CONVOLUTION TRANSFORMS RELATED TO NON-HARMONIC FOURIER SERIES

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1. Introduction. Widder has pointed out (**2**, p. 219) in connection with Wiener's fundamental work on the operational calculus (**1**, pp. 557–584), that the convolution transform

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\,\phi(t)\,dt \tag{1.1}$$

will be inverted by the operator $DE(D)$, where $D = d/dx$, and

$$1/wE(w) = \int_{-\infty}^{\infty} \exp(-xw)\,G(x)\,dx,$$

where a suitable interpretation must be found for $E(D)$. Cases where $E(w)$ is entire have been considered by Widder (**2**, pp. 217–249; **3**, pp. 7–60), Hirschman and Widder (**4**, pp. 659–696; **8**, pp. 135–201), and the author (**5**).

The most general method of interpreting $E(D)$ is as $\lim_{n\to\infty} P_n(D)$, where $P_n(w)$ is a polynomial of degree n, the method requiring a knowledge of $f(x)$ only for real values of its argument. However in cases where more is known about $E(w)$ (**4**, p. 692; **5**, pp. 174–183; **6**, p. 219), it is possible to represent $E(w)$ as an integral, when the computations are simpler, but it is necessary to have $f(x)$ defined for complex arguments.

The purpose of this article is to consider convolution transforms for which the invertor function $E(w)$ is entire, is not necessarily even, and can be represented by a Fourier-Lebesgue integral. The real numbers which are taken to be the zeros of $E(w)$ are a generalization of the non-harmonic Fourier exponents discussed by Levinson (**7**, pp. 47–57). The classical Stieltjes transform (**3**), and the generalized form of it (**5**), are particular cases. The assumptions made about the zeros of $E(w)$ are sufficient to establish all properties needed, and no integrability condition is postulated for $E(u)$.

2. Definitions. We suppose throughout that

$$\lambda_n = \rho + n - \delta + 2\delta\alpha_n,\ \mu_n = n - \delta + 2\delta\beta_n \qquad (n = 1, 2, \ldots), \tag{2.1}$$
$$0 \leqslant \alpha_n, \beta_n \leqslant 1,\quad 0 \leqslant \delta < \tfrac{1}{4},\quad 0 \leqslant \rho < 1 - 2\delta;$$

$$E(w) = \prod_{1}^{\infty} (1 - w/\lambda_n)(1 + w/\mu_n); \tag{2.2}$$

$$G(z) = \lim_{R\to\infty} (2\pi i)^{-1} \int_{c-iR}^{c+iR} \exp(zw)\,dw/wE(w), \qquad 0 < c < \lambda_1. \tag{2.3}$$

The symbols A, A_k denote absolute constants throughout.

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3. Some properties of $E(w)$. The numbers λ_n, μ_n are those used by Levinson (**7**, pp. 47–57) in his work on non-harmonic Fourier series. With the notation $w = u + iv = r \exp(i\phi)$, Levinson's methods may be used to establish the following inequalities:

(3.1) $$|E(w)| \leqslant A \exp(\pi|v|)/r^{\rho+1-4\delta}, \qquad (r \geqslant 1);$$

(3.2) $$|E(w)| \geqslant A \exp(\pi|v|)/r^{\rho+1-4\delta}, \textit{ provided that}$$

(3.21) $r \geqslant 1$, $|w - r_n| \geqslant \Delta > 0$, $r_n = \lambda_n$ *or* $-\mu_n$;

(3.3) $$|E'(r_n)|^{-1} \leqslant A r^{\rho+1+4\delta};$$

(3.4) *there exists a constant* q, $1 < q \leqslant 2$, *such that* $E(u) \in L^q(-\infty, \infty)$.

For the behaviour of $E(w)$ along the imaginary axis, we establish the more precise inequalities

(3.5) $$|E(iv)| \leqslant A \exp(\pi|v|)/|v|^{\rho+1-2\delta}, \quad |E(iv)| \geqslant A \exp(\pi|v|)/|v|^{\rho+1+2\delta};$$

(3.6) $$|\text{amp } E(iv\theta)/E(iv)| < A(1 - \theta),$$

where $0 < \theta \leqslant 1$, *and the constant is independent of v;*

(3.7) $|E(iv\theta)/E(iv)|$ *is a decreasing function of* $|v|$, $0 < \theta < 1$.

