# POLYTOPES WITH CENTRALLY SYMMETRIC FACES

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*In honour of Professor H. S. M. Coxeter on his sixtieth birthday*

**Introduction.** If a convex polytope $P$ is centrally symmetric, and has the property that all its faces (of every dimension) are centrally symmetric, then $P$ is called a *zonotope*. Zonotopes have many interesting properties which have been investigated by Coxeter and other authors (see **(4**, §2.8 and §13.8**)** and **(5)** which contains a useful bibliography). In particular, it is known **(5**, §3**)** that a zonotope is completely characterized by the fact that all its two-dimensional faces are centrally symmetric. The purpose of this paper is to generalize these results, investigating the properties of polytopes all of whose $j$-dimensional faces are centrally symmetric for some given value of $j$. We shall prove four theorems, the statements of which will be given in this introductory section; proofs will appear in later sections of the paper.

For brevity we shall write *$d$-polytope* to mean a $d$-dimensional closed convex polytope in Euclidean space $E^n$ $(n \geqslant d)$, *$j$-face* to mean a (closed) $j$-dimensional face of such a polytope, and *$r$-flat* to mean an $r$-dimensional affine subspace of $E^n$. Our first theorem generalizes a result of A. D. Alexandrov **(1)**:

THEOREM 1. *If every $j$-face of a $d$-polytope $P$ is centrally symmetric, where $j$ is some integer satisfying $2 \leqslant j \leqslant d$, then the $k$-faces of $P$ are also centrally symmetric for all $k$ such that $j \leqslant k \leqslant d$.*

Here we are regarding $P$ as a $d$-face of itself, so the theorem implies that, under the given conditions, $P$ is a centrally symmetric polytope.

Let $P$ be any given $d$-polytope in $E^d$, and $R$ be any $r$-flat passing through the origin $o$. Let $\pi_R$ denote orthogonal projection on to $R$, so that $\pi_R(P)$ is an $r$-polytope in $R$. Then since, for each $j$ satisfying $0 \leqslant j \leqslant r - 1$, the $j$-faces of $\pi_R(P)$ are the images under $\pi_R$ of faces of $P$ whose dimension is at least $j$, we deduce the following: If $j$ and $r$ are given integers satisfying $2 \leqslant j \leqslant r \leqslant d$, and if $P$ is a $d$-polytope with centrally symmetric $j$-faces, then the $j$-faces of $\pi_R(P)$ are also centrally symmetric. However, for certain $r$-flats $R$ we can assert much more:

THEOREM 2. *If every $j$-face of a $d$-polytope $P \subset E^d$ is centrally symmetric, where $j$ is some integer satisfying $2 \leqslant j \leqslant d$, and $R$ is a $(d - j + 1)$-flat orthogonal to any $(j - 1)$-face $F^{j-1}$ of $P$, then $\pi_R(P)$ is a zonotope.*

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In the proof of this theorem we shall show that the vertices of $\pi_R(P)$ are the images under $\pi_R$ of $(j-1)$-faces of $P$, each of which is either congruent to $F^{j-1}$ or to the reflection of $F^{j-1}$ in a point. In this way it can be shown that the $(d-1)$-faces of $P$ lie in a number of "zones" analogous to the zones of faces of zonotopes.

The next theorem relates to projection on to $r$-flats which are not orthogonal to any face of $P$. Let $Q$ represent the set of $j$-flats through $o \in E^d$ which are parallel to the $j$-faces of $P \subset E^d$ for all $j$ satisfying $1 \leqslant j \leqslant d-1$. Then an $s$-flat $S$ $(0 \leqslant s \leqslant d-1)$ through $o$ is said to be in *general position* with respect to $P$ if $S$ meets each $j$-flat of $Q$ in a flat of $\max(0, s+j-d)$ dimensions. If $R$ is an $r$-flat $(1 \leqslant r \leqslant d)$, following the terminology of **(8)**, we shall call $\pi_R(P)$ a *regular r-projection of P* if and only if the $(d-r)$-flat through $o$ which is orthogonal to $R$ is in general position with respect to $P$. (In the important special case $r = d-1$, a regular projection results if the line orthogonal to $R$ is not parallel to any proper face of $P$.) The polytope $P$ is called *r-equiprojective* if, for $0 \leqslant j \leqslant r-1$, the number $f_j(\pi_R(P))$ of $j$-faces of $\pi_R(P)$ has the same value for all regular projections $\pi_R(P)$. For example, it is a familiar fact that every regular 2-projection of a 3-cube $C^3$ is a hexagon, so $C^3$ is 2-equiprojective. Less familiar is the fact (which, so far as the author is aware, has not previously been mentioned in the literature) that every $d$-dimensional zonotope is $r$-equiprojective for each value of $r$ satisfying $1 \leqslant r \leqslant d$ (compare **(10**, §4**)**). It is this property of zonotopes which we generalize.

