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EQUILIBRIUM POINTS FOR A SYSTEM INVOLVING m-ACCRETIVE OPERATORS

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Abstract Let E be a real uniformly smooth Banach space and let A be a nonlinear ϕ-strongly quasi-accretive operator with range $R(A)$ and open domain $D(A)$ in E. For a given $f \in E$, let A satisfy the evolution system $\mathrm{d}u(t)/\mathrm{d}t + Au(t) = f$, $u(0) = u_0$. We establish the strong convergence of the Ishikawa and Mann iterative methods with appropriate error terms recently introduced by Xu to the equilibrium points of this system. Related results deal with the strong convergence of the iterative methods to the fixed points of ϕ-strong pseudocontractions defined on open subsets of E.

Keywords: uniformly smooth spaces; accretive operators; pseudocontractive operators; Ishikawa and Mann iterative methods; fixed points; equilibrium points

AMS 2000 *Mathematics subject classification:* Primary 47H06; 47H15; 47H17

1. Introduction and preliminaries

Let E be a real normed space. A mapping U with domain $D(U)$ and range $R(U)$ in E is called *accretive* if the inequality

$$\|x - y\| \leqslant \|x - y + s(Ux - Uy)\| \tag{1.1}$$

holds for every $x, y \in D(U)$ and for all $s > 0$. For a Banach space E we shall denote by J the normalized duality map from E to 2^{E^*} defined by $Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}$, where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. As a consequence of a result of Kato [**24**], the accretive condition (1.1) can be expressed in terms of the normalized duality map as follows. For each $x, y \in D(U)$, there exists $j(x-y) \in J(x-y)$ such that $\langle Ux-Uy, j(x-y) \rangle \geqslant 0$. A mapping $T : D(T) \subseteq E \to E$ is called *strongly accretive* if for all $x, y \in D(T)$ there exist a constant $k > 0$ and $j(x - y) \in J(x - y)$ such that $\langle Tx - Ty, j(x - y) \rangle \geqslant k\|x - y\|^2$. The map T is called ϕ*-strongly accretive* if there exist $j(x - y) \in J(x - y)$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for each $x, y \in E$:

$$\langle Tx - Ty, j(x - y) \rangle \geqslant \phi(\|x - y\|)\|x - y\|. \tag{1.2}$$

The operator T is called *m-accretive* if it is accretive and $(I+rT)(D(T))=E$ for all $r>0$, where I denotes the identity operator on $D(T)$. Moreover, if the nullspace of T, $N(T):=\{x\in D(T):Tx=0\}$, is non-empty and the relations (1.1) and (1.2) hold for any $x\in D(T)$ but $y\in N(T)$, then the corresponding operator T is called *quasi-accretive*, *quasi ϕ-strongly accretive*, respectively.

Closely related to the class of accretive operators is the class of *pseudocontractive* operators. An operator T with domain $D(T)$ and range $R(T)$ in E is called *strongly pseudocontractive* if for all $x,y\in D(T)$, there exist $j(x-y)\in J(x-y)$ and a constant $t>1$ such that $\langle Tx-Ty,j(x-y)\rangle\leqslant t^{-1}\|x-y\|^2$. If $t=1$, then T is called *pseudocontractive*. The map T is called *ϕ-strongly pseudocontractive* if for all $x,y\in D(T)$, there exist $j(x-y)\in J(x-y)$ and a strictly increasing function $\phi:[0,\infty)\to[0,\infty)$ with $\phi(0)=0$ such that

$$\langle Tx-Ty,j(x-y)\rangle\leqslant\|x-y\|^2-\phi(\|x-y\|)\|x-y\|. \tag{1.3}$$

T is called *ϕ-hemicontractive* if relation (1.3) holds for all $x\in D(T)$ and $x^*\in N(I-T)$. It follows from inequalities (1.2) and (1.3) that T is ϕ-strongly pseudocontractive if and only if $(I-T)$ is ϕ-strongly accretive, so that the mapping theory for accretive operators is intimately connected with the fixed-point theory for pseudocontractions.

The notion of accretive operators was introduced in 1967 by Browder [**1**] and Kato [**24**]. An early fundamental result in the theory of accretive operators, due to Browder, states that the initial-value problem

$$\frac{\mathrm{d}u}{\mathrm{d}t}+Au=0,\quad u(0)=u_0, \tag{1.4}$$

is solvable if A is *locally* Lipschitzian and accretive on E. This result was subsequently generalized by Martin [**28**] to the *continuous* accretive operators. It is well known (see, for example, [**10**, **39**]) that many physically significant problems can be modelled in terms of an initial-value problem of the form (1.4), where T is either accretive, ϕ-strongly accretive or ϕ-strongly quasi-accretive. Typical examples of how such evolution equations arise are found in models involving either the heat, the wave or the Schrödinger equation. If u is independent of t,

$$Au=0, \tag{1.5}$$

and the solution of this equation corresponds to the equilibrium points of system (1.4). Consequently, considerable research effort has been devoted, especially within the last 10 years or so, to developing constructive techniques for the determination of the kernels of accretive operators in Banach spaces (see, for example, [**2**, **3**, **5**–**23**, **25**, **26**, **29**–**33**, **35**, **37**, **39**–**41**]).