Proof of (3.1). Let $\Re(w) > 0$, and N be the integer defined by

(3.8) $$(\rho + N - \tfrac{1}{2}) \cos\phi \leqslant r < (\rho + N + \tfrac{1}{2}) \cos\phi.$$

On considering separately the factors in (2.2) for which $1 \leqslant n < N$, $n = N$ and $n \geqslant N + 1$, as Levinson does, we get

(3.9) $$|E(w)| \leqslant \left|\frac{\Gamma(\rho + N + 1 + \delta)\, \Gamma(\rho + N + 1 - \delta - w)}{\Gamma(\rho + N + 1 - \delta)\, \Gamma(\rho + N + 1 + \delta - w)}\right| \cdot \left|\frac{\Gamma(\rho + 1 - \delta)\, \Gamma(1 - \delta)(\lambda_N - w)}{\Gamma(\rho + 1 - \delta - w)\, \Gamma(1 - \delta + w)(\rho + N - \delta - w)}\right|.$$

By Stirling's theorem the first factor in (3.9) does not exceed

(3.10) $$A_1(\rho + N - \tfrac{1}{2})^{2\delta}/|\rho + N + 1 - \delta - w|^{2\delta};$$

while the second factor does not exceed

(3.11) $$\frac{A_2}{r^{\rho+1-2\delta}}\left|\frac{(\lambda_N - w)\sin\pi(w + \delta - \rho)}{\pi(w + \delta - N - \rho)}\right|.$$

Now when $|w + \delta - N - \rho| < \frac{1}{2}$, $|\sin \pi(w + \delta - \rho)/\pi(w + \delta - N - \rho) < A_3$; and by (2.1), $|\lambda_N - w| < 1$. When $|w + \delta - N - \rho| \geqslant \frac{1}{2}$, $|\sin \pi(w + \delta - \rho)| \leqslant A_4 \exp(\pi|v|)$, and $|(\lambda_N - w)/(w + \delta - N - \rho)| \leqslant 1 + 2\delta\alpha_N/|w + \delta - N - \rho| < 2$. Thus in all cases the second factor in (3.9) does not exceed

(3.12) $$A_5 \exp(\pi|v|)/r^{\rho+1-2\delta}.$$

We prove now that

(3.13) $$bd\{|\rho + N + 1 - \delta - w| \cos\phi\} > 0.$$

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For by (3.21), (3.8), when $0 \leqslant |\phi| \leqslant \pi/4$,

$$|\rho + N + 1 - \delta - w| \cos \phi \geqslant 2^{-\frac{1}{2}}(\rho + N + 1 - \delta - u) > 2^{-\frac{1}{2}}[(\rho + N + \tfrac{1}{2}) \sin^2\phi + \tfrac{1}{2} - \delta] \geqslant 2^{-\frac{1}{2}}(\tfrac{1}{2} - \delta);$$

and when $\pi/4 < |\phi| < \pi/2$,

$$|\rho + N + 1 - \delta - w| \cos \phi \geqslant (\rho + N + 1 - \delta - u)/(\rho + N + \tfrac{1}{2}) \geqslant \sin^2\phi + (\tfrac{1}{2} - \delta)/(\rho + N + \tfrac{1}{2}) \geqslant \tfrac{1}{2},$$

by (3.8) and $r \geqslant 1$.

From (3.9), (3.10), and (3.12),

$$\begin{aligned} |E(w)| &\leqslant A_6(\rho + N - \tfrac{1}{2})^{2\delta}\exp(\pi|v|)/r^{\rho+1-2\delta}|\rho + N + 1 - \delta - w|^{2\delta}, \\ &\leqslant A_7 r^{2\delta}\exp(\pi|v|)/r^{\rho+1-2\delta}\{|\rho + N + 1 - \delta - w| \cos \phi\}^{2\delta}, \\ &\leqslant A_8\exp(\pi|v|)/r^{\rho+1-4\delta} \end{aligned}$$

by (3.13). This proves (3.1) for $\Re(w) > 0$; and the assertion is seen to be true for $\Re(w) < 0$ by applying the same argument to $E(-w)$.

Remark on the proof of (3.4). Levinson's method may be used to show that when $\rho + N \leqslant u \leqslant \rho + 2N$,

$$(3.14) \qquad |E(u)| \leqslant \frac{A_9 N^{4\delta-\rho-1}}{(\rho + 2N + 1 - u)^{2\delta}} \prod_N^{2N} \left| \frac{\rho + n - \delta + \alpha_n - u - 2i}{\rho + n - \delta - u - 2i} \right|^{2\delta},$$

$$(3.15) \qquad \int_{\rho+N}^{\rho+2N} |E(u)|^q du \leqslant A_{10}/N^{q(\rho+1)-1-2q\delta},$$

provided that

$$(3.16) \qquad 0 \leqslant 2q\delta < 1.$$

If in addition

$$(3.17) \qquad q(\rho + 1 - 2\delta) > 1,$$

it follows from (3.15) that $E(u) \in L(0, \infty)$. The final conclusion follows by considering $E(-u)$ in the same way.

