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A DECOMPOSITION THEOREM FOR POSITIVE SUPERHARMONIC FUNCTIONS

BY

SIRKKA-LIISA ERIKSSON-BIQUE

ABSTRACT. Let X be a harmonic space in the sense of C. Constantinescu and A. Cornea. We show that, for any subset E of X, a positive superharmonic function u on X has a representation $u = p + h$, where p is the greatest specific minorant of u satisfying $p = R_p^E$ and $h = R_h^{X \setminus E}$. This result is a generalization of a theorem of M. Brelot. We also state some characterizations of extremal superharmonic functions.

Introduction. Let X be a harmonic space in the sense of C. Constantinescu and A. Cornea [6], p. 30. The hyperharmonic sheaf on X is denoted by $\mathcal{U}$ and the set of positive hyperharmonic functions on an open set U by $\mathcal{U}^+(U)$. The reduced function of a positive hyperharmonic function u on X relative to a subset E of X is given by

$$R_u^E = \inf\{v \in \mathcal{U}^+(X) : v \geqq u \text{ on } E\}.$$

The function $\hat{R}_u^E = \liminf R_u^E$ is the balayage of $u \in \mathcal{U}^+(X)$. In this work we mostly use only properties of reduced and balayage functions ([6], §4, 5 or [2], VI).

Our main theorem (Theorem 1.2) states that any positive superharmonic function u on X has a representation $u = h + p$, where $h = R_h^{X \setminus E}$ and p is the greatest specific minorant of u satisfying $p = R_p^E$. M. Brelot proved this result in special Brelot spaces ([5], Theorem 5) and asked, whether it is true in harmonic spaces of C. Constantinescu and A. Cornea. In fact, this result also holds in balayage-spaces presented in [2]. Moreover our decomposition applies to the solutions of linear elliptic or parabolic partial differential equations of second order, since they form a harmonic space. For the Laplace equation it seems that the decomposition is new for an arbitrary set E.

It is natural to think that our result is connected to the decomposition of R.-M. Hervé. According to R.-M. Hervé [10], Theorem 5, a positive superharmonic function u on X can be written as $u = h_E + p_E$, where h_E is the greatest specific minorant of u harmonic on an open set E and p_E is a potential on X. We are able to show that $h_E \succeq h$ and $p_E = R_{p_E}^E$. Moreover, the decomposition of R.-M. Hervé can be obtained

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from our result and a decomposition of F. Riesz ([12] or [6], Theorem 2.2.2), which states that a positive superharmonic function on an open set E is the sum of its greatest harmonic minorant on E and a potential on E.

Extremal harmonic and superharmonic functions give an integral representation of positive superharmonic functions in some harmonic spaces. It is a well-known result that extremal superharmonic functions are harmonic except possibly at one point ([6], Proposition 11.4.3). We show that in some cases from an extremal superharmonic function on an open subset of X we can obtain an extremal superharmonic function on the whole space X. We also prove some characterizations of extremal superharmonic functions similar to those given in [7], [8], [13]. For example, a positive superharmonic function is extremal if and only if, for any finely open subset E of X, the sets E and $X\backslash E$ are not both thin relative to u. Applying this we verify a limit theorem for extremal superharmonic functions similar to L. Naïm [11], Theorem 8.17, and K. Gowrisankaran [7], Theorem 1.2.

1. **A decomposition theorem for positive superharmonic functions.** In the sequel, let X be a harmonic space in the sense of C. Constantinescu and A. Cornea [6], p. 30. We denote by $\mathcal{S}^+(X)$ the set of positive superharmonic functions on X. The specific order ($\preceq$) in $\mathcal{S}^+(X)$ is defined by

$$u \preceq v \text{ if } v = u + u' \text{ for some } u' \in \mathcal{S}^+(X).$$

Lattice operations with respect to the specific order are denoted by $\curlywedge$ and $\curlyvee$.