Two well-known iteration schemes—the *Mann iteration scheme* (see, for example, [**27**]) and the *Ishikawa iteration scheme* (see, for example, [**23**])—have successfully been employed to approximate the kernels of accretive *self-maps*. However, if a map T is such that its range does not intersect its domain (and this occurs frequently in many applications), then neither the Mann nor the Ishikawa scheme may be well defined. In fact, equation (1.5) may not even have a solution. Take for example $T : [0,1] \to \mathbb{R}$ defined by $Tx = \frac{1}{2}x + 1$. Clearly, T is a contraction map but has no fixed point in $[0,1]$. Hence $A := (I - T) : [0,1] \to \mathbb{R}$ is strongly accretive and $Au = 0$ has no solution. In general, if $D(A) \cap R(A) = \emptyset$, some extra condition on A must be imposed for equation (1.5) to have a solution. One such condition which has been studied extensively and for which fixed-point theorems have been established is that of 'weak inwardness' (see, for example, [**4**]). For weakly inward maps no generality is lost if the domain is assumed to be an open set (see, for example, [**4**, ch. IV]). In [**2**], Bruck studied the problem of iteratively approximating the solution of $Au = f$ for a given $f \in H$, where $A := (I + T)$ and $T : H \to H$ is a monotone operator on a Hilbert space H. (In Hilbert spaces accretive operators are called *monotone*.) He proved the following theorem.

Theorem B (see page 1259 in [2]). *Let T be a multivalued monotone operator with open domain $D(T)$ in a Hilbert space H and $f \in R(I + T)$. Then there exist a neighbourhood $N \subseteq D(T)$ of $x^* = (I + T)^{-1}f$ and a real number $\sigma_1 > 0$ such that for any $\sigma \geqslant \sigma_1$, any initial guess $x_1 \in N$ and any single-valued section T_0 of T, the sequence $\{x_n\}$ generated from x_1 by*

$$x_{n+1} = x_n - (n + \sigma)^{-1}(x_n + T_0 x_n - f)$$

remains in $D(T)$ and converges to x^* with $\|x_n - x^*\| = O(n^{-1/2})$.

Let E be a real Banach space. The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) := \tfrac{1}{2} \sup\{\|x + y\| + \|x - y\| - 2 : \|x\| \leqslant 1,\ \|y\| \leqslant \tau\}.$$

For $q > 1$, E is called *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho_E(\tau) \leqslant c\tau^q$, and is called *uniformly smooth* if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

Clearly, every q-uniformly smooth Banach space is uniformly smooth. Moreover, it is well known (see, for example, [**36**, **38**]) that Hilbert spaces are 2-uniformly smooth, while

$$L_p (\text{or } \ell_p) \quad \text{or} \quad W_m^p \text{ is } \begin{cases} p\text{-uniformly smooth} & \text{if } 1 < p \leqslant 2, \\ 2\text{-uniformly smooth} & \text{if } p \geqslant 2. \end{cases}$$

Since its publication in 1973, several authors have extended Theorem B to more general Banach spaces (see, for example, [**10**, **11**, **13**, **15**, **40**]). The most general result of type Theorem B now known appears to be the following recent theorem of one of the authors.

Theorem C (see Theorem 4.1 on page 56 in [11]). *Let E be a real q-uniformly smooth Banach space, $q > 1$. Suppose A is a set-valued strongly accretive operator with open domain $D(A)$ in E and $f \in Ax$ has a solution $x^* \in D(A)$. Then there exist a neighbourhood $B \subseteq D(A)$ of x^* and real number $r_1 > 0$ such that for any $r > r_1$, any* *initial guess* $x_1 \in B$, *any single-valued selection* A_0 *of* A *and some real sequence* $\{c_n\}_{n=1}^{\infty}$, *the sequence* $\{x_n\}_{n=1}^{\infty}$ *generated from* x_1 *by*

$$x_{n+1} = x_n + c_n(f - A_0 x_n), \quad n \geqslant 1,$$

remains in $D(A)$ *and converges strongly to* x^* *with* $\|x_n - x^*\| = O(n^{(q-1)/q})$.

In 1995, Liu [**25**] first introduced what he called Ishikawa and Mann iteration processes 'with errors' for nonlinear strongly accretive mappings. Recently, Yuguang Xu [**37**] objected to the definition of Liu (see [**37**]) and then introduced the following definitions.

(A) Let K be a non-empty convex subset of E and $T : K \to K$ a mapping. For any given $x_0 \in K$, the sequence $\{x_n\}$, defined iteratively by

$$x_{n+1} = a_n x_n + b_n T y_n + c_n u_n; \quad y_n = a'_n x_n + b'_n T x_n + c'_n v_n, \quad n \geqslant 0,$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in K, and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$ are sequences in $(0, 1)$ such that $a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n$ for all integers $n \geqslant 0$, is called the *Ishikawa iteration sequence with errors*.