It is evident that there always exists a number q, $1 < q \leqslant 2$, satisfying (3.16) and (3.17), for example $q = (1 - \Delta)^{-1}$, where $2\delta < \Delta < 1 - 2\delta$.

Proof of (3.5). Since

$$\left|\left(1 - \frac{iv}{\rho + n + \delta}\right)\left(1 + \frac{iv}{n + \delta}\right)\right| \leqslant \left|\left(1 - \frac{iv}{\lambda_n}\right)\left(1 + \frac{iv}{\mu_n}\right)\right| \leqslant \left|\left(1 - \frac{iv}{\rho + n - \delta}\right)\left(1 + \frac{iv}{n - \delta}\right)\right|,$$

it follows that

$$\left|\frac{\Gamma(\rho + 1 + \delta)\, \Gamma(1 + \delta)}{\Gamma(\rho + 1 + \delta - iv)\, \Gamma(1 + \delta + iv)}\right| \leqslant |E(iv)| \leqslant \left|\frac{\Gamma(\rho + 1 - \delta)\, \Gamma(1 - \delta)}{\Gamma(\rho + 1 - \delta - iv)\, \Gamma(1 - \delta + iv)}\right|,$$

and (3.5) then follows from a classical property (**9**, p. 259) of the Γ-function.

Proof of (3.6). We give details for the case $v > 0$. Writing

$$\phi_n = \operatorname{amp}\{(1 - iv\theta/\lambda_n)(1 + iv\theta/\mu_n)/(1 - iv/\lambda_n)(1 + iv/\mu_n)\},$$

and using the inequalities for λ_n, μ_n and ρ in (2.1), we have

$$\begin{aligned}\phi_n &= \arctan\left(\frac{v(1-\theta)\lambda_n}{v^2\theta + \lambda_n^2}\right) - \arctan\left(\frac{v(1-\theta)\mu_n}{v^2\theta + \mu_n^2}\right),\\ &< \arctan\left(\frac{v(1-\theta)(\rho + n + \delta)}{v^2\theta + (\rho + n - \delta)^2}\right) - \arctan\left(\frac{v(1-\theta)(n-\delta)}{v^2\theta + (n+\delta)^2}\right),\\ &< \arctan\left(\frac{v(1-\theta)(n+1-\delta)}{v^2\theta + (n-\delta)^2}\right) - \arctan\left(\frac{v(1-\theta)(n-\delta)}{v^2\theta + (n+\delta)^2}\right),\\ &= \arctan\frac{v(1-\theta)\{v^2\theta + n^2(1+4\delta) + 2n\delta(1-2\delta) + \delta^2\}}{[v^2\theta + (n-\delta)^2][v^2\theta + (n+\delta)^2] + v^2(1-\theta)^2(n+1-\delta)(n-\delta)}.\end{aligned}$$

On observing that

$$0 < v^2\theta + n^2(1 + 4\delta) + 2n\delta(1 - 2\delta) + \delta^2 < 2[v^2\theta + (n + \delta)^2],$$

we see that $\phi_n < \arctan\{2v(1 - \theta)/[v^2\theta + (n - \delta)^2]\}$. A similar argument applied to $-\phi_n$ gives

$$-\phi_n < \arctan\{3v(1 - \theta)/[v^2\theta + (n - \delta)^2]\};$$

and thus

$$|\phi_n| < \arctan\{3v(1 - \theta)/[v^2\theta + (n - \delta)^2]\}.$$

It then follows easily that $|\operatorname{amp}[E(iv\theta)/E(iv)]| < A(1 - \theta)$, the constant being independent of v.

Proof of (3.7). Let Λ be the region consisting of the w-plane from which the points $v = 0$, $|u| \geqslant 1 - \delta$ have been removed. Then the series

$$\sum_1^\infty \left(\frac{w}{\lambda_n - w} - \frac{\theta w}{\lambda_n - \theta w}\right), \quad \sum_1^\infty \left(\frac{w}{\mu_n - w} - \frac{\theta w}{\mu_n - \theta w}\right)$$

converge absolutely and uniformly in any compact subset of Λ, and

$$\begin{aligned}\frac{d}{dv}\log\left[\frac{E(iv\theta)}{E(iv)}\right] &= i\sum_1^\infty \left(\frac{\lambda_n + iv}{\lambda_n^2 + v^2} - \frac{\theta(\lambda_n + iv\theta)}{\lambda_n^2 + v^2\theta^2}\right)\\ &\quad - i\sum_1^\infty \left(\frac{\mu_n - iv}{\mu_n^2 + v^2} - \frac{\theta(\mu_n - iv\theta)}{\mu_n^2 + v^2\theta^2}\right),\end{aligned}$$

