of PBIB designs given in Theorem2 contain several designs which are useful for statistical experiments.

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