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ON SHIFT OPERATORS

BY
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ABSTRACT. A definition of an isometric shift operator on a Banach space is given which extends the usual definition of a shift operator on a separable Hilbert space. It is shown that there is no such shift on many of the common Banach spaces of continuous functions. The associated ideas of a semi-shift and a backward shift are also introduced and studied in the case of continuous function spaces.

1. **Introduction.** If H is a separable Hilbert space and $\{\phi_n\}_{n=1}^{\infty}$ an orthonormal basis for H, the operator S on H defined by $S\phi_n = \phi_{n+1}$, $n = 1, 2, \ldots$, is called a (unilateral) *shift*. Such operators and their generalizations (e.g. weighted shifts) are among the most important and well-studied operators on Hilbert spaces. Not surprisingly, then, extensions of the notion of a shift operator to general Banach spaces have been introduced and studied by various authors (e.g. [1], [3], [4], [5], [6]). We introduce here another definition which, while similar to that of [1], retains the isometric aspect inherent in the Hilbert space shift while yet being "basis free" and hence meaningful in a very general context.

DEFINITION. An operator S on a Banach space X will be called a *shift operator* on X if

1) S is an isometry,
2) $\mathrm{Codim}(\mathrm{Ran}\ S) = 1$,
3) $\cap_{n=1}^{\infty} \mathrm{Ran}\ S^n = 0$.

One easily checks that an operator S on a separable Hilbert space is a shift according to this definition if and only if there is an orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$ for which $S\phi_n = \phi_{n+1}$ for all n. Hence our definition is a natural extension of the notion of a shift operator to more general spaces which allows the geometry of the underlying space to play a significant role in the theory of these operators. In this paper we begin a study of the existence of such shift operators on various Banach spaces of continuous functions. We also introduce the related ideas of a semi-shift and a backward shift on general Banach spaces and

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consider again questions of existence of such operators on continuous function spaces. Finally, a number of interesting unsolved problems are mentioned.

2. **Shifts on Function Spaces.** We intend to examine the problem of existence for shift operators on various Banach spaces of continuous functions. It is well-known that such shifts exist on certain of these spaces. For example, let K_0 denote the one-point compactification of the positive integers and $C(K_0)$ the space of all real valued continuous functions on K_0 (with supremum norm, as usual). Then $C(K_0)$ is easily identified with the sequence space c of all convergent real sequences on which the operator S defined by $S(\{a_n\}_{n=1}^{\infty}) = (0, a_1, a_2, \ldots)$ is obviously a shift. The fact which emerges from our discussion below is that this example is, in a sense, the only one possible. More particularly, we show that if a space of continuous functions supports a shift operator then the underlying space must be composed of infinitely many components, thereby ruling out the possibility of a shift on many of the common function spaces. This is the content of

THEOREM 2.1. *Let K be a compact Hausdorff space having a finite number of components. Then there is no shift operator on $C(K)$.*

PROOF. Suppose $S:C(K) \to C(K)$ is a shift. We consider the two possibilities:

CASE (i). *No component of K consists of a single point* (and thus no point of K is open).

CLAIM. If $f \in C(K)$ and $|f(t)| = 1$ for $t \in K$, then $|(SF)(t)| = 1$ for all $t \in K$.

For, suppose $|S(f)(t_0)| = 1 - \epsilon < 1$ for some $t_0 \in K$. Since $S(f) \in C(K)$ there is then an open set $U(t_0)$ about t_0 so that if $t \in U(t_0)$ then $|S(f)(t)| \leqq 1 - \epsilon/2$. Moreover, since t_0 is not an open set there is at least one other point t_1 in $U(t_0)$.

Now K is a Hausdorff space so there are disjoint open sets $U_0(t_0)$ and $U_1(t_1)$, both of which are in $U(t_0)$. Since t_0 and t_1 are closed subsets of K, and hence compact, by Urysohn's Lemma [7, p. 40] there exist non-zero continuous functions f_0 and f_1 on K whose supports lie in the sets $U_0(t_0)$ and $U_0(t_1)$, respectively.

By assumption the codimension of the range of S in $C(K)$ is one, implying there is a non-zero linear combination of f_0 and f_1 which is in Ran S. Call this function $g(t)$, and assume $||g|| = 1$.