$$\Re\left\{\frac{d}{dv}\log\frac{E(iv\theta)}{E(iv)}\right\} = -\sum_1^\infty \left[\frac{v\lambda_n^2(1-\theta^2)}{(\lambda_n^2 + v^2)(\lambda_n^2 + v^2\theta^2)} + \frac{v\mu_n^2(1-\theta^2)}{(\mu_n^2 + v^2)(\mu_n^2 + v^2\theta^2)}\right] < 0.$$

It follows that $|E(iv\theta)/E(iv)|$ is a decreasing function of $|v|$.

4. Representation of the operator. Let p be the index conjugate to q, so that $p \geqslant 2$. By (3.4), the function

$$k(y) = \underset{R\to\infty}{\overset{(p)}{\lim}} (2\pi)^{-\frac{1}{2}} \int_{-R}^{R} E(u)\exp(-iuy)\,du$$

exists and belongs to $L^p(-\infty, \infty)$. On considering the contour integral $\int E(w) \exp(-iyw)\, dw$ taken round the boundary of the semi-circular disc $|w| \leqslant R$, $v \geqslant 0$, we see that the part of the integral taken round the arc of the semi-circle is $O(R^{-\beta})$, $(R \to \infty)$ when $|y| > \pi$, where $\beta = \rho + 1 - 4\delta > 0$ in (3.1). Thus

$$(4.1) \qquad k(y) = 0 \qquad p.p. \text{ in } |y| > \pi.$$

Since $E(u)$ is continuous

$$(4.2) \qquad E(u) = \lim_{R\to\infty} (2\pi)^{-\frac{1}{2}} \int_{-R}^{R} k(y)[1 - |y|/R] \exp(iuy)\, dy,$$
$$= (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} k(y) \exp(iuy)\, dy,$$

and $k(y) \in L^2(-\pi, \pi)$ by (4.1) and $p \geqslant 2$.

It is easily seen that for complex w, the integral

$$(2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} k(y) \exp(iyw)\, dy$$

defines an entire function, and by (4.2) we may write

$$(4.3) \qquad E(w) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} k(y) \exp(iyw)\, dy.$$

In proving the inversion theorem we shall make use of the function $E(\theta w)$, $0 < \theta \leqslant 1$, for which we prove

$$(4.4) \qquad |E(\theta r_n)| \leqslant A(1 - \theta) r_n,$$

where r_n stands for λ_n or $-\mu_n$ and the constant is independent of n.

Let $m(y, \alpha) = \exp(-iy\alpha\theta) - \exp(-iy\alpha)$. Then for $|y| < \pi$, and $0 < \theta \leqslant 1$,

$$|m(y, \alpha)| = |\alpha| \left| \int_{\theta y}^{y} \exp(-it\alpha)\, dt \right| \leqslant \pi |\alpha| (1 - \theta);$$
$$|E(-\theta\mu_k) - E(-\mu_k)| = (2\pi)^{-\frac{1}{2}} \left| \int_{-\pi}^{\pi} k(y)\, m(y, \mu_k)\, dy \right| \leqslant A(1 - \theta)\, \mu_k,$$

by the Schwarz inequality.

With the usual interpretation of $\exp(aD)$ as a shift operator, we have formally

$$DE(D).f(x) = \lim_{\theta\to 1} DE(\theta D).f(x),$$
$$= \lim_{\theta\to 1} (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} k(y) \exp(iy\theta D)\, dy.f'(x),$$
$$= \lim_{\theta\to 1} (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} k(y) f'(x + iy\theta)\, dy.$$

We therefore *define* the operation $DE(D).f(x)$ by

$$(4.5) \qquad DE(D).f(x) = \lim_{\theta\to 1} (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} k(y) f'(x + iy\theta)\, dy.$$

5. Properties of the nucleus. Denoting the strip $|y| < \pi$ of the z-plane by B, we prove the following propositions:

(5.1) *the integral* (2.3) *defining* $G(z)$ *converges absolutely when* $z \in B$, *converges uniformly when* z *belongs to a compact subset of* B, *and therefore defines a function analytic in* B;

$$(5.2) \qquad G(z) = \begin{cases} 1 + O[\exp(-x\mu_1)], & (x \to \infty \text{ in } B), \\ O[\exp(x\lambda_1)], & (x \to -\infty \text{ in } B); \end{cases}$$

$$(5.3) \qquad G'(z) = \begin{cases} O[\exp(-x\mu_1)], & (x \to \infty \text{ in } B), \\ O[\exp(x\lambda_1)], & (x \to -\infty \text{ in } B); \end{cases}$$