(B) If, with the same notation and definitions as in (A), $b'_n = c'_n = 0$ for all integers $n \geqslant 0$, then the sequence $\{x_n\}$, now defined by $x_0 \in K$, $x_{n+1} = a_n x_n + b_n T x_n + c_n u_n$, $n \geqslant 0$, is called the *Mann iteration sequence with errors*.

It is our purpose in this paper to consider iteration processes of the *Ishikawa and Mann types with errors in the sense of Yuguang* [**37**] and prove theorems much more general than Theorem C in the sense that our *theorems hold for the much more general class of* ϕ*-strongly accretive operators* and in *the more general real uniformly smooth spaces*. Moreover, our method is of independent interest.

2. Main results

In the sequel we shall need the following lemma.

Lemma X–R (see [38, 39]). *Let* E *be a real uniformly smooth Banach space. Then, for every* $x, y \in E$, *and some positive constants* C *and* D, *we have*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x)\rangle + D \max\{\|x\| + \|y\|, \tfrac{1}{2}C\}\rho_E(\|y\|). \tag{2.1}$$

2.1. Convergence theorems for *Φ*-strongly accretive operator equations

Theorem 2.1. *Let* E *be a real uniformly smooth Banach space and let* $T : D(T) \subseteq E \to E$ *be a* ϕ*-strongly accretive operator with open domain* $D(T) \subseteq E$. *Suppose the equation* $Tx = f$ *has a solution* $x^* \in D(T)$ *for a given* $f \in E$. *Define* $S : D(T) \to E$ *by* $Sx = x - Tx + f$. *Then there exist a real number* $\mu \geqslant 1$ *and a neighbourhood* $B \subseteq D(T)$ *of* x^* *such that starting with arbitrary* $x_0, u_0, v_0 \in B$, *the sequence* $\{x_n\}_{n=0}^{\infty}$ *defined iteratively by*

$$y_n = a_n x_n + b_n S x_n + c_n u_n, \quad n \geqslant 0, \tag{2.2}$$

$$x_{n+1} = a'_n x_n + b'_n S y_n + c'_n v_n, \quad n \geqslant 0, \tag{2.3}$$

remains in B and converges strongly to x^*, where

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$;

(ii) $b_n = b'_n = 1/(2(\mu + n))$, $n \geqslant 0$;

(iii) $c_n = c'_n = 1/(2(n + \mu)^2)$, $n \geqslant 0$; *and*

(iv) $\{u_n\}$ *and* $\{v_n\}$ *are arbitrary bounded sequences in* E.

Proof. Set $\beta_n = b_n + c_n$, $\alpha_n = b'_n + c'_n$, then (2.2) and (2.3) become

$$y_n = (1 - \beta_n)x_n + \beta_n S x_n + c_n(u_n - Sx_n), \quad n \geqslant 0, \tag{2.4}$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n + c'_n(v_n - Sy_n), \quad n \geqslant 0. \tag{2.5}$$

Observe that x^* is a fixed point of S and that $(I - S)$ is accretive and, hence, is locally bounded at each interior point of its effective domain. Choose $r > 0$ such that $B = \bar{B}_r(x^*) = \{x \in E : \|x - x^*\| \leqslant r\}$ is contained in $D(T) = D(S)$, and $(I - S)(B)$ is bounded. Let $d := \operatorname{diam}[(I - S)(B)]$, and $M_1 = \max\{\sup \|u_n - x^*\|, \sup \|v_n - x^*\|\}$. Then, for all $x \in B$,

$$\|u_n - Sx\| \leqslant \|u_n - x^*\| + \|x - x^*\| + \|x - Sx\| \leqslant M_1 + r + d.$$

Similarly, $\|v_n - Sx\| \leqslant M_1 + r + d$. Set $d^* := 2(r + d) + M_1$, $M = D \max\{d^*, \frac{1}{2}C\}$, where D and C are the constants appearing in inequality (2.1). Since E is uniformly smooth,

$$\frac{\rho_E(d^*\tau)}{\tau} \to 0 \quad \text{as } \tau \to 0.$$

Thus we can choose $\tau_0 > 0$ such that for all $0 < \tau < \tau_0$, we have

$$\frac{\rho_E(d^*\tau)}{\tau} \leqslant \frac{\phi(\frac{1}{2}r)r^2}{3M(1 + \phi(r) + r)}.$$

Furthermore, the uniform continuity of j on bounded subsets of E implies that given

$$\epsilon := \frac{\phi(\frac{1}{2}r)r^2}{6d(1 + \phi(r) + r)},$$

we can choose $\delta_\epsilon > 0$ such that $\|x - y\| < \delta_\epsilon$ implies $\|j(x) - j(y)\| \leqslant \epsilon$. Choose $\delta_0 > 0$ such that $\delta_0 < \delta_\epsilon$; define

$$\mu := \max\left\{1, \frac{1}{\tau_0}, \frac{8d^*}{r}, \frac{d^*}{\delta_0}, \frac{24d^*(1 + \phi(r) + r)}{\phi(\frac{1}{2}r)r}\right\}$$

and generate the sequence $\{x_n\}$ iteratively by (2.4) and (2.5). We show that $\{x_n\}$ is well defined and is in B. To do this we first prove that $y_n \in B$ whenever $x_n \in B$. Suppose, for