THEOREM 3. *If every r-face of a d-polytope P is centrally symmetric, where r is some integer satisfying* $2 \leqslant r \leqslant d$, *then P is r-equiprojective.*

This theorem cannot be strengthened. If $P$ has centrally symmetric $j$-faces, and $j > r$, then in general $P$ will not be $r$-equiprojective. For example, the regular 24-cell $Q^4$ has centrally symmetric 3-faces (octahedra) and so is 3-equiprojective, a fact that is illustrated by the models of the regular 3-projections shown in **(4**, Plate VI**)**. On the other hand its 2-faces (triangles) are not centrally symmetric and it is not 2-equiprojective since some regular 2-projections are hexagons (see **(4**, Fig. 14.3c**)**) and others are 12-gons.

Theorems 1 and 2 imply that if for some value of $j$ $(2 \leqslant j \leqslant d)$ the $j$-faces of $P$ are centrally symmetric, then $P$ will be $r$-equiprojective for all $r$ satisfying $j \leqslant r \leqslant d$. This is a special property of polytopes which have centrally symmetric faces, for it is not generally true that $j$-equiprojective polytopes are also $r$-equiprojective for all $r \geqslant j$. For example, if $T_1$ and $T_2$ are two triangles in $E^4$ such that $T_1 \cap T_2$ is a point in the relative interior of each triangle, then the "prism" $T_1 + T_2$ (vector addition) is 2-equiprojective because all its regular 2-projections are hexagons, but it is not 3-equiprojective since some of its regular 3-projections have eight vertices and others have nine vertices.

The concept of equiprojectivity is of some intrinsic interest, since no characterization of $r$-equiprojective $d$-polytopes is known, even for $r = 2$, $d = 3$. Further, it has recently been shown **(8**, §3; **10**, §4**)** that the angles of equiprojective polytopes have interesting invariance properties. We recall that with each $j$-face $F^j$ of a $d$-polytope $P$ $(0 \leqslant j \leqslant d-1)$ is associated a well-defined real number $\phi(P, F^j)$ called the *interior angle* of $P$ at the face $F^j$. (For a formal definition, see (**8**, §2).) If we sum the interior angles at all the $j$-faces of $P$, we obtain the *jth angle sum* of $P$, denoted by $\phi_j(P)$. This angle sum is *affine invariant* if $\phi_j(P) = \phi_j(TP)$ for all non-singular affine transformations $T$ of $E^d$. It is known (**8**, §3) that all the angle sums $\phi_j(P)$ $(0 \leqslant j \leqslant d-1)$ of $P$ are affine invariant if and only if $P$ is $(d-1)$-equiprojective. For example, since a 3-cube $C^3$ is 2-equiprojective, its angle sums $\phi_0(C^3) = 1$, $\phi_1(C^3) = 3$, $\phi_2(C^3) = 3$ are affine invariants, leading to familiar facts about the vertex angles and dihedral angles of a parallelepiped in $E^3$. Since a regular 24-cell $Q^4$ is 3-equiprojective, all its angle sums are affine invariant; in fact, using (**8**, Theorem (10)) we see that $\phi_0(Q^4) = 3$, $\phi_1(Q^4) = 24$, $\phi_2(Q^4) = 32$, and $\phi_3(Q^4) = 12$. These figures enable us to calculate the interior angles of $Q^4$ in a very simple manner.