(5.4) *when* $z_0 \in B$, *there is a constant* R_0 *such that the integrals*

$$\int_{-\infty}^{-R_0} \left| \frac{d}{dt} \frac{G(z-t)}{G(z_0-t)} \right| dt, \quad \int_{R_0}^{\infty} \left| \frac{d}{dt} \frac{G(z-t)}{G(z_0-t)} \right| dt$$

converge uniformly when z *belongs to any compact subset of* B.

Proof of (5.1). By (2.3), (3.2),

$$\left| \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\exp(zw)\, dw}{wE(w)} \right| \leqslant A_1 \exp(cx) \int_{-\infty}^{\infty} \exp[-yv - (\pi - \epsilon)|v|]\, dv$$
$$\leqslant A_2 \exp(cx),$$

since $|y| < \pi$ in B. This inequality is sufficient to establish the assertions of (5.1).

Proof of (5.2), (5.3). On account of the classical properties of Dirichlet series it is sufficient to show that

$$(5.5) \qquad G(z) = \begin{cases} 1 - \sum_1^\infty \exp(-z\mu_n)/\mu_n E'(-\mu_n), & (x > 0,\ |y| < \pi), \\ - \sum_1^\infty \exp(z\lambda_n)/\lambda_n E'(\lambda_n), & (x < 0,\ |y| < \pi), \end{cases}$$

the Dirichlet series converging absolutely in the indicated regions.

Details are given for the case $x < 0$. Designating the points $c - iR$, $c + R\cot\beta - iR$, $c + R\cot\beta + iR$ and $c + iR$ by A, B, C and D respectively, where $0 < \beta < \frac{1}{2}\pi$, let L be the contour formed by the linear segments AB and CD and the circular arc $|w - c| = R$ joining to B to C. Consider

$$I = \int_L \exp(zw)\, dw/wE(w).$$

By using (3.2) and $x < 0$, $0 < \beta < \frac{1}{2}\pi$, $|y| < \pi$, and estimating the integrals along AB, BC and CD separately, we see that $I = o(1)$ as $R \to \infty$. The second equation in (5.5) then follows from the definition of $G(z)$ and the calculus of residues.

Proof of (5.4). When $x \neq 0$, $G(z)$ is represented by the absolutely convergent Dirichlet series (5.5), and it is well known that functions so defined can have but a finite number of zeros. Since z_0 is given, and $X_1 \leqslant x \leqslant X_2$

in the compact subset of B, we may choose R_0 so that $G(z_0 - t)$ does not vanish for $|t| \geqslant R$. It then follows easily that

$$\left|\frac{d}{dt}\frac{G(z-t)}{G(z_0-t)}\right| = \begin{cases} O[\exp(t\mu_1)] & (t \to -\infty), \\ O\{\exp[-t(\lambda_2 - \lambda_1)]\} & (t \to \infty), \end{cases}$$

where the constants are independent of z. These estimates are sufficient for the proof.

6. Properties of the transform. The following theorem gives properties of the functions $f(x)$ and $\phi(t)$ in (1.1) which will be used later.

THEOREM I. *Let $\phi(t) \in L(0, R)$ for any R and be such that the integral* (1.1) *converges for at least one z in B, and let $\Phi(t) = \int_0^t \phi(u)du$: then*
(6.1) *the integral* (1.1) *converges for all z in B, and defines a function analytic in B;*

$$\Phi(t) = \begin{cases} o(\exp t\lambda_1) & (t \to \infty), \\ o(\exp - t\Delta) & (t \to -\infty), \end{cases} \tag{6.2}$$

for any positive Δ.

Proof of (6.1). On account of (5.4), the method of Widder-Hirschmann may be used (**4**, pp. 691–692).

Proof of (6.2). It follows from the representation (5.5) of $G(z)$ for $x \neq 0$ that $G(z)$ and $G'(z)$ have at most a finite number of zeros. Let A be a positive number such that neither $G(z)$ nor $G'(z)$ vanishes for $|x| \geqslant A$. To prove the first assertion, define $\psi(t) = \int_0^t G(-A - u)\, d\Phi(u)$. Then by hypothesis, $\psi(\infty)$ is finite, and

$$\begin{aligned} \Phi(t)\exp(-t\lambda_1) &= \exp(-t\lambda_1)\int_0^t d\psi(u)/G(-A-u), \\ &= \exp(-t\lambda_1)\left[\frac{\psi(t)}{G(-A-t)} - \int_0^t \frac{\psi(u)\, G'(-A-u)\, du}{G^2(-A-u)}\right], \\ &= o(1) \end{aligned}$$

as $t \to \infty$ by l'Hopital's rule, and (5.5).