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contradiction, that there exists $n \geqslant 0$ such that $x_n \in B$ and $y_n \notin B$. Then $\|x_n - x^*\| \leqslant r$ and $\|y_n - x^*\| > r$. Hence

$$\begin{aligned}\|x_n - x^*\| &\geqslant \|y_n - x^*\| - \beta_n\|x_n - Sx_n\| - c_n\|u_n - Sx_n\| \\ &> r - 2d^*\beta_n \quad (\text{since } \|x_n - Sx_n\| \leqslant d^*,\ \|u_n - Sx_n\| \leqslant d^* \text{ and } c_n \leqslant \beta_n) \\ &\geqslant r - \frac{2d^*}{\mu} \geqslant \tfrac{1}{2}r,\end{aligned}$$

so that $\phi(\|x_n - x^*\|) > \phi(\frac{1}{2}r)$. Since T is ϕ-strongly accretive, inequality (1.2) implies that for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\begin{aligned}\langle (I-S)x - (I-S)y, j(x-y)\rangle &\geqslant \phi(\|x-y\|)\|x-y\| \\ &\geqslant \frac{\phi(\|x-y\|)}{1 + \phi(\|x-y\|) + \|x-y\|}\|x-y\|^2 \\ &= r(x,y)\|x-y\|^2, \end{aligned} \tag{2.6}$$

where

$$r(x,y) = \frac{\phi(\|x-y\|)}{1+\phi(\|x-y\|)+\|x-y\|} \in [0,1), \quad \forall x, y \in D(T).$$

Using (2.6) and inequality (2.1) we obtain

$$\begin{aligned}\|y_n - x^*\|^2 &= \|x_n - x^* - \beta_n(x_n - Sx_n) + c_n(u_n - Sx_n)\|^2 \\ &\leqslant \|x_n - x^*\|^2 - 2\beta_n\langle x_n - Sx_n, j(x_n - x^*)\rangle + 2c_n\langle u_n - Sx_n, j(x_n - x^*)\rangle \\ &\quad + D\max\{\|x_n - x^*\| + \|\beta_n(x_n - Sx_n) + c_n(u_n - Sx_n)\|, \tfrac{1}{2}C\} \\ &\quad \times \rho_E(\|\beta_n(x_n - Sx_n) + c_n(u_n - Sx_n)\|) \\ &\leqslant [1 - 2\beta_n r(x_n, x^*)]\|x_n - x^*\|^2 + \frac{2\beta_n d^* r}{(\mu + n + 1)} + D\max\{d^*, \tfrac{1}{2}C\}\rho_E(d^*\beta_n) \\ &\leqslant [1 - \beta_n r(x_n, x^*)]\|x_n - x^*\|^2 + \frac{2d^* r\beta_n}{(\mu+n+1)} + M\beta_n\frac{\rho_E(d^*\beta_n)}{\beta_n} \\ &\leqslant [1 - \beta_n r(x_n, x^*)]r^2 + 2\frac{d^* r\beta_n}{\mu} + M\beta_n\frac{\rho_E(d^*\beta_n)}{\beta_n}, \end{aligned} \tag{2.7}$$

since $\beta_n r(x_n - x^*) \in [0,1)$. Since $\frac{1}{2}r < \|x_n - x^*\| \leqslant r$, we have

$$r(x_n, x^*) = \frac{\phi(\|x_n - x^*\|)}{(1 + \phi(\|x_n - x^*\| + \|x_n - x^*\|)} \geqslant \frac{\phi(\frac{1}{2}r)}{(1 + \phi(r) + r)},$$

so that inequality (2.7) implies

$$r^2 < r^2 - \beta_n\left[\frac{\phi(\frac{1}{2}r)r^2}{(1+\phi(r)+r)} - \frac{2d^* r}{\mu} - M\frac{\rho_E(d^*\beta_n)}{\beta_n}\right] \leqslant r^2,$$

a contradiction, so that $y_n \in B$. We now prove that $x_n \in B$, $\forall n \geqslant 0$. The proof is by induction. By our choice $x_0 \in B$. Suppose $x_n \in B$. Then $y_n \in B$. We prove that