Then the support of $g(t)$ is in $U(t_0)$, and $g(t) = S(h)(t)$ for some function $h(t) \in C(K)$ with $||h(t)|| = 1$. Since $|f(t)| = 1$ for all t it must be that either $||(f + h)|| = 2$ or $||f - h|| = 2$, and hence that $||Sf + g|| = 2$ or $||Sf - g|| = 2$.

But this is impossible since $|Sf(t)| \leqq 1 - \epsilon/2$ for all $t \notin U(t_0)$ while $g(t) = 0$ for $t \notin U(t_0)$. Therefore if $|f(t)| = 1$ for all $t \in K$, then $|S(f)(t)| = 1$ for all $t \in K$ also and the claim is established.

Now let $K = \cup_{i=1}^n K_i$, where $\{K_i\}_{i=1}^n$ is the set of components of K. If $f(t) \in C(K)$ and $|f(t)| = 1$ for all that $t \in K$, then it must be that for every $i = 1, 2, \ldots, n$, $f(t) = 1$ or $f(t) = -1$ for all $t \in K_i$. That is, we can associate with every such $f(t)$ the ordered n-tuple $(\epsilon_1, \ldots, \epsilon_n)$ where $|\epsilon_i| = 1$ for all i and $f(t) = \epsilon_i$ for all $t \in K_i$. Since there are only 2^n such n-tuples, there are only 2^n functions $f(t) \in C(K)$ with $|f(t)| = 1$ for all t. Hence the set $\{S^n(1)\}_{n=0}^{\infty}$ is, by our observation above, a finite set. In particular $S^i(1) = S^j(1)$ for some $0 \leqq i < j$, implying $S^{j-i}(S^i(1)) = S^i(1)$. If we let $f(t) = S^i(1)$ and $j - i = k$, then we have $S^k(f) = f$, so $S^{nk}(f) = f$ for all $n = 1, 2, \ldots$. That is, $f \in \text{Ran}(S^{nk})$ for $n = 1, 2, \ldots$, and hence $f \in \text{Ran } S^m$ for all $m = 1, 2, \ldots$ since $\text{Ran}(S^{m+1}) \subset \text{Ran } S^m$ for all m. But $|f(t)| = 1$ for all $t \in K$ and $\cap_{m=1}^{\infty} \text{Ran } S^m = \{0\}$ by assumption, a contradiction. Therefore there can be no shift on $C(K)$ in the case where no component of K is a single point.

CASE (ii) *At least one component of K is a single point.*

Clearly we may assume that K itself is an infinite set (or else no isometry having a range of codimension 1 can exist on $C(K)$, and we're done).

Therefore assume the components of K are the sets $\{K_1, \ldots, K_n\}$ where $\{K_1, K_2, \ldots, K_m\}$ $(1 \leqq m < n)$ are points and $\{K_{m+1}, \ldots, K_n\}$ are infinite sets.

If we set $J = \cup_{i=1}^m K_i$ and $M = \cup_{i=m+1}^n K_i$ then any $f \in C(K)$ can be uniquely written as $f = f_1 + f_2$, where $f_1 \in C(K)$ has support in M and $f_2 \in C(K)$ is supported on J. We also note that $C(J)$ is isometrically isomorphic to ℓ_m^{∞}, an m-dimensional space.

Now suppose

$$f_0(x) = \begin{Bmatrix} 1 & , & \text{if } x \in M \\ f_2(x), & & \text{if } x \in J \end{Bmatrix},$$

where $f_2(x)$ is arbitrary in $C(J)$ and $\|f_2\|_{\infty} \leqq 1$. If $|S(f_0)(t_0)| < 1$ for some $t_0 \in M$ then there is an open set $U(t_0) \subset M$ on which $|S(f_0)(t)| \leqq 1 - \epsilon$ for some $\epsilon > 0$. Since M is infinite we can use the same argument as in the proof of Case (i) above to find, for any integer $k \geqq 2$, a set of k linearly independent functions $\{q_i\}_{i=1}^k$ in $C(K)$ whose supports all lie in $U(t_0)$ and which are in the range of S. Choose any $k > m$, and suppose $q_i(t) = S(g_i)$ $(i = 1, 2, \ldots, k)$ is such a set of functions. By the above, each g_i can be written as $g_i = g_1^{(i)} + g_2^{(i)}$ where $\{g_2^{(i)}\}_{i=1}^k$ is a set of functions supported on J, hence a set of k functions in an m-dimensional space $(k > m)$. Thus there is a linear combination $g(t) = (\sum_{i=1}^k c_i g_i)(t) = \sum_{i=1}^k c_i(g_1^{(i)}(t) + g_2^{(i)}(t))$ for which $\|g\| = 1$ and $\sum_{i=1}^k c_i(g_2^{(i)}(t)$