To prove the second assertion, write $\psi(t) = \int_0^t G(A - u)\, d\Phi(u)$; and in the same way, $\Phi(t)\exp(t\Delta) = o(1)$, $(t \to -\infty)$.

It is convenient at this point to establish some properties of the function

$$K(x, \theta) = \theta(2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} k(y)\, G'(x + iy\theta)\, dy. \tag{6.3}$$

These properties are:

$$K(x, \theta) = \begin{cases} (1 - \theta)\, O[\exp(-x\mu_1), & (x \to \infty), \\ (1 - \theta)\, O[\exp(x\lambda_1)], & (x \to -\infty), \end{cases} \tag{6.4}$$

with similar estimates for $K'(x, \theta)$;

(6.5) $K(x, \theta) = O[(1 - \theta)^{-1}]$ *uniformly in x as $\theta \to 1$;*

(6.6) *when x is positive,* $\lim_{\theta \to 1} \int_0^x K(t, \theta)\, dt = \frac{1}{2} = \lim_{\theta \to 1} \int_{-x}^0 K(t, \theta)\, dt.$

Proof of (6.4). Details are given for the case $x > 0$.

$$|K(x,\theta)| = \left|\theta(2\pi)^{-\frac{1}{2}}\int_{-\pi}^{\pi} k(y)\,dy\,(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\exp[w(x+iy\theta)]\,dw/E(w)\right|$$

$$= \left|(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}E(\theta w)\exp(xw)\,dw/E(w)\right|, \qquad by\ (4.3);$$

$$= \left|\sum_1^\infty E(-\theta\mu_n)\exp(-x\mu_n)/E'(-\mu_n)\right|,$$

$$\leqslant A_1(1-\theta)\sum_1^\infty \mu_n\exp(-x\mu_n)/|E'(-\mu_n)|, \qquad by\ (4.4);$$

and as $x > 0$ and (3.3) guarantee the convergence of this series, our assertion is proved.

Proof of (6.5). By (4.3), (6.3) and Cauchy's theorem,

$$K(x,\theta) = \theta(2\pi)^{-\frac{1}{2}}\int_{-\pi}^{\pi} k(y)\,dy\,(2\pi i)^{-1}\int_{-i\infty}^{i\infty}\exp[w(x+iy\theta)]dw/E(w),$$

$$= \theta(2\pi)^{-1}\int_{-\infty}^{\infty}\exp(ixv)\,E(iv\theta)\,dv/E(iv).$$

Using the fact that for $0 < \theta < 1$,

$$|(1-iv\theta/\lambda_n)/(1-iv\lambda_n)| \text{ and } |(1+iv\theta/\mu_n)/(1+iv/\mu_n)|$$

are less than unity, we have

$$\left|\frac{1-iv\theta/(\rho+n-\delta)}{1-iv/(\rho+n-\delta)}\right| \leqslant \left|\frac{1-iv\theta/\lambda_n}{1-iv/\lambda_n}\right| \leqslant \left|\frac{1-iv\theta/(\rho+n+\delta}{1-iv/(\rho+n+\delta}\right|,$$

with similar inequalities involving μ_n. Hence

$$\left|\frac{\Gamma(\rho+1-\delta-iv)\,\Gamma(1-\delta+iv)}{\Gamma(\rho+1-\delta-iv\theta)\,\Gamma(1-\delta+iv\theta)}\right| \leqslant \left|\frac{E(iv\theta)}{E(iv)}\right|$$
$$\leqslant \left|\frac{\Gamma(\rho+1+\delta-iv)\,\Gamma(1+\delta+iv)}{\Gamma(\rho+1+\delta-iv\theta)\,\Gamma(1+\delta+iv\theta)}\right|,$$

and by (9, p. 259),

(6.7) $$|E(iv\theta)/E(iv)| \sim A\exp[-\pi|v|(1-\theta)] \qquad (|v|\to\infty).$$

Since $0 < \theta < 1$, this is sufficient to prove our result.