If the $d$-polytope $P$ is $(d-2)$-equiprojective, then it is known that its angle deficiencies are affine invariant (see (**10**) for the definitions and proof). In the case of polytopes with centrally symmetric faces, a more powerful assertion is possible, which is given in the final theorem:

THEOREM 4. *If, for some value of j satisfying* $2 \leqslant j \leqslant d-1$, *all the j-faces of a d-polytope P are centrally symmetric, then for any k-face* $F^k$ *of P* $(j < k \leqslant d)$, *all the angle sums* $\phi_j(F^k)$ $(0 \leqslant j \leqslant k-1)$ *are affine invariant.*

Theorem 4 is an immediate consequence of the above assertions and of Theorem 3 applied to each $k$-face $F^k$ of $P$. We shall now give proofs of the first three theorems.

**Proof of Theorem 1.** We begin by recalling the following classical result:

LEMMA 1. *If all the* $(d-1)$*-faces of a d-polytope P* $(d \geqslant 3)$ *are centrally symmetric, then P is centrally symmetric.*

The first proof of this lemma was given by A. D. Alexandrov in 1933 (**1**) for the case $d = 3$, and he states that his proof "extends easily to any number of dimensions" without giving any details. Other proofs for the case $d = 3$ will be found in (**3**; **4**; **5**). Here a new proof will be given for the general case, which is, even for $d = 3$, simpler than each of the proofs just mentioned.

Let $R$ be any $(d-1)$-flat in $E^d$ which does not intersect the given polytope $P \subset E^d$. A $(d-1)$-face $F^{d-1}$ of $P$ will be said to be *remote* from $R$ if the line segment joining any relative interior point $z$ of $F^{d-1}$ to $\pi_R(z)$ intersects the interior of $P$. It is clear that this definition is independent of the choice of $z$, and that $\pi_R(P)$ is the union of the images under orthogonal projection on to $R$ of all those $(d-1)$-faces of $P$ which are remote from $R$. For any $F^{d-1}$, the polytope $\pi_R(F^{d-1}) \subset \pi_R(P)$ is centrally symmetric, so $\pi_R(P)$ is the union of centrally symmetric $(d-1)$-polytopes which are non-overlapping, that is, two such polytopes intersect in at most boundary points of each. By a theorem of Minkowski (**7**, §6) this is sufficient to establish that $\pi_R(P)$ is centrally symmetric. Thus $\pi_R(P)$ is centrally symmetric for each $R$, and by a theorem of Blaschke and Hessenberg (**2**, §61; **9**) this implies that $P$ is centrally symmetric. Thus the lemma is proved.

The proof of Theorem 1 now follows immediately. If the $j$-faces of $P$ are centrally symmetric, then since they are the $j$-faces of the $(j+1)$-faces of $P$, the lemma shows that the $(j+1)$-faces of $P$ are centrally symmetric. Thus $k-j$ applications of the lemma will establish that, for $j \leqslant k \leqslant d$, all the $k$-faces of $P$ are centrally symmetric, and so Theorem 1 is proved.

**Proof of Theorem 2.** We require the following lemma:

LEMMA 2. *Let $P$ be a convex polytope in $E^d$ with centrally symmetric $j$-faces for some value of $j \geqslant 2$, and $R$ be a $(d-j+1)$-flat perpendicular to some $(j-1)$-face $F^{j-1}$ of $P$. Then if $\pi_R(P)$ is a $(d-j+1)$-polytope, it has the property that for $j-1 \leqslant s \leqslant d$ each of its $(s-j+1)$-faces is the image under $\pi_R$ of some $s$-face of $P$. In particular, $P$ is a $d$-polytope.*

*Proof.* The proof is by induction on $d$.

If $d = j$, $P$ is a centrally symmetric $d$-polytope and $\pi_R(P)$ is a line segment (1-polytope). Clearly the two vertices of $\pi_R(P)$ (the end points of the line segment) are the images under $\pi_R$ of $F^{j-1}$ and of the face $*F^{d-1}$ which is the image of $F^{j-1}$ under reflection in the centre of $P$. Hence the lemma is true in this case.