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$= 0$ for all t. Note, then, that $g(t)$ has its support in M and the support of $S(g)$ is in $U(t_0)$. It follows that since $\|g\| = 1$ and $f_0(t) = 1$ for all $t \in M$ we must have $\|f_0 + g\| = 2$ or $\|f_0 - g\| = 2$. But $\|Sf_0 + Sg\|$ and $\|Sf_0 - Sg\|$ are both $\leqq 2 - \epsilon$ (since $\|Sf_0\| = 1$ with $|Sf_0(t)| \leqq 1 - \epsilon$ for all $t \in U(t_0)$). This contradicts the fact that S is an isometry, so we see that if $\|f_0\| = 1$ and $f_0(t) = 1$ for $t \in M$, then $|S(f_0)(t)| = 1$ for all $t \in M$.

In particular this is true when f_0 is the characteristic function of M (which is in $C(K)$). Then if $g(t)$ is any function whose support is in J and $\|g\| \leqq 1$ we have that $f = f_0 + g$ is in $C(K)$ with $\|f\| = 1$, so $1 = \|S(f)\| = \|Sf_0 + Sg\|$. Similarly $\|Sf_0 - Sg\| = 1$. By the above, $|S(f_0)(t)| = 1$ for all $t \in M$, so it must be that $S(g)(t) = 0$ for all $t \in M$. That is, if $g(t) \in C(K)$ has support in J, then so does $S(g)$. This implies that S induces a mapping from $C(J)$ into $C(J)$, where $C(J) = \ell_m^\infty$, a finite dimensional space. Since S is an isometry and hence one-to-one, this induced mapping must be onto $C(J)$, implying that the set of all functions in $C(K)$ with support in J is contained in the range of S^n for all $n = 1, 2, \ldots$, a contradiction to the definition of a shift, and the theorem is proved in Case (ii) as well.

In particular we have:

COROLLARY 2.2. *There is no shift operator on the space* $C[a, b]$.

In actuality the same result holds (with essentially the same proof) for many of the important Banach spaces of continuous functions on an interval, as we now show.

THEOREM 2.3. *Let F be a Banach space consisting of continuous functions on an interval $[a, b]$ for which*

(1) $\|f\|_F = \|f\|_\infty + \alpha(f)$ *for all* $f \in F$, *where* $\|\cdot\|_F$ *denotes the norm on* F, $\|\cdot\|_\infty$ *denotes the supremum norm, and* α *is a semi-norm on* F;

(2) $1 \in F$ *and* $\alpha(1) = 0$;

(3) *Given any interval* $I \subset [a, b]$ *there exists an infinite dimensional subspace of* F *all of whose members have support in* I.

Then there is no shift on F.

PROOF. Suppose $S:F \to F$ is a shift. Note that for any $f \in F$, $\alpha(1 + f) \leqq \alpha(1) + \alpha(f) = \alpha(f) = \alpha((1 + f) - 1) \leqq \alpha(1 + f) + \alpha(1) = \alpha(1 + f)$. That is, $\alpha(1 + f) = \alpha(f)$ for all $f \in F$. Hence if $f \in F$ and $\|1 + f\|_\infty = 1 + \|f\|_\infty$ then

$$\begin{aligned}\|1 + f\|_F &= \|1 + f\|_\infty + \alpha(1 + f)\\ &= 1 + \|f\|_\infty + \alpha(f)\\ &= 1 + \|f\|_F.\end{aligned}$$

Similarly, if $\|1 - f\|_\infty = 1 + \|f\|_\infty$ we also have $\|1 - f\|_F = 1 + \|f\|_F$.

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Thus, given $f \in F$ either $\|1 + f\|_F$ or $\|1 - f\|_F$ equals $1 + \|f\|_F$, so *either* $\|S(1) + S(f)\|_F$ *or* $\|S(1) - S(f)\|_F = 1 + \|f\|_F$.