Proof of (6.6). Write $I = \int_0^x K(t,\theta)\,dt$, where $x \neq 0$. Then

$$I = \theta(2\pi)^{-\frac{1}{2}}\int_0^x dt\int_{-\pi}^{\pi}k(y)\,G'(t+iy\theta)\,dy,$$

$$= \theta(2\pi)^{-\frac{1}{2}}\int_{-\pi}^{\pi}k(y)[G(x+iy\theta)-G(iy\theta)]\,dy,$$

the interchange of the integrations being justified, since $k(y) \in L^2(-\pi,\pi)$ and

$$|G'(t+iy\theta)| = |(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} \exp[w(t+iy\theta)]\,dw/E(w)|,$$
$$= |(2\pi)^{-1}\int_{-\infty}^{\infty} \exp(-vy\theta + ivt)\,dv/E(iv)|,$$
$$= \int_{-\infty}^{\infty} \exp[-\pi|v|(1-\theta)]\,O[|v|^{\rho+1+2\delta}]\,dv, \text{ by (3.5)},$$
$$= O[(1-\theta)^{-\rho-2-2\delta}].$$

Thus

$$I = \theta(2\pi)^{-\frac{1}{2}}\int_{-\pi}^{\pi} k(y)dy(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\{\exp[w(x+iy\theta)]-\exp(iy\theta w)\}dw/wE(w).$$

Again by (4.3) and the absolute convergence of the inner integral for $|y| < \pi$, we may interchange the integrations, and get

$$I = \frac{\theta}{2\pi}\int_{-\infty}^{\infty}\frac{[\exp(ixv) - 1]\,E(iv\theta)}{ivE(iv)}\,dv, \tag{6.8}$$

the application of Cauchy's theorem being justified by the analyticity of $[\exp(xw) - 1]/w$ at the origin.

We observe next that $E(\theta w)/E(w)$ is real when w is real. Hence $I = P - Q$, where

$$Q = \frac{\theta}{2\pi}\int_0^{\infty}\frac{1-\cos xv}{v}\,\Im\left[\frac{E(iv\theta)}{E(iv)}\right]dv,$$
$$P = \frac{\theta}{2\pi}\int_0^{\infty}\frac{\sin xv}{v}\,\Re\left[\frac{E(iv\theta)}{E(iv)}\right]dv.$$

It is then sufficient to show that

$$Q \to 0, \qquad \theta \to 1, \tag{6.9}$$
$$P \to \tfrac{1}{2}, \qquad \theta \to 1. \tag{6.10}$$

To prove (6.9), it is sufficient to consider

$$Q_1 = \int_1^{\infty}\frac{1-\cos xv}{v}\,\Im\left[\frac{E(iv\theta)}{E(iv)}\right]dv.$$

We then have

$$|Q_1| = \left|\int_1^{\infty}\frac{1-\cos xv}{v}\left|\frac{E(iv\theta)}{E(iv)}\right|\sin\operatorname{amp}\left[\frac{E(iv\theta)}{E(iv)}\right]dv\right|,$$
$$\leqslant A_1(1-\theta)\int_1^{\infty} v^{-1}\exp[-\pi v(1-\theta)]\,dv, \qquad \textit{by } (3.6) \textit{ and } (6.7),$$
$$= A_1(1-\theta)\int_{\pi(1-\theta)}^{\infty} t^{-1}\exp(-t)\,dt < A_2(1-\theta)^{\frac{1}{2}}\Gamma(\tfrac{1}{2}).$$

Thus Q_1, and consequently Q tends to zero as θ tends to unity.

To prove (6.10), we observe that on account of (3.7),

$$\int_0^{\infty}\frac{\sin xv}{v}\left|\frac{E(iv\theta)}{E(iv)}\right|dv$$

converges uniformly in $\frac{1}{2} \leqslant \theta \leqslant 1$; and from (3.6), that $\Re[E(iv\theta)/E(iv)]$ is positive when θ is close to unity. It is therefore sufficient to prove that $C(\theta) \to 0$ $(\theta \to 1)$, where

$$C(\theta) = \int_0^\infty \frac{\sin xv}{v}\left\{\left|\frac{E(iv\theta)}{E(iv)}\right| - \Re\left[\frac{E(iv\theta)}{E(iv)}\right]\right\} dv.$$

But

$$\begin{aligned} |C(\theta)| &\leqslant \int_0^\infty \left|\frac{E(iv\theta)}{E(iv)}\right| \left\{1 - \cos \operatorname{amp}\left[\frac{E(iv\theta)}{E(iv)}\right]\right\} dv, \\ &\leqslant A_3(1-\theta)^2 \int_0^\infty \exp[-\pi v(1-\theta)]\, dv, \\ &= O[(1-\theta)], \qquad \text{by (3.6).} \end{aligned}$$

7. The inversion theorems. The main result is

THEOREM II. *Let $\phi(t) \in L(0, R)$ for any R and be such that the integral* (1.1) *converges for at least one z in the strip B: then if $f(z)$ is defined by* (1.1) *and $DE(D).f(x)$ by* (4.5),

$$DE(D).f(x) = \tfrac{1}{2}[\phi(x+) + \phi(x-)],$$

whenever the right-hand side has a meaning.