$x_{n+1} \in B$. Assume for contradiction that $x_{n+1} \notin B$. Then $\|x_{n+1} - x^*\| > r$. Thus we have the following estimates,

$$\begin{aligned}\|x_n - x^*\| &\geqslant \|x_{n+1} - x^*\| - \alpha_n\|x_n - Sy_n\| - c'_n\|v_n - Sy_n\| \\ &> r - 2d^*\alpha_n \quad (\text{since } c'_n \leqslant \alpha_n,\ \|x_n - Sy_n\| \leqslant d^* \text{ and } \|v_n - Sy_n\| \leqslant d^*),\end{aligned}$$

and

$$\begin{aligned}\|y_n - x^*\| &\geqslant \|x_n - x^*\| - \beta_n\|x_n - Sx_n\| - c_n\|u_n - Sx_n\| \\ &\geqslant r - 2d^*\alpha_n - 2d^*\alpha_n = r - 4d^*\alpha_n \geqslant \tfrac{1}{2}r,\end{aligned}$$

so that $\phi(\|y_n - x^*\|) > \phi(\frac{1}{2}r)$. Thus,

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\quad = \|x_n - x^* - \alpha_n(x_n - Sy_n) + c'_n(v_n - Sy_n)\|^2 \\ &\quad \leqslant \|x_n - x^*\|^2 - 2\alpha_n\langle x_n - Sy_n, j(x_n - x^*)\rangle + 2c'_n\langle v_n - Sy_n, j(x_n - x^*)\rangle \\ &\qquad + D\max\{\|x_n - x^*\| + \|\alpha_n(x_n - Sy_n) + c'_n(v_n - Sy_n)\|, \tfrac{1}{2}C\} \\ &\qquad \times \rho_E(\|\alpha_n(x_n - Sy_n) + c'_n(v_n - Sy_n)\| \\ &\quad \leqslant \|x_n - x^*\|^2 - 2\alpha_n\langle x_n - y_n, j(x_n - x^*)\rangle \\ &\qquad - 2\alpha_n\langle y_n - Sy_n, j(x_n - x^*)\rangle + \frac{2\alpha_n d^* r}{(\mu + n + 1)} + M\rho_E(d^*\alpha_n) \\ &\quad \leqslant \|x_n - x^*\|^2 - 2\alpha_n\beta_n\phi(\|x_n - x^*\|)\|x_n - x^*\| + 2c_n\alpha_n\langle u_n - Sx_n, j(x_n - x^*)\rangle \\ &\qquad - 2\alpha_n\langle y_n - Sy_n, j(x_n - x^*)\rangle + \frac{2\alpha_n d^* r}{(\mu + n + 1)} + M\alpha_n\frac{\rho_E(d^*\alpha_n)}{\alpha_n}. \qquad (2.8)\end{aligned}$$

Observe that

$$\begin{aligned}\langle y_n - Sy_n, j(x_n - x^*)\rangle &= \langle y_n - Sy_n, j(x_n - x^*) - j(y_n - x^*)\rangle + \langle y_n - Sy_n, j(y_n - x^*)\rangle \\ &\geqslant r(y_n, x^*)\|y_n - x^*\|^2 + \langle y_n - Sy_n, j(x_n - x^*) - j(y_n - x^*)\rangle. \qquad (2.9)\end{aligned}$$

$$\begin{aligned}\|y_n - x^*\| &= \|x_n - x^* - \beta_n(x_n - Sx_n) + c_n(u_n - Sx_n)\| \\ &\geqslant |\|x_n - x^*\| - \|\beta_n(x_n - Sx_n) - c_n(u_n - Sx_n)\||,\end{aligned}$$

so that

$$\begin{aligned}\|y_n - x^*\|^2 \geqslant \|x_n - x^*\|^2 &- 2\|x_n - x^*\|\|\beta_n(x_n - Sx_n) - c_n(u_n - Sx_n)\| \\ &+ \|\beta_n(x_n - Sx_n) - c_n(u_n - Sx_n)\|^2. \qquad (2.10)\end{aligned}$$

Using (2.10) in (2.9) yields

$$\begin{aligned}&\langle y_n - Sy_n, j(x_n - x^*)\rangle \\ &\quad \geqslant r(y_n, x^*)\|x_n - x^*\|^2 - 2r(y_n, x^*)\|x_n - x^*\|\|\beta_n(x_n - Sx_n) - c_n(u_n - Sx_n)\| \\ &\qquad + r(y_n, x^*)\|\beta_n(x_n - Sx_n) - c_n(u_n - Sx_n)\|^2 \\ &\qquad + \langle y_n - Sy_n, j(x_n - x^*) - j(y_n - x^*), \qquad (2.11)\end{aligned}$$

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and using this in (2.8) we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \leqslant [1 - 2\alpha_n r(y_n, x^*)]\|x_n - x^*\|^2 + 4\alpha_n\|x_n - x^*\|\|\beta_n(x_n - Sx_n) - c_n(u_n - Sx_n)\| \\
&\qquad + 2c_n\alpha_n\langle u_n - Sx_n, j(x_n - x^*)\rangle + \frac{2\alpha_n d^* r}{(\mu + n + 1)} + M\alpha_n\frac{\rho_E(d^*\alpha_n)}{\alpha_n} \\
&\qquad - 2\alpha_n\langle y_n - Sy_n, j(x_n - x^*) - j(y_n - x^*)\rangle \\
&\quad \leqslant [1 - 2\alpha_n r(y_n, x^*)]\|x_n - x^*\|^2 + 4\alpha_n\beta_n d^* r + \frac{2d^* r\alpha_n}{(\mu + n + 1)} + \frac{2d^* r\alpha_n}{(\mu + n + 1)} \\
&\qquad + M\alpha_n\frac{\rho_E(d^*\alpha_n)}{\alpha_n} + 2d\alpha_n\|j(x_n - x^*) - j(y_n - x^*)\| \\
&\quad \leqslant [1 - 2\alpha_n r(y_n, x^*)]\|x_n - x^*\|^2 + \frac{8\alpha_n d^* r}{(\mu + n)} \\
&\qquad + M\alpha_n\frac{\rho_E(d^*\alpha_n)}{\alpha_n} + 2d\alpha_n\|j(x_n - x^*) - j(y_n - x^*)\|. \qquad (2.12)
\end{aligned}$$