A subsemigroup $\mathcal{V}$ of $\mathcal{S}^+(X)$ is called a specific ideal if for any $u \in \mathcal{S}^+(X)$ the condition $u \preceq v$ for some $v \in \mathcal{V}$ implies $u \in \mathcal{V}$. An element $p \in \mathcal{S}^+(X)$ is called a specific projection of $u \in \mathcal{S}^+(X)$ on $\mathcal{V} \subseteq \mathcal{S}^+(X)$ if

$$p = \operatorname{sp}\max\{x \in \mathcal{V} : x \preceq u\},$$

where sp max is the maximum relative to the specific order. If $\mathcal{V}$ is a specific ideal and p is the specific projection of u on $\mathcal{V}$, we easily see that $(u-p) \curlywedge v = 0$ for any $v \in \mathcal{V}$. Specific ideals and projections are studied by M. Arsove and H. Leutwiler in algebraic potential theory [1].

The fine topology on X is the coarsest topology on X in which any hyperharmonic function on any open set of X is continuous ([6], p. 116). Open or closed sets with respect to the fine topology are called finely open or finely closed, respectively.

Let E be an arbitrary subset of X. We use the notation $\mathcal{S}_E$ for the set of positive superharmonic functions s on X satisfying $s = R_s^E$.

LEMMA 1.1. *The set $\mathcal{S}_E$ is a specific ideal in $\mathcal{S}^+(X)$. If E is finely open, the specific projection p of $u \in \mathcal{S}^+(X)$ on $\mathcal{S}_E$ is $p = u \curlywedge R_u^E$. In the general case, the specific projection of $u \in \mathcal{S}^+(X)$ on $\mathcal{S}_E$ for an arbitrary subset E of X is*

$$p = \bigwedge_{U \in \mathcal{B}} u \curlywedge R_u^U,$$

where $\mathcal{B}$ is the collection of finely open sets containing $E \cap \{x \in X \mid u(x) < \infty\}$.

PROOF. Note that

$$(1.1) \qquad R^E_{s+t} = R^E_s + R^E_t \quad \text{and} \quad \hat{R}^E_{s+t} = \hat{R}^E_s + \hat{R}^E_t$$

for all $s, t \in \mathcal{S}^+(X)$ by [6], Theorem 4.2.1. Using this we easily see that $\mathcal{S}_E$ is a subsemigroup of $\mathcal{S}^+(X)$.

Suppose that $u \in \mathcal{S}^+(X)$ and $u \preceq v$ for some $v \in \mathcal{S}_E$. The equality (1.1) results in

$$u + u' = v = R^E_v = R^E_u + R^E_{u'},$$

for some $u' \in \mathcal{S}^+(X)$. Since $u \geqq R^E_u$ and $u' \geqq R^E_{u'}$, we have $u = R^E_u$. Hence $\mathcal{S}_E$ is a specific ideal.

Let u be a positive superharmonic function on X. Assume that E is finely open. Then we have

$$\hat{R}^E_u = R^E_u = R^E_{R^E_u}$$

by [6], Corollary 5.1.3, and therefore $R^E_u \in \mathcal{S}_E$. Since $\mathcal{S}_E$ is a specific ideal, $u \curlywedge R^E_u \in \mathcal{S}_E$. Assuming $v + v' = u$ for $v \in \mathcal{S}_E$ and $v' \in \mathcal{S}^+(X)$ we obtain

$$R^E_u = R^E_v + R^E_{v'} = v + R^E_{v'}$$

by (1.1). Hence $v \preceq R^E_u$, which yields $v \preceq u \curlywedge R^E_u$. This accomplishes the proof that $u \curlywedge R^E_u$ is the specific projection of u on $\mathcal{S}_E$ for any finely open set E contained in X.