Now assume, as inductive hypothesis, that the lemma is true for polytopes in $E^{d-1}$ with centrally symmetric $j$-faces for some value of $j$ satisfying

$$2 \leqslant j \leqslant d-1.$$

Let $P$ be a convex polytope in $E^d$ with centrally symmetric $j$-faces, $F^{j-1}$ be the chosen $(j-1)$-face of $P$, and $T$ be the $(j-1)$-flat containing $F^{j-1}$ and perpendicular to the $(d-j+1)$-flat $R$. Let $H$ be any $(d-j)$-flat in $R$ which supports $\pi_R(P)$, contains the vertex $\pi_R(F^{j-1})$, and intersects $\pi_R(P)$ in a $(d-j)$-face $G^{d-j}$. Then $H$ is perpendicular to $T$, and the $(d-1)$-flat spanned by $H$ and $T$ supports $P$ and so intersects $P$ in some face $F \supset F^{j-1}$. The inductive hypothesis shows that every $(s-j+1)$-face of $G^{d-j}$ is, for $j-1 \leqslant s \leqslant d-1$, the image under $\pi_R$ of an $s$-face of $F$ (and so, in particular, $F$ is a $(d-1)$-face of $P$). Thus every vertex of $G^{d-j}$ is the image under $\pi_R$ of a $(j-1)$-face of $P$. Let ${F_1}^{j-1}$ be one of these faces. Then repeating the above argument using ${F_1}^{j-1}$ instead of $F^{j-1}$, and some $(d-j)$-face ${G_1}^{d-j}$ of $\pi_R(P)$ which contains $\pi_R({F_1}^{j-1})$ other than $G^{d-j}$, we see that the properties of $G^{d-j}$ established above are true for ${G_1}^{d-j}$ also. In particular, this shows that $P$ contains two $(d-1)$-faces which do not lie in the same $(d-1)$-flat, and so $P$ is $d$-dimensional. If we now repeat the same argument $f_{d-j}(\pi_R(P))$ times (once for each $(d-j)$-face of $\pi_R(P)$), we see that for $j - 1 \leqslant s \leqslant d - 1$ every $(s - j + 1)$-face of $\pi_R(P)$ is the image under $\pi_R$ of some $s$-face of $P$. Finally $\pi_R(P)$ is the image under $\pi_R$ of the $d$-polytope $P$, so the statement is true for $s = d$ also. Hence the induction is completed and the lemma is true generally.

Theorem 2 is now proved by noticing that every face of $\pi_R(P)$ whose dimension is at least 2 is the image under $\pi_R$ of some face of $P$ whose dimension is at least $j + 1$. Thus every face of $\pi_R(P)$ is centrally symmetric and therefore $\pi_R(P)$ is a zonotope.

If $G^1$ is any edge of $\pi_R(P)$, then the end points of $G^1$ are the images under $\pi_R$ of two $(j - 1)$-faces $F_1^{j-1}$ and $F_2^{j-1}$ of $P$. These are parallel faces of the $j$-face $F^j$ of $P$ such that $\pi_R(F^j) = G^1$. Hence $F_2^{j-1}$ is the image of $F_1^{j-1}$ under reflection in the centre of $F^j$. Thus the $(j - 1)$-faces of $P$ that project into the vertices of $\pi_R(P)$ are either congruent to $F^{j-1}$ or to the reflection of $F^{j-1}$ in a point, as asserted in the Introduction. A typical zone on $P$ consists of those $(d - 1)$-faces which project into the $(d - j)$-faces of $\pi_R(P)$. In particular, we have proved that the number of $(d - 1)$-faces in any zone on $P$ is equal to the number of $(d - j)$-faces of a $(d - j + 1)$-dimensional zonotope. The latter can be calculated from projective diagrams as described in **(9)**.