Suppose $S(1)$ is not a constant on $[a, b]$. Then $S(1) = h(t) \in F$, where $|h(t)| < \|h\|_\infty$ for all t in some interval $I \subset [a, b]$. By (3) above (and the fact that codim(Ran S) = 1 in F) there is a function $g \in$ Ran S so that $\|g\|_F = 1$ and the support of g lies in I. In particular, then, both $\|h + g\|_\infty$ and $\|h - g\|_\infty$ are less than $\|h\|_\infty + \|g\|_\infty$. Therefore if $g = Sf$ for some f with $\|f\|_F = 1$ and $\|S(1) + S(f)\|_F = 1 + \|f\|_F$ then we have

$$\begin{aligned} 1 &= \|f\|_F = \|S(1) + S(f)\|_F = \|h + g\|_F \\ &= \|h + g\|_\infty + \alpha(h + g) < \|h\|_\infty + \|g\|_\infty + \alpha(h) + \alpha(g) \\ &= \|h\|_F + \|g\|_F = \|S(1)\|_F + \|Sf\|_F = 1 + \|f\|_F, \end{aligned}$$

a contradiction.

The other possibility, that $\|S(1) - S(f)\|_F = 1 + \|f\|_F$, leads to the same contradiction. Hence it must be that $S(1)$ is a (non-zero) constant function on $[a, b]$, so $1 \in$ Ran S^n for all $n \geqq 1$, and S is not a shift, a contradiction which then establishes the theorem.

As an example of a space of functions to which Theorem 2.3 applies we mention:

COROLLARY 2.4. *There is no shift on the space* $C^{(n)}[a, b]$ $(n \geqq 1)$ *of all real valued functions on* $[a, b]$ *having n continuous derivatives there.*

PROOF. The norm on $C^{(n)}[a, b]$ is given by $\|f\| = \sum_{i=0}^n \|f^{(i)}\|_\infty = \|f\|_\infty + \sum_{i=1}^n \|f^{(i)}\|_\infty$, where $\alpha(f) = \sum_{i=1}^n \|f^{(i)}\|_\infty$ clearly satisfies the conditions of Theorem 2.3.

Though Corollary 2.2 shows there is no shift on the space $C[a, b]$, it is interesting to note that this non-existence is a direct consequence of the fact that we require the range of a shift to have finite codimension. If we drop this requirement we get a quite different result.

DEFINITION. An operator T on a Banach space X is called a *semi-shift* if

1) T is an isometry, and
2) $\cap_{n=1}^\infty$ Ran $T^n = 0$.

THEOREM 2.5. *There is a semi-shift on* $C[0, 1]$.

PROOF. Let $\{\phi_n\}_{n=0}^\infty$ denote the usual Schauder basis for $C[0, 1]$ (see, e.g., [8, p. 11]). Define the operator $T: C[0, 1] \to C[0, 1]$ by:

$$T(\phi_0) = \phi_1 + \frac{1}{2}\phi_2,$$

$$T(\phi_1) = \phi_1 - \frac{1}{2}\phi_2,$$

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$$T(\phi_2) = \phi_4, \text{ and}$$

$$T(\phi_{2^n+j}) = \phi_{2^{n+1}+2^n+j} \quad \text{for } n \geqq 1 \text{ and } 1 \leqq j \leqq 2^n.$$

Then by definition of the functions $\{\phi_n\}_{n=0}^{\infty}$ we see that

$$(T\phi_0)(t) = \begin{Bmatrix} 2t, & 0 \leqq t \leqq \frac{1}{2} \\ 1, & \frac{1}{2} \leqq t \leqq 1 \end{Bmatrix}, \ (T\phi_1)(t) = \begin{Bmatrix} 0, & 0 \leqq t \leqq \frac{1}{2} \\ 2t - 1, & \frac{1}{2} \leqq t \leqq 1 \end{Bmatrix},$$

and $(T\phi_{2^n+j})(t) = 0$ if $t \leqq [0, 1/2]$ for all $n = 1, 2, \ldots,$ and all $1 \leqq j \leqq 2^n$. Moreover if $h:[1/2, 1] \to [0, 1]$ is the homeomorphism defined by $h(t) = 2t - 1$ then it is obvious from the above and from the definitions of the functions $\{\phi_n\}_{n=0}^{\infty}$ and $\{T\phi_n\}_{n=0}^{\infty}$ that if $t \in [1/2, 1]$ then

$$\sum_{n=0}^{N} a_n T\phi_n(t) = \sum_{n=0}^{N} a_n \phi_n(h(t)) \quad \text{for all} \quad \{a_n\}_{n=0}^{N}.$$