Hence

$$\begin{aligned}
r^2 < \|x_{n+1} - x^*\|^2 &\leqslant [1 - \alpha_n r(y_n, x^*)]r^2 + \frac{8d^* r\alpha_n}{\mu} + M\alpha_n\frac{\rho_E(d^*\alpha_n)}{\alpha_n} \\
&\qquad + 2d\alpha_n\|j(x_n - x^*) - j(y_n - x^*)\| \\
&\leqslant r^2 - \alpha_n\biggl[\frac{\phi(\frac{1}{2}r)r^2}{(1 + \phi(r) + r)} - \frac{8d^* r}{\mu} - M\frac{\rho_E(d^*\alpha_n)}{\alpha_n} \\
&\qquad - 2d\|j(x_n - x^*) - j(y_n - x^*)\|\biggr] \leqslant r^2,
\end{aligned}$$

a contradiction, so that $x_{n+1} \in B$. Hence $x_n \in B$, $\forall n \geqslant 0$.

We now prove that $\lim_{n\to\infty} x_n = x^*$. From (2.12) we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leqslant [1 - 2\alpha_n r(y_n, x^*)]\|x_n - x^*\|^2 + \frac{8d^* r\alpha_n}{(\mu + n)} \\
&\qquad + 2\alpha_n d\|j(x_n - x^*) - j(y_n - x^*)\| + M\frac{\alpha_n\rho_E(d^*\alpha_n)}{\alpha_n} \\
&= [1 - 2\alpha_n r(y_n, x^*)]\|x_n - x^*\|^2 + \alpha_n\lambda_n, \qquad (2.13)
\end{aligned}$$

where

$$\lambda_n = \frac{8d^* r}{(\mu + n)} + 2d\|j(x_n - x^*) - j(y_n - x^*)\| + M\frac{\rho_E(d^*\alpha_n)}{\alpha_n}.$$

Observe that $\liminf \|y_n - x^*\| = \liminf \|x_n - x^*\|$. Let $\liminf \|y_n - x^*\| = \delta \geqslant 0$. We prove that $\delta = 0$. Assume for contraction that $\delta > 0$. Then there exists a positive integer N_0 such that $\|y_n - x^*\| \geqslant \frac{1}{2}\delta$ and $\|x_n - x^*\| \geqslant \frac{1}{2}\delta$, $\forall n \geqslant N_0$. Since $\lim \lambda_n = 0$, we can choose a positive integer N_1 such that

$$\lambda_n \leqslant \frac{\delta^2\phi(\frac{1}{2}\delta)}{4(1 + \phi(r) + r)}, \quad \forall n \geqslant N_1.$$

Hence, for all $n \geqslant N = \max\{N_0, N_1\}$ we have

$$\|x_{n+1} - x^*\|^2 \leqslant \|x_n - x^*\|^2 - \frac{\alpha_n \delta^2 \phi(\frac{1}{2}\delta)}{4(1+\phi(r)+r)} - \alpha_n \left[\frac{\delta^2 \phi(\frac{1}{2}\delta)}{4(1+\phi(r)+r)} - \lambda_n\right],$$

so that

$$\frac{\delta^2 \phi(\frac{1}{2}\delta)}{4(1+\phi(r)+r)} \alpha_n \leqslant \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2, \quad \forall n \geqslant N.$$

Thus

$$\frac{\delta^2 \phi(\frac{1}{2}\delta)}{4(1+\phi(r)+r)} \sum_{j=N}^{n} \alpha_j \leqslant \|x_N - x^*\|,$$

so that $\sum_{n=0}^{\infty} \alpha_n < \infty$ contradicting $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Hence $\liminf \|y_n - x^*\| = \liminf \|x_n - x^*\| = 0$, so that there exists a subsequence $\{\|x_{n_j} - x^*\|\}_{n=0}^{\infty}$ of the sequence $\{\|x_n - x^*\|\}$ such that $\lim_{j\to\infty} \|x_{n_j} - x^*\| = 0$. It follows that given any $\epsilon > 0$ there exists a positive integer j_0 such that

$$\|x_{n_j} - x^*\| < \epsilon, \quad \forall j \geqslant j_0 \quad (\forall n_j \geqslant n_{j_0}).$$

Since $\lim \lambda_n = \lim \alpha_n = \lim \beta_n = 0$, there exists a positive integer N_2 such that

$$\lambda_n \leqslant \frac{2\epsilon^2 \phi(\frac{1}{2}\epsilon)}{(1+\phi(r)+r)}, \quad \alpha_n \leqslant \frac{\epsilon}{8d^*}, \quad \beta_n \leqslant \frac{\epsilon}{8d^*} \quad \forall n \geqslant N_2.$$