**Proof of Theorem 3.** The following lemma corresponds to the case $r = d - 1$ of the theorem:

LEMMA 3. *If every $(d - 1)$-face of the $d$-polytope $P$ is centrally symmetric, then $P$ is $(d - 1)$-equiprojective.*

*Proof.* Since $P$ is centrally symmetric, its $(d - 1)$-faces fall into

$$\tfrac{1}{2}f_{d-1}(P) = s + 1$$

parallel pairs which may be denoted by $F_0^{d-1}$, $*F_0^{d-1}$; $F_1^{d-1}$, $*F_1^{d-1}$; ... ; $F_s^{d-1}$, $*F_s^{d-1}$, where $*F_i^{d-1}$ is the reflection of $F_i^{d-1}$ in the centre of $P$. Each pair $F_i^{d-1}$, $*F_i^{d-1}$ defines a unique $(d - 1)$-flat $U_i$ through the origin $o$ and parallel to each of these $(d - 1)$-faces. $U_0, \ldots, U_s$ intersect the unit $(d - 1)$-sphere centred at $o$ in $s + 1$ "great spheres" which form the boundaries of the spherical polytopes of a honeycomb on $S^{d-1}$. The interiors of these spherical polytopes will be called *regions* and will be denoted by $J_1, \ldots, J_t$. The set $Q$ associated with $P$ that was defined in the Introduction consists of $U_0, \ldots, U_s$, together with some of the intersections of these $(d - 1)$-flats. From this it will be apparent that a projection $\pi_H(P)$ on to a $(d - 1)$-flat $H$ is regular if and only if the unit normal $n$ of $H$ belongs to one of the regions $J_i$ (and not to any of the $U_i$). Further, as was shown in **(8**, §2**)**, a $j$-face $F^j$ of $P$ will project into a $j$-face $G^j$ of $\pi_H(P)$ if and only if $n$ lies in a certain (open) spherical polytope on $S^{d-1}$ bounded by parts of the $(d - 1)$-flats $U_{i_1}, \ldots, U_{i_k}$ that are parallel to the $(d - 1)$-faces of $P$ incident with $F^j$. In this way we see that for all $n$ in the same region $J_i$ (or in the region antipodal to $J_i$ on $S^{d-1}$) the corresponding regular projections are all combinatorially equivalent; see (**8**, proof of (10)). Thus if $n_1$ and $n_2$ belong to the same region $J_i$, and $H_1$ and $H_2$ denote the $(d-1)$-flats with normals $n_1$ and $n_2$, then $f_j(\pi_{H_1}(P)) = f_j(\pi_{H_2}(P))$ for $0 \leqslant j \leqslant d-1$. Consequently, in order to prove the lemma, it is only necessary to show that $f_j(\pi_{H_1}(P)) = f_j(\pi_{H_2}(P))$ when $n_1$ and $n_2$ belong to different regions, and it is sufficient to show that this is so when $n_1$ and $n_2$ belong to adjacent regions, that is, regions which are separated by exactly one of the $(d-1)$-flats, say $U_0$. Further, we may suppose without loss of generality that $n_1$ and $n_2$, though lying on opposite sides of $U_0$, are arbitrarily close to one another, and their orthogonal projections on to $U_0$ coincide.

Let $F_j$ be any $j$-face of $P$ which is not incident with $F_0^{d-1}$ or $*F_0^{d-1}$. Then, as remarked above, $F^j$ will project into a $j$-face of $\pi_H(P)$ if and only if the normal $n$ of $H$ belongs to a certain spherical polytope $\Pi$ whose boundary consists of parts of $U_{i_1}, \ldots, U_{i_k}$ (but *not* of $U_0$). Since $n_1$ and $n_2$ are separated only by $U_0$, we deduce that they both belong to $\Pi$ or neither does so. Thus $F^j$ projects into a $j$-face of $\pi_{H_1}(P)$ and a $j$-face of $\pi_{H_2}(P)$, or does not project into a $j$-face of either. Hence, writing $f_j^{(0)}(\pi_H(P))$ for the number of $j$-faces of $\pi_H(P)$ that are the projections of $j$-faces of $P$ which are not incident with $F_0^{d-1}$ or $*F_0^{d-1}$, we deduce that

$$f_j^{(0)}(\pi_{H_1}(P)) = f_j^{(0)}(\pi_{H_2}(P)) \qquad (0 \leqslant j \leqslant d-2).$$