Suppose next that E is an arbitrary subset of X. Since the set $F = \{x \in X : u(x) < \infty\}$ is dense in X we have

$$\hat{R}^E_u = \hat{R}^{E \cap F}_u.$$

Applying [6], Proposition 4.2.1, we conclude

$$R^{E\cap F}_u = \inf_{U \in \mathcal{B}} R^U_u.$$

In order to shorten the notations, set $p = \bigwedge_{U \in \mathcal{B}} (u \curlywedge R^U_u)$. If U, V belongs to $\mathcal{B}$ and $U \subseteq V$ then

$$u \curlywedge R^U_u = R^U_{u \curlywedge R^U_u} \leqq R^V_{u \curlywedge R^U_u} \leqq u \curlywedge R^U_u,$$

whence $u \curlywedge R^U_u \in \mathcal{S}_V$. Combining this with $u \curlywedge R^U_u \preceq u$, we see that

$$u \curlywedge R^U_u \preceq \text{ sp } \max\{v \in \mathcal{S}_V : v \preceq u\} = u \curlywedge R^V_u.$$

Thus the family $(u \curlywedge R^U_u)_{U \in \mathcal{B}}$ is specifically decreasing and therefore

$$p = \underset{U \in \mathcal{B}}{\curlywedge} (u \curlywedge R^U_u)$$

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by [6], Proposition 4.1.4. Hence Lemma 1.1 assures us that $p = R_p^U$ for all $U \in \mathcal{B}$. This results in the equality

$$p = \bigwedge_{U \in \mathcal{B}} R_p^U.$$

Put $G = \{x \in X : p(x) < \infty\}$ and denote by $\mathcal{F}$ the family of finely open subsets of X containing $G \cap E$. Since p is finite on a dense set, we have $\hat{R}_p^E = \hat{R}_p^{E \cap G}$. Moreover, $R_p^V = R_p^{V \cap F}$ for any set $V \in \mathcal{F}$. Applying again [6], Proposition 4.2.1, we obtain

$$R_p^{E \cap G} = \inf_{V \in \mathcal{F}} R_p^V = \inf_{V \in \mathcal{F}} R_p^{V \cap F} \geqq \inf_{U \in \mathcal{B}} R_p^U = p.$$

Note that the inequality follows from $V \cap F \in \mathcal{B}$ for all $V \in \mathcal{F}$. Hence $p = R_p^E$, and so $p \in \mathcal{S}_E$. On the other hand, if $x \preceq u$ and $x = R_x^E = R_x^{E \cap F}$, then $x = R_x^U$ for all $U \in \mathcal{B}$. This implies $x \preceq u \curlywedge R_u^U$ for all $U \in \mathcal{B}$ and therefore

$$x \preceq \curlywedge_{U \in \mathcal{B}} (u \curlywedge R_u^U) = p.$$

Consequently, p is the specific projection of u on $\mathcal{S}_E$. This completes the proof.

Let X be an $\mathcal{S}$-harmonic space possessing a countable base and satisfying the axiom of polarity ([6], p. 219). Then

$$\hat{R}_{\hat{R}_u^E}^E = \hat{R}_u^E$$

for any subset E of X by [6], Theorem 9.2.1. Using the same arguments as in the finely open case of Lemma 1.1, we see that the specific projection of any positive superharmonic function u on $\mathcal{S}_E$ is $u \curlywedge \hat{R}_u^E$ for all subsets E of X.

Our main theorem is a generalization of the result of M. Brelot [5], Theorem 5.

THEOREM 1.2. *Every positive superharmonic function u on X can be represented as*

$$u = p + h \tag{1.2}$$

where p is the specific projection of u on $\mathcal{S}_E$ and $h = R_h^{X \setminus E}$.

PROOF. Let u be a positive superharmonic function on X. Set p equal to the specific projection of u on $\mathcal{S}_E$ and $h = u - p$. Here $u - p$ means the unique superharmonic function p' satisfying $u = p + p'$.

In order to show the assertion $h = R_h^{X \setminus E}$, we first assume that E is finely open. Then we have

$$\hat{R}_h^E = R_h^E = R_{R_h^E}^E \tag{1.3}$$

by [6], Corollary 5.1.3. On account of [6], Proposition 5.3.4, we have $h \preceq R_h^E + \hat{R}_h^{X \setminus E}$. Since the specific Riesz-decomposition property holds in $\mathcal{S}^+(X)$ ([3], Theorem 2.1.5), we obtain $h = h_1 + h_2$ for some $h_1 \preceq R_h^E$ and $h_2 \preceq \hat{R}_h^{X \setminus E}$. Lemma 1.1 and (1.3) ensure

$h_1 = R^E_{h_1}$, whence $p + h_1 \in \mathcal{S}_E$ and $p + h_1 \preceq u$. But this leads to $h_1 = 0$, since p is the specific projection of u on $\mathcal{S}_E$. Therefore we have

$$h = h_2 \preceq \hat{R}^{X\setminus E}_h \leqq R^{X\setminus E}_h \leqq h$$

and, so $h = R^{X\setminus E}_h$.