Since $\lim_{j\to\infty} n_j = \infty$, we can choose j_* such that $n_{j_*} \geqslant \max\{n_{j_0}, N_2\}$ so that $\|x_{n_{j_*}} - x^*\| < \epsilon$, and

$$\lambda_n \leqslant \frac{2\epsilon^2 \phi(\frac{1}{2}\epsilon)}{(1+\phi(r)+r)}, \quad \alpha_n \leqslant \frac{\epsilon}{8d^*}, \quad \beta_n \leqslant \frac{\epsilon}{8d^*} \quad \forall n \geqslant n_{j_*}.$$

We prove that

$$\|x_{n_{j_*}+p} - x^*\| < \epsilon, \quad \text{for all integers } p \geqslant 1.$$

The proof is by induction. For $p = 1$, we prove that $\|x_{n_{j_*}+1} - x^*\| < \epsilon$. Suppose for contradiction that $\|x_{n_{j_*}+1} - x^*\| \geqslant \epsilon$. Then

$$\|x_{n_{j_*}} - x^*\| \geqslant \|x_{n_{j_*}+1} - x^*\| - \alpha_{n_{j_*}} \|x_{n_{j_*}} - Sy_{n_{j_*}}\| - c'_{n_{j_*}} \|v_{n_{j_*}} - Sy_{n_{j_*}}\| \geqslant \epsilon - 2\alpha_{n_{j_*}} d^* \geqslant \tfrac{3}{4}\epsilon,$$

and

$$\|y_{n_{j_*}} - x^*\| \geqslant \|x_{n_{j_*}} - x^*\| - \beta_{n_{j_*}} \|x_{n_{j_*}} - Sx_{n_{j_*}}\| - c_{n_{j_*}} \|u_{n_{j_*}} - Sx_{n_{j_*}}\| \geqslant \tfrac{3}{4}\epsilon - 2d^* \beta_{n_{j_*}} \geqslant \tfrac{1}{2}\epsilon.$$

Hence

$$r(y_{n_{j_*}}, x^*) \geqslant \frac{\phi(\frac{1}{2}\epsilon)}{(1+\phi(r)+r)},$$

so that it follows from (2.13) that

$$\epsilon^2 \leqslant \|x_{n_{j_*}+1} - x^*\|^2 < \epsilon^2 - 2\alpha_{n_{j_*}} r(y_{n_{j_*}}, x^*)\epsilon^2 + \alpha_{n_{j_*}} \lambda_{n_{j_*}} \leqslant \epsilon^2,$$

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a contradiction, so that $\|x_{n_{j_*}+1} - x^*\| < \epsilon$. Assume now that $\|x_{n_{j_*}+p_0} - x^*\| < \epsilon$ for some $p_0 > 1$. We prove that $\|x_{n_{j_*}} + (p_0+1) - x^*\| < \epsilon$. Assume for contradiction that $\|x_{n_{j_*}+(p_0+1)} - x^*\| \geqslant \epsilon$. Then

$$\|x_{n_{j_*}+p_0} - x^*\| \geqslant \|x_{n_{j_*}+(p_0+1)} - x^*\| - \alpha_{n_{j_*}+p_0}\|x_{n_{j_*}+p_0} - Sy_{n_{j_*}+p_0}\| \\ - c'_{n_{j_*}+p_0}\|v_{n_{j_*}+p_0} - Sy_{n_{j_*}+p_0}\| \geqslant \tfrac{3}{4}\epsilon,$$

and

$$\begin{aligned}\|y_{n_{j_*}+p_0} - x^*\| &\geqslant \|x_{n_{j_*}+p_0} - x^*\| - \beta_{n_{j_*}+p_0}\|x_{n_{j_*}+p_0} - Sx_{n_{j_*}+p_0}\| \\ &\quad - c_{n_{j_*}+p_0}\|u_{n_{j_*}+p_0} - Sx_{n_{j_*}+p_0}\| \\ &\geqslant \tfrac{3}{4}\epsilon - 2\beta_{n_{j_*}+p_0}d^* \geqslant \tfrac{1}{2}\epsilon.\end{aligned}$$

From (2.13) we obtain

$$\epsilon^2 \leqslant \|x_{n_{j_*}+(p_0+1)} - x^*\|^2 < \epsilon^2 - 2\epsilon^2\alpha_{n_{j_*}+p_0} r(y_{n_{j_*}+p_0}, x^*) + \alpha_{n_{j_*}+p_0}\lambda_{n_{j_*}+p_0} \leqslant \epsilon^2,$$

a contraction, so that $\|x_{n_{j_*}+(p_0+1)} - x^*\| < \epsilon$. Hence $\|x_{n_{j_*}+p} - x^*\| < \epsilon$, for all integers $p \geqslant 1$, and this implies $\lim \|x_n - x^*\| = 0$, completing the proof of Theorem 2.1. □