Hence for any $\{a_n\}_{n=0}^{N}$ we have

$$\begin{aligned}
\left\| \sum_{n=0}^{N} a_n T\phi_n \right\| &= \max\left\{ |a_0|, \sup_{t \in [1/2,1]} \left| \sum_{n=0}^{N} a_n T\phi_n(t) \right| \right\} \\
&= \max\left\{ |a_0|, \sup_{t \in [1/2,1]} \left| \left(\sum_{n=0}^{N} a_n \phi_n \right) h(t) \right| \right\} \\
&= \max\left\{ |a_0|, \sup_{s \in [0,1]} \left| \sum_{n=0}^{N} a_n \phi_n(s) \right| \right\} \\
&= \max\left\{ |a_0|, \left\| \sum_{n=0}^{N} a_n \phi_n \right\| \right\} \\
&= \left\| \sum_{n=0}^{N} a_n \phi_n \right\| \left(\text{since } \left\| \sum_{n=0}^{N} a_n \phi_n \right\| \geqq |a_0| \right).
\end{aligned}$$

Thus T is an isometry. From the definition of the set $\{T\phi_n\}_{n=0}^{\infty}$ we see that for $n \geqq 1$,

$$T^n \phi_0 = \phi_1 - \frac{1}{2}\phi \ldots . - \frac{1}{2}\phi_{2^{n-1}} + \frac{1}{2}\phi_{2^n},$$

$$T^n \phi_1 = \phi_1 - \frac{1}{2}\phi_2 \ldots . - \frac{1}{2}\phi_{2^{n-1}} - \frac{1}{2}\phi_{2^n},$$

and $T^n\phi_i = \phi^{(n)}_{\alpha(i)}$ for $i \geqq 2$, where $\alpha^{(n)}(i) \geqq 2^{n+1}$ for each $i \geqq 2$ and for all $n \geqq 1$. If $f_0 = \sum_{i=0}^{\infty} a_i\phi_i$ is in Ran T^n for all $n \geqq 1$ then $f_0 = T^n(g_n) = T^n(\sum_{i=0}^{\infty} b_i^{(n)}\phi_i)$ for all $n = 1, 2, \ldots$. That is,

$$\sum_{i=0}^{\infty} a_i\phi_i = b_0^{(n)}\left[\phi_1 - \frac{1}{2}\phi_2 - \ldots - \frac{1}{2}\phi_{2^{n-1}} + \frac{1}{2}\phi_{2^n}\right]$$

$$+ b_1^{(n)}\left[\phi_1 - \frac{1}{2}\phi_2 - \ldots - \frac{1}{2}\phi_{2^{n-1}} - \frac{1}{2}\phi_{2^n}\right]$$

$$+ b_2^{(n)}\phi_{\alpha^{(n)}(2)} + b_3^{(n)}\phi_{\alpha^n(3)} + \ldots$$

$$= \left[(b_0^{(n)} + b_1^{(n)})\phi_1 - \frac{1}{2}(b_0^{(n)} + b_1^{(n)})\phi_2\right.$$

$$\left. - \ldots - \frac{1}{2}(b_0^{(n)} + b_1^{(n)})\phi_{2^{n-1}} + \frac{1}{2}(b_0^{(n)} - b_1^{(n)})\phi_{2^n}\right]$$

$$+ \sum_{i=2}^{\infty} b_i^{(n)}\phi_{\alpha^{(n)}(i)}, \quad \text{where } \alpha^{(n)}(i) \geqq 2^{n+1}.$$

It follows that $a_i = 0$ if $0 \leqq i \leqq 2^n$ but $i \notin \{2^k\}_{k=0}^{\infty}$, and that $a_2 = a_4 = \ldots = a_{2^{n-1}} = -(1/2)a_1$. Since this is true for all $n \geqq 1$ it must be that $a_i = 0$ for all $i \geqq 0$, and hence that $f_0 = 0$. Therefore $\cap_{n=1}^{\infty}$ Ran $T^n = 0$ and T is a semi-shift.