Assume secondly that E is an arbitrary subset of X. Denote by C the set $\{x \in X\setminus E : h(x) < \infty\}$. Since h is finite on a dense subset of X we have

$$\hat{R}^{X\setminus E}_s = \hat{R}^C_s, \hat{R}^E_s = \hat{R}^{X\setminus C}_s$$

for all $s \in \mathcal{S}^+(X)$. Let $\mathcal{B}$ be the family of finely open sets containing C. Using the preceding part of the proof, we find for any $U \in \mathcal{B}$ the functions $p_U \in \mathcal{S}^+(X)$ and $h_U \in \mathcal{S}^+(X)$ satisfying the conditions $u = p_U + h_U$, $p_U = R^U_{p_U}$ and $h_U = R^{X\setminus U}_{h_U}$. Since $X\setminus U \subset X\setminus C$ we have

$$h_U = R^{X\setminus U}_{h_U} \leqq R^{X\setminus C}_{h_U} \leqq h_U.$$

This yields

$$h_U = R^{X\setminus C}_{h_U} = R^E_{h_U}$$

for any $U \in \mathcal{B}$. Hence $h_U \in \mathcal{S}_E$, and so $h_U \preceq p$. This leads by Lemma 1.1 to $R^U_h = h$ for all $U \in \mathcal{B}$. Finally, applying [6], Proposition 4.2.1, we conclude

$$h \geqq R^{X\setminus E}_h \geqq R^C_h = \inf_{U\in\mathcal{B}} R^U_h = h.$$

Hence $h = R^{X\setminus E}_h$, completing the proof.

COROLLARY 1.3. *Let p and h be as in Theorem 1.2. Then the specific projection of h on $\mathcal{S}_E$ is zero. Furthermore the functions p and h are harmonic on $X\setminus cl\ E$ and $int\ E$, respectively.*

The preceding result follows from [6], Proposition 5.3.1, since $p \preceq \hat{R}^E_u$ and $h \preceq \hat{R}^{X\setminus E}_u$.

2. **Comparison of Theorem 1.2 with other decomposition theorems.** We compare Theorem 1.2 with the decomposition theorem of R.-M. Hervé [10], Theorem 12.2, stated in Brelot spaces. We also show how one obtains the decompostion theorem of R.-M. Hervé using Theorem 1.2 and the decomposition theorem of F. Riesz, [12].

For the sake of completeness, we first prove the decomposition theorem of R.-M. Hervé in a harmonic space X defined by C. Constantinescu and A. Cornea [6], p. 30.

THEOREM 2.1. *(R.-M. Hervé) Let E be an open subset of a harmonic space X and u be a positive superharmonic function on X. Then u has a decomposition*

$$u = h_E + p_E, \tag{2.1}$$

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where $h_E \in \mathcal{S}^+(X)$ *is the greatest specific minorant of* u *harmonic on* E *and* $p_E \in \mathcal{S}^+(X)$ *is a potential on* X.

The ideas of the proof are the same as in Theorem 1.2. In fact, we only need to verify the following result:

LEMMA 2.2. *Let* E *be an open subset of* X. *Denote by* $\mathcal{U}_E$ *the set of positive superharmonic functions on* X *which are harmonic on* E. *Then* $\mathcal{U}_E$ *is a specific ideal and every positive superharmonic function admits the specific projection on* $\mathcal{U}_E$.