For

$$\begin{aligned} DE(D).f(x) &= \lim_{\theta\to 1} (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} k(y)\, dy \int_{-\infty}^{\infty} G'(x-t+iy\theta)\, \phi(t)\, dt, \\ &= \lim_{\theta\to 1} \int_{-\infty}^{\infty} \phi(t)\, dt\, (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} k(y)\, G'(x-t+iy\theta)\, dy, \\ &= \lim_{\theta\to 1} \int_{-\infty}^{\infty} K(x-t, \theta)\, \phi(t)\, dt, \end{aligned}$$

the interchange of the integrations being justified by the uniform convergence of (1.1) in any compact subset of B, and the fact that $k(y) \in L^2(-\pi, \pi)$. It is sufficient to prove

$$\int_{-\infty}^{x} K(x-t, \theta)\, \phi(t)\, dt \to \tfrac{1}{2}\phi(x-), \tag{7.1}$$

$$\int_{x}^{\infty} K(x-t, \theta)\, \phi(t)\, dt \to \tfrac{1}{2}\phi(x+), \tag{7.2}$$

as $\theta \to 1$.

We give details for (7.1). Let $T > 0$, and write

$$\int_{x-T}^{x} K(x-t,\theta)[\phi(t)-\phi(x-)]dt = \int_0^T K(t,\theta)[\phi(x-t)-\phi(x-)]dt = J(0, T).$$

Then

$$\begin{aligned} |J[0, \pi(1-\theta)]| &\leqslant \int_0^{\pi(1-\theta)} |\phi(x-t) - \phi(x-)||K(t,\theta)|dt \\ &\leqslant A(1-\theta)^{-1} \int_0^{\pi(1-\theta)} |\phi(x-t) - \phi(x-)|dt \\ &= o(1), \end{aligned} \tag{7.3}$$

as $\theta \to 1$, by (6.5). Next by (6.4),

$$(7.4)\qquad |J[\pi(1-\theta), T]| \leqslant A(1-\theta)\int_{\pi(1-\theta)}^{T} |\phi(x-t)-\phi(x-)|\exp(-t\mu_1)dt = O(1-\theta).$$

Thus by (6.6), (7.3) and (7.4)

$$\lim_{\theta\to 1}\int_{x-T}^{x} K(x-t,\theta)\,\phi(t)\,dt = \tfrac{1}{2}\phi(x-).$$

It remains to prove that $\int_{-\infty}^{x-T} K(x-t,\theta)\,\phi(t)\,dt \to 0$ as $\theta\to 1$. As this integral need not converge absolutely, we write it as

$$[K'(x-t,\theta)\,\Phi(t)]_{-\infty}^{T} + \int_{-\infty}^{x-T} K'(x-t,\theta)\,\Phi(t)\,dt.$$

By (6.2) and (6.4) the integrated term $= o(1)$; and for the same reason

$$\left|\int_{-\infty}^{x-T} K'(x-t,\theta)\,\Phi(t)\,dt\right| = \left|\int_{T}^{\infty} K'(t,\theta)\,\Phi(x-t)\,dt\right| = O(1-\theta).$$

Since (7.2) may be proved in the same way, the theorem is complete.

The proof of the following theorem is similar:

Theorem III. *Let $f(z) = \int_{-\infty}^{\infty} G(z-t)\,d\alpha(t)$, where $\alpha(t)$ is a normalized function of bounded variation in any finite interval: then if this integral converges for any z in B, it converges for all such z, converges uniformly in any compact subset of B, and defines a function analytic in B. Also*

$$\lim_{\theta\to 1}(2\pi)^{-\frac{1}{2}}\int_{x_1}^{x_2} dx\int_{-\pi}^{\pi} k(y)\,f'(x+iy\theta)\,dy = \alpha(x_2)-\alpha(x_1).$$

8. Remarks. In the proof of (5.4) we have used the fact that from its representation (5.5) as a Dirichlet series, the nucleus $G(z)$ has but a finite number of zeros. Hirschmann and Widder (**8**, p. 159) have shown that a more general nucleus has no zeros on the real axis, and it is certainly true that $G(iy) \neq 0$ for $|y| < \pi$. The proof that $G(z)$ does not vanish in B seems to be connected with properties of functions defined by Dirichlet series with coefficients of alternating sign, and will be dealt with elsewhere.

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