2.2. Convergence theorems for *m*-accretive operator equations

Corollary 2.2. *Let E be a real uniformly smooth Banach space and let $T : D(T) \subseteq E \to E$ be an m-accretive operator with open domain $D(T)$. Let x^* denote the solution of the equation $x + Tx = f$, $f \in E$. Define $S : D(T) \to E$ by $Sx = f - Tx$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{u_n\}$ and $\{v_n\}$ be as in Theorem 2.1. Then there exist a real number $\mu \geqslant 1$ and a neighbourhood $B \subseteq D(T)$ of x^* such that starting with arbitrary $x_0, u_0, v_0 \in B$, the sequence $\{x_n\}_{n=0}^\infty$ defined iteratively by*

$$\begin{aligned} y_n &= a_n x_n + b_n S x_n + c_n u_n, \quad n \geqslant 0, \\ x_{n+1} &= a'_n x_n + b'_n S y_n + c'_n v_n, \quad n \geqslant 0, \end{aligned}$$

remains in B and converges strongly to x^*.

Proof. As in the proof of Theorem 2.1, set $\beta_n = b_n + c_n$, and $\alpha_n = b'_n + c'_n$, then

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n S x_n + c_n(u_n - Sx_n), \quad n \geqslant 0, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S y_n + c'_n(v_n - Sy_n), \quad n \geqslant 0. \end{aligned}$$

Observe that x^* is a fixed point of S, and, since T is accretive, $(I - S)$ is also accretive. The corollary follows from Theorem 2.1. □

2.3. Convergence theorems for *Φ*-strong pseudocontractions

Corollary 2.3. *Let E be a real uniformly smooth Banach space and let $T : D(T) \subseteq E \to E$ be a ϕ-strong pseudocontraction with open domain $D(T) \subseteq E$. Suppose T has*

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a fixed point $x^* \in D(T)$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{u_n\}$ and $\{v_n\}$ be as in Theorem 2.1. Then there exist a real number $\mu \geqslant 1$ and a neighbourhood $B \subseteq D(T)$ of x^* such that starting with arbitrary $x_0, u_0, v_0 \in B$, the sequence $\{x_n\}_{n=0}^{\infty}$, defined iteratively by

$$y_n = a_n x_n + b_n T x_n + c_n u_n, \quad n \geqslant 0,$$
$$x_{n+1} = a'_n x_n + b'_n T y_n + c'_n v_n, \quad n \geqslant 0,$$

remains in B and converges strongly to x^*.

Proof. As in the proof of Theorem 2.1, set $\beta_n = b_n + c_n$, and $\alpha_n = b'_n + c'_n$, then

$$y_n = (1-\beta_n)x_n + \beta_n T x_n + c_n(u_n - T x_n), \quad n \geqslant 0,$$
$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + c'_n(v_n - T y_n), \quad n \geqslant 0.$$

Since T is ϕ-strongly pseudocontractive, $(I-T)$ is accretive. The corollary follows from Theorem 2.1. □

2.4. Convergence theorems for quasi-accretive operators

An operator $T : D(T) \subseteq E$ is called *quasi-accretive* if $N(T) := \{x \in D(T) : Tx = 0\} \neq \emptyset$ and for all $x \in D(T)$ and $x^* \in N(T)$, there exists $j(x-x^*) \in J(x-x^*)$ such that

$$\langle Tx - Tx^*, j(x-x^*)\rangle \geqslant 0,$$

and is called *strongly quasi-accretive* if there exists $k > 0$ such that

$$\langle Tx - Tx^*, j(x-x^*)\rangle \geqslant k\|x-x^*\|^2.$$

T is called *ϕ-strongly quasi-accretive* if $N(T) \neq \emptyset$ and for all $x \in D(T)$ and $x^* \in N(T)$ there exist $j(x-x^*) \in J(x-x^*)$ and a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Tx^*, j(x-x^*)\rangle \geqslant \phi(\|x-x^*\|)\|x-x^*\|.$$

It is easy to see that our proof of Theorem 2.1 carries over to the class of ϕ-strongly quasi-accretive operators, while the proof of Corollary 2.2 carries over to quasi-accretive operators, if the operators are locally bounded at each interior point of their effective domains.

2.5. Convergence theorems for *Φ*-hemicontractions

An operator $T : D(T) \subseteq E \to E$ is called a *ϕ-hemicontraction* (see, for example, [**29**]) if $F(T) := \{x \in D(T) : x \in Tx\} \neq \emptyset$ and for all $x \in D(T)$, $x^* \in F(T)$, there exist $j(x-x^*) \in J(x-x^*)$ and a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - x^*, j(x-x^*)\rangle \leqslant \|x-x^*\|^2 - \phi(\|x-x^*\|)\|x-x^*\|.$$

An example in [**14**] shows that the class of ϕ-strong pseudocontractions with non-empty fixed-point sets is a proper subclass of the class of ϕ-hemicontractions.

It is easy to see that Corollary 2.3 carries over to ϕ-hemicontractions T for which $(I-T)$ is locally bounded at each interior point of its effective domain.

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