PROOF. The first statement is obvious. In order to prove the second one, let u be a positive superharmonic function on X. If $h_1 \in \mathcal{S}^+(X)$ and $h_2 \in \mathcal{S}^+(X)$ are harmonic on E and specifically smaller than u, then $h_1 \curlyvee h_2$ is harmonic on E and specifically smaller than u. Hence the set

$$\mathcal{P}_u = \{x \in \mathcal{U}_E : x \preceq u\}$$

is specifically directed upwards. This implies

$$\sup \mathcal{P}_u = \curlyvee \mathcal{P}_u$$

by [6], Proposition 4.1.4. Therefore $\sup \mathcal{P}_u$ is the specific projection of u on $\mathcal{U}_E$.

Using Theorem 1.2, we are able to establish an additional property for the potential part of the decomposition (2.1).

THEOREM 2.3. *Let* E *be an open subset of a harmonic space* X *([6], p. 30). Then the potential part* p_E *of the decomposition (2.1) of* u *satisfies the condition* $R^E_{p_E} = p_E$.

PROOF. From Theorem 1.2 it follows that $u = p+h$, where p is the specific projection on $\mathcal{S}_E = \{s \in \mathcal{S}^+(X) : s = R^E_s\}$ and $h = R^{X\setminus E}_h$ is harmonic on E. This leads to $h \preceq h_E$, and so

$$u = p_E + h_E = h + p \preceq h_E + p.$$

Hence we have $p \succeq p_E$. Since $p = R^E_p$, Lemma 1.1 results in $p_E = R^E_{p_E}$.

COROLLARY 2.4. *Let* E *be an open subset of a harmonic space* X ([6], $p.$30) *and* u *be a positive superharmonic function on* X. *If* p *is the specific projection of* u *on* $\mathcal{S}_E$ *and* $h \in \mathcal{S}^+(X)$ *satisfies* $u = p + h$, *then* $h \preceq h_E$ *and* $p \succeq p_E$, *where* h_E *and* p_E *are the same as in the decomposition* (2.1).

We recall the decomposition theorem of F. Riesz.

THEOREM 2.5. *Let* E *be an open subset of a harmonic space* X ([6], $p.$30). *If* u *is a superharmonic function on* E *possessing a subharmonic minorant then* u *has the representation*

$$u = h_r + p_r, \tag{2.2}$$

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where h_r is the greatest harmonic minorant of u on E and p_r is a potential on E.

Let E be an open set contained in a harmonic space X and u a positive superharmonic function on X. Then the functions h_r and p_r defined in (2.2) can be approximated from above by positive superharmonic functions on X. We define

$$Rh_r = \inf\{v \in \mathcal{S}^+(X) : v \geqq h_r \text{ on } E\}$$

and similarly Rp_r. Since $u \geqq Rh_r$ and $u \geqq Rp_r$ both functions Rh_r and Rp_r are superharmonic functions harmonic on $X \backslash cl\ E$ ([6], Proposition 2.2.3).

Note that $Rh_r = h_r$ on E. Indeed, $Rh_r \leqq u$ and Rh_r is harmonic on E by [6], Proposition 2.2.3. However, Rp_r is not generally equal to p_r on E.

THEOREM 2.6. *Let E be an open subset of a harmonic space X* ([6], *p.* 30) *and u be a positive superharmonic function on X. Denote by h_r the greatest harmonic minorant of u on E. Then $u \curlywedge Rh_r$ is the greatest specific minorant of u harmonic on E if and only if $R_u^E = u$.*

PROOF. Assume that $u \curlywedge Rh_r$ is the greatest specific minorant of u harmonic on E. On account of Theorem 2.1 and 2.3 we have

$$u = u \curlywedge Rh_r + p_E,$$

where $p_E = R_{p_E}^E$. Since $Rh_r = h_r$ on E, it follows what $R_{Rh_r}^E = Rh_r$. Therefore Lemma 1.1 results in

$$R_{u \curlywedge Rh_r}^E = u \curlywedge Rh_r.$$

Hence we obtain

$$R_u^E = R_{u \curlywedge Rh_r}^E + R_{p_E}^E = u \curlywedge Rh_r + p_E = u.$$

Conversely, assume that $u = R_u^E$. Then the function $u \curlywedge Rh_r$ is a candidate for the greatest specific minorant of u harmonic on E. Suppose that $s \in \mathcal{S}^+(X)$ is harmonic on E and specifically smaller than u. Then $h_r = s + f$ for some harmonic function f on E and $R_s^E = s$ by lemma 1.1. This leads to