On the other hand, suppose $F^j$ is a $j$-face of $P$ and $F^j \subset F_0^{d-1}$. The case $F^j \subset *F_0^{d-1}$ can be dealt with in a similar manner. Let us choose a relative interior point of $F^j$ as origin, and suppose $U_0$ has the equation $\langle x, u_0\rangle = 0$ with $u_0$ chosen as the inward normal so that $P$ lies in the half-space $\langle x, u_0\rangle \geqslant 0$. Let $v$ be any vector in $U_0$ such that the $(d-1)$-flat $\langle x, v\rangle = 0$ supports $F_0^{d-1}$, intersects it in $F^j$, and $F_0^{d-1}$ lies in the half-space $\langle x, v\rangle \geqslant 0$. We shall show that $F^j$ projects into a $j$-face of $\pi_{H_1}(P)$ if and only if $v$ can be chosen to satisfy the above conditions and so that $\langle n_1, u_0\rangle \neq 0$ and $\langle n_1, v\rangle \neq 0$ are of opposite sign. To establish this we consider two cases:

I. Let $\langle n_1, u_0\rangle$, $\langle n_1, v\rangle$ be of opposite sign. Then define $\epsilon > 0$ by the equation $\langle n_1, u_0 + \epsilon v\rangle = 0$. We have remarked that $n_1$ may be taken arbitrarily close to $U_0$ (so that $\langle n_1, u_0\rangle$ can be made arbitrarily small) and so we may suppose, without loss of generality, that $0 < \epsilon < \beta/2\alpha$, where

$$\beta = \min\{\langle x, u_0\rangle : x \in \text{vert } P\backslash\text{vert } F_0^{d-1}\} > 0,$$

$$\alpha = \max\{|\langle x, v\rangle| : x \in \text{vert } P\backslash\text{vert } F_0^{d-1}\} \geqslant 0.$$

Here vert $P$ means the set of vertices of $P$, and $\beta/2\alpha$ is to be interpreted as $+\infty$ if $\alpha = 0$. Then Grünbaum has shown (**6**, Theorem 3.1.5) that

$$\langle x, u_0 + \epsilon v\rangle = 0$$

is a supporting $(d-1)$-flat of $P$ which intersects $P$ in $F^j$. As this supporting hyperplane also contains $n_1$, we deduce that $F^j$ projects into a $j$-face of $\pi_{H_1}(P)$, as was to be shown.

II. Let $F^j$ be such that, with $u_0$ defined as above, $\langle n_1, u_0\rangle$ and $\langle n_1, v\rangle$ have the same sign for all $v \in U_0$ such that the $(d-1)$-flat $\langle x, v\rangle = 0$ supports $F_0^{d-1}$, intersects $F_0^{d-1}$ in $F^j$, and $F_0^{d-1}$ lies in the half-space $\langle x, v\rangle \geqslant 0$. Let $\pi_{H_1}(F^j) = G^j$, and $H^*$ be any $(d-2)$-flat in $H_1$ through $G^j$. If $H$ is the $(d-1)$-flat spanned by $H^*$ and $n_1$, then we shall show that $H$ cannot support $P$ for any choice of $H^*$, and so $G^j$ is not a $j$-face of $\pi_{H_1}(P)$.

To begin with, if $H \cap U_0$ is not a supporting $(d-2)$-flat of $F_0^{d-1}$ in $U_0$, then points of $F_0^{d-1}$ will lie on both sides of $H \cap U_0$, and hence on both sides of $H$. Thus $H$ does not support $F_0^{d-1}$ and so does not support $P$.

Secondly, if $H \cap U_0$ is a supporting $(d-2)$-flat of $F_0^{d-1}$, then it may be written $\langle x, v\rangle = 0$, with $F_0^{d-1}$ lying in the half-space $\langle x, v\rangle \geqslant 0$. $H$ has the equation $\langle x, u_0 + \epsilon v\rangle = 0$, where $\epsilon$ is chosen so that $\langle n_1, u_0 + \epsilon v\rangle = 0$. From the hypothesis that $\langle n_1, u_0\rangle$ and $\langle n_1, v\rangle$ are necessarily of the same