QUESTION. If X is a separable Banach space, does there always exist a semi-shift on X?

3. **Backward Shifts.** If X is a Hilbert space and $\{\phi_n\}_{n=1}^{\infty}$ an orthonormal basis for X, the operator T on X defined by $T(\phi_1) = 0$ and $T\phi_n = \phi_{n-1}$ for $n \geqq 2$ is called a *backward shift* (and is, in fact, the adjoint of a shift operator on X). Once again it is easy to check that an operator T on X is a backward shift if and only if it satisfies the properties:

1) dim(Ker T) = 1,

2) the induced operator $\hat{T}:X/\text{Ker } T \to X$ defined by $\hat{T}(x + \text{Ker } T) = Tx$ is an isometry,

3) $\cup_{n=1}^{\infty}$ Ker T^n is dense in X.

Consequently we introduce the following generalization of this idea:

DEFINITION. An operator T on a Banach space X is called a *backward shift* if it has the properties (1), (2), and (3) above.

Once again the operator T on c defined by $T(\{a_n\}_{n=1}^{\infty}) = (a_2, a_3, \ldots)$ is an example of such a backward shift on a space of continuous functions, and once again this example is the prototype of the only such case in which this can occur. To see that this is so it is worthwhile to first prove the following rather general result.

THEOREM 3.1. *If X is a Banach space whose unit ball has at least one, but at most finitely many, extreme points, then there is no backward shift on X.*

PROOF. Suppose $T:X \rightarrow X$ were a backward shift. Let $M = \text{Ker } T$. Since $\dim M = 1$ we have $M = \{\lambda m_0 | \lambda \in R\}$, where $m_0 \neq 0$ is in M. By assumption, $\hat{T}:X/M \rightarrow X$ is an isometry so $||Tx|| = ||\hat{T}(x + M)|| = |\,||x + M||\,| = \inf||x + m|| = \text{dist}(x, M)$ for all $x \in X$. In particular, then, $||T|| \leqq 1$.

Let $\{e_i\}_{i=1}^{N}$ be the set of an extreme point of the unit ball of X. Since $\hat{T}$ is an isometry of X/M onto X there exists an $x \in X$ for which $\hat{T}(x + M) = e_1$, where $|\,||x + M||\,| = 1$. Note that $Tx = e_1$ also. Since a one-dimensional subspace is proximinal in X, $||x + \lambda m_0|| = |\,||x + M||\,| = 1$ for some $\lambda \in R$ and if $x_1 = x + \lambda m_0$ then $||x_1|| = 1$ and $Tx_1 = e_1$. If we let $A = \{\lambda \in R|\; ||x_1 + \lambda m_0|| = 1\}$ then A is a non-empty bounded subset of R and hence $\sup A = b$ exists. Clearly A is closed (since $f(\lambda) = ||x_1 + \lambda m_0||$ is continuous on R), so $b \in A$. That is, $||x_1 + bm_0|| = 1$, $T(x_1 + bm_0) = e_1$, and b is the largest number with these properties.

Let $x_0 = x_1 + bm_0$, so $||x_0|| = 1$ and $Tx_0 = e_1$.

CLAIM. x_0 is an extreme point of the unit ball of X.

If not, $x_0 = 1/2y_1 + 1/2y_2$ for $||y_1|| = ||y_2|| = 1$ and $y_i \neq x_0$, $i = 1, 2$. Then $e_1 = Tx_0 = 1/2Ty_1 + 1/2Ty_2$ where $||Ty_i|| \leqq 1$ for $i = 1, 2$ since $||T|| \leqq 1$. Since e_1 is an extreme point of the unit ball it follows that $Ty_1 = Ty_2 = Tx_0 = e_1$, and hence that $y_1 = x_0 + m_1$ and $y_2 = x_0 + m_2$ for $m_i \in M$ with $m_i \neq 0$, $i = 1, 2$.