$$Rh_r \leqq R_s^E + Rf = s + Rf.$$

Since $u = s + s' \geqq s + f$ on E for some $s' \in \mathcal{S}^+(X)$, we have $s' \geqq Rf$ and further $s + Rf \leqq u$. But $s + Rf$ is also harmonic on E by [6], Proposition 2.2.3, whence $s + Rf = h_r = Rh_r$ on E. From $Rh_r \leqq s + Rf$ it follows that $Rh_r = s_1 + s_2$ for some $s_1, s_2 \in \mathcal{S}^+(X)$ satisfying $s_1 \leqq s$ and $s_2 \leqq Rf$. Then the equality

$$Rh_r = s_1 + s_2 = s + Rf$$

holds on E. Using the properties $s_1 \leqq s$ and $s_2 \leqq Rf$ we obtain $s_1 = s$ on E and $s_2 = Rf \geqq f$ on E. Hence $s_1 \geqq R_s^E = s$ and $s_2 \geqq Rf$. Consequently, $s_1 = s$ and

$s_2 = Rf$, which leads to $Rh_r = s + Rf$. Therefore $s \preceq u \curlywedge Rh_r$. This establishes that $u \curlywedge Rh_r$ is the greatest specific minorant of u harmonic on E, completing the proof.

Now we are ready to state the relationship between the decomposition of R.-M. Hervé and the decomposition of F. Riesz and Theorem 1.2.

THEOREM 2.7. *Let E be an open subset of a harmonic space X ([6], p. 30) and u be a positive superharmonic function on X. Denote by p the specific projection of u on $\mathcal{S}_E = \{v \in \mathcal{S}^+(X) : v = R_v^E\}$ and $h = u - p$. Then the greatest specific minorant h_E of u harmonic on E is given by*

$$h_E = h + p \curlywedge Rs = u \curlywedge (Rs + h),$$

where s is the greatest harmonic minorant of p on E.

PROOF. According to Corollary 1.3, h is harmonic on E. Moreover, $p \curlywedge Rs$ is harmonic on E, since $p \curlywedge Rs \preceq Rs$ and Rs is harmonic on E by [6], Proposition 2.2.3. Hence $h + p \curlywedge Rs$ is a candidate for the greatest specific minorant of u harmonic on E. Suppose that $f \in \mathcal{S}^+(X)$ is harmonic on E and specifically smaller than u. Then $f \preceq h + p$ and therefore there exist f_1 and f_2 in $\mathcal{S}^+(X)$ such that $f = f_1 + f_2$ and $f_1 \preceq h, f_2 \preceq p$ by [2], Theorem 2.1.5. Theorem 2.6 asserts that $p \curlywedge Rs$ is the greatest specific minorant of p harmonic on E. Since f_2 is harmonic on E and $f_2 \preceq p$ we have $f_2 \preceq p \wedge Rs$, whence

$$f = f_1 + f_2 \preceq h + p \curlywedge Rs.$$

Consequently, $h + p \curlywedge Rs$ is the greatest specific minorant of u harmonic on E, finishing the proof.

Note that by Theorem 2.7 the decompositions of R.-M. Hervé and Theorem 1.2 are equal if and only if $p \curlywedge Rs = 0$.

3. **Extremal superharmonic functions.** Extremal harmonic and superharmonic functions play an important role in finding an integral representation of superharmonic functions. For references we mention M. Brelot [4], K. Gowrisankaran [7], [8], [9], C. Constantinescu and A. Cornea [6] and M. Sieveking [13].

In the sequel, let X be a harmonic space in the sense of [6], p. 30. We recall the definition of extremal superharmonic functions.

DEFINITION 3.1. *A positive superharmonic function $u (\not\equiv 0)$ is called extremal if every specific minorant v of u satisfies $v = \alpha u$ for some positive $\alpha \leqq 1$.*

There exist many characterizations of extremal harmonic functions given by K. Gowrisankaran [7], [8]. We use those ideas and Theorem 1.2 to characterize extremal superharmonic functions.

Note that extremal superharmonic functions are either potentials or harmonic functions on X. This fact follows easily from Theorem 2.5.

In some cases from an extremal superharmonic function on an open set we can get an extremal superharmonic function on the whole space X.

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THEOREM 3.2 *Let s be an extremal superharmonic function on an open set U contained in X. If there exists a superharmonic function on X majorizing s on U, the function*

$$Rs = \inf\{w \in \mathcal{S}^+(X) : w \geqq s \text{ on } U\}$$

is extremal.

PROOF. Let s be extremal superharmonic function on U majorized by a superharmonic function on X. Then Rs is superharmonic by [6], Proposition 2.2.3. Suppose that $Rs = s_1 + s_2$ for some $s_1, s_2 \in \mathcal{S}^+(X)$. Since $s \leqq s_1 + s_2$ on U, by [6], Theorem 5.1.1, there exist positive superharmonic functions s' and s'' on U such that $s = s' + s''$ and $s' \leqq s_1, s'' \leqq s_2$. Using extremality of s, we see that $s' = \alpha s$ and $s'' = (1-\alpha)s$ for some positive $\alpha \leqq 1$. From $s_1 \geqq s'$ and $s_2 \geqq s''$ it follows that $s_1 \geqq \alpha Rs$ and $s_2 \geqq (1-\alpha)Rs$. Hence the equality $Rs = s_1 + s_2 = \alpha Rs + (1-\alpha)Rs$ asserts that $s_1 = \alpha Rs$ and $s_2 = (1-\alpha)Rs$. Therefore Rs is an extremal superharmonic function on X.

We use the following definition of K. Gowrisankaran [7], p. 313:

DEFINITION 3.3. *A subset E of X is called thin relative to $u \in \mathcal{S}^+(X)$ if $R_u^E \not\equiv u$.*

Let $E \subset X$ be given. M. Brelot noted in [**5**], p. 299, that E and its complement $X \backslash E$ are not both thin relative to an extremal superharmonic function. Next we state some characterizations of extremal superharmonic functions.

THEOREM 3.4. *Let u be a positive superharmonic functions on a harmonic space X in the sense of [6], p. 30. Then the following statements are mutually equivalent:*

(i) *u is extremal;*
(ii) *The family $\mathcal{F}_u = \{E \subset X : R_u^{X\backslash E} \not\equiv u\}$ is a filter;*
(iii) *For any subset E of X the sets E and $X \backslash E$ are not both thin relative to u;*
(iv) *For any finely open set E the sets E and $X \backslash E$ are not both thin relative to u.*

PROOF. Assume that a superharmonic function u is extremal. Let U and V be arbitrary sets in $\mathcal{F}_u$. In order to prove (ii) it is enough to show that $U \cap V \in \mathcal{F}_u$. On the contrary, suppose that $R_u^{(X\backslash U)\cup(X\backslash V)} = u$. By [6], Proposition 5.3.4, we have

$$u = \hat{R}_u^{X\backslash(U\cap V)} \preceq \hat{R}_u^{X\backslash U} + \hat{R}_u^{X\backslash V}$$

and therefore $u + s = \hat{R}_u^{X\backslash U} + \hat{R}_u^{X\backslash V}$ for some $s \in \mathcal{S}^+(X)$. Applying (3], Theorem 2.1.5, we obtain $\hat{R}_u^{X\backslash U} = x_1 + s_1$ for some $x_1, s_1 \in \mathcal{S}^+(X)$ such that $x_1 \preceq u$ and $s_1 \preceq s$. Then extremality of u yields $x_1 = \alpha u$ for some positive real number $\alpha < 1$. In case $\alpha > 0$ we see that

$$\hat{R}_u^{X\backslash U}(x) = \alpha u(x) + s_1(x) > \alpha R_u^{X\backslash U}(x) + s_1(x) \geqq R_u^{X\backslash U}(x) \geqq \hat{R}_u^{X\backslash U}(x)$$

for some $x \in X$, which is impossible. Hence $\alpha = 0$ and $\hat{R}_u^{X\backslash V} = u + s_2$ for some $s_2 \in \mathcal{S}^+(X)$ such that $s_2 \preceq s$. This leads to $s_2 = 0$ and $\hat{R}_u^{X\backslash V} = u$, which is a contradiction. Thus $U \cap V \in \mathcal{F}_u$. Consequently (ii) holds.

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It is clear that (ii) implies (iii) and (iii) implies (iv). Lastly we show that (iv) implies (i). Suppose that $v \in \mathcal{S}^+(X)$ is a specific minorant of u. Denote by α the smallest real number satisfying $v \leqq \alpha u$. Let β be an arbitrary positive real number such that $\beta < \alpha$. We consider the set

$$E_\beta = \{x \in X : v(x) > \beta u(x)\}.$$

Since E_β is finely open, we have $u = R_u^{E_\beta}$ or $u = R_u^{X \backslash E_\beta}$. Assume first that $u = R_u^{X \backslash E_\beta}$. The condition $v \preceq u$ implies $v = R_v^{X \backslash E_\beta}$ by Lemma 1.1. Since

$$X \backslash E_\beta = \{x \in X : v(x) \leqq \beta u(x)\}$$

we see that

$$\beta u \geqq R_v^{X \backslash E_\beta} = v,$$

which is a contradiction. Hence $u = R_u^{E_\beta}$ for all positive $\beta < \alpha$, and so $\beta u = R_{\beta u}^{E_\beta}$ for all positive $\beta < \alpha$. Therefore $v \geqq \beta u$ for all positive $\beta < \alpha$, which yields $v \geqq \alpha u$. Since $v \leqq \alpha u$, we have $v = \alpha u$. Consequently, u is extremal finishing the proof.

The equivalence $(i) \Leftrightarrow (ii)$ in strongly harmonic spaces is proved by Sieveking [13], p. 21.

The preceding theorem enables us to show a limit theorem for extremal superharmonic functions similar to K. Gowrisankaran [7], Theorem 1.3, or L. Naïm [11], Theorem 8.17.

THEOREM 3.5. *Let u be an extremal superharmonic function and v be an arbitrary positive superharmonic function. Denote by D the set where v/u is defined. Then v/u has a finite limit along $\mathcal{F}_u \mid D = \{U \cap D : U \in \mathcal{F}_u\}$. Moreover, this limit is equal to*

$$\sup\{\beta : \beta u \leqq v\} = \inf_{x \in D} \frac{v(x)}{u(x)}.$$

PROOF. Note that the sets $E_0 = \{x \in X : u(x) = 0\}$ and $E_\infty = \{x \in X : u(x) = \infty\}$ are both thin relative to u. Therefore $D = (X \backslash E_0) \cap (X \backslash E_\infty)$ belongs to $\mathcal{F}_u$. This means that $D \cap U$ is nonempty for any $U \in \mathcal{F}_u$. Denote $\alpha = \sup\{\beta : \beta u \leqq v\}$. Then α is finite, since $\bigwedge_{n \in \mathbb{N}} v/n = 0$. Set, for any $\varepsilon > 0$,

$$E_\varepsilon = \{x \in X : v(x) \leqq (\varepsilon + \alpha) u(x)\}.$$

On account of $R_u^{X \backslash E_\varepsilon} \leqq v/(\varepsilon + \alpha)$, we see that $X \backslash E_\varepsilon$ is thin relative to u. Hence $E_\varepsilon \in \mathcal{F}_u$ and

$$\limsup_{\mathcal{F}_u | D} \frac{v}{u} \leqq \alpha + \varepsilon.$$

The number ε being arbitrary, we have

$$\limsup_{\mathcal{F}_u | D} \frac{v}{u} \leqq \alpha.$$

Finally, it is easy to notice that

$$\alpha \leqq \inf_{x \in D} \frac{v(x)}{u(x)} \leqq \liminf_{\mathcal{F}_u|D} \frac{v}{u} \leqq \limsup_{\mathcal{F}_u|D} \frac{v}{u} = \alpha,$$

completing the proof.

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Department of Mathematics
University of Joensuu
SF-80101 Joensuu, Finland