But then $x_0 = y_1 - m_1$ and $x_0 = y_2 - m_2$, so $x_0 = (1/2y_1 + 1/2y_2) - 1/2(m_1 + m_2) = x_0 - 1/2(m_1 + m_2)$. It follows that $m_1 + m_2 = 0$, and hence that $y_1 = x_0 + m_1$ and $y_2 = x_0 - m_1$ (by the above), where $m_1 \neq 0$ and $||y_1|| = ||y_2|| = 1$. Since $m_1 = \lambda_1 m_0$ we then have

$$1 = ||x_0 + m_1|| = ||x_1 + bm_0 + \lambda_1 m_0||, \quad \text{and}$$

$$1 = ||x_0 - m_1|| = ||x_1 + bm_0 - \lambda_1 m_0||.$$

That is, $1 = ||x_1 + (b + \lambda_1)m_0|| = ||x_1 + (b - \lambda_1)m_0||$. However then $b + \lambda_1$ and $b - \lambda_1$ are both in A, and since $\lambda_1 \neq 0$ one of these two is larger than b, contradicting the definition of $b = \sup A$. Thus it must be that x_0 is an extreme point of the unit ball of X for which $Tx_0 = e_1$.

A similar argument establishes the fact that given any extreme point e_i of the

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unit ball there is another extreme point z_i for which $Tz_i = e_i$. Since the set of extreme points is finite it is obvious that this extreme point z_i is unique for each $i = 1, 2, \ldots, N$. Thus is must be that $Te_i = e_{\pi(i)}$ for $i = 1, 2, \ldots, N$, where π denotes some permutation of the set $\{1, 2, \ldots, N\}$.

Now suppose that for some n there is some $w \in \text{Ker } T^n$ for which $\|e_1 - w\| < 1$. Then since $\|T\| \leqq 1$, $\|T^n\| \leqq 1$ also and we have

$$1 > \|T^n e_1 - T^n w\| = \|T^n e_1\| = \|e_{\pi^n(1)}\| = 1,$$

a contradiction. Hence $\|e_1 - w\| \geqq 1$ for all $w \in \text{Ker } T^n$ and any $n \geqq 1$, implying $\cup_{n=1}^{\infty} \text{Ker } T^n$ is not dense in X and thereby contradicting property (3) of a backward shift. The theorem follows.

Corollary 3.2. *If K is a compact Hausdorff space having only finitely many components, then $C(K)$ has no backward shift.*

Proof. Suppose $f \in C(K)$ is an extreme point of the unit ball in $C(K)$. If $|f(t_0)| < 1$ for some $t_0 \in K$ then there is an open set $U(t_0) \subset K$ and an $\epsilon > 0$ such that $|f(t)| \leqq 1 - \epsilon$ for all $t \in U(t_0)$. By Urysohn's Lemma there is a non-zero continuous function $g(t)$ whose support is in $U(t_0)$ and for which $\|g\| < \epsilon$ [7, p. 40].

If we let $f_1 = f + g$ and $f_2 = f - g$ then $\|f_i\| = 1$, $f = 1/2 f_1 + 1/2 f_2$, and $f_i \neq f$, $i = 1, 2$. That is, f is not an extreme point of the unit ball in $C(K)$, a contradiction.

Hence $|f(t)| = 1$ for all $t \in K$, implying $f(t) = 1$ or $f(t) = -1$ for all t in any component of K. Since there are only a finite number of components of K, this says there are only a finite number of extreme points of the unit ball in $C(K)$, so by Theorem 3.1 there is no backward shift on $C(K)$.

4. **Unsolved Problems.** While the results given here show that many of the common Banach spaces of continuous functions fail to have shifts and/or backward shifts on them, still they do not apply to several important such spaces. For example they shed no light on the question of whether or not there is a shift on the space $L^{\infty}(\mu)$ which is known to be isometric to a space $C(T)$ for some compact Hausdorff space T [2, p. 445]. Consequently we pose the following problems:

1) Characterize those compact Hausdorff spaces K with the property that $C(K)$ admits a shift (or a backward shift).

Conjecture. If K has at least one infinite component there is no shift or backward shift on $C(K)$.

2) Is there a shift (or backward shift) on $L^{\infty}(\mu)$? What about $L^1(\mu)$?

3) One can show there is a semi-shift on the space $K(\ell^2)$ of all compact operators on the Hilbert space ℓ^2. Does there exist a shift on $K(\ell^2)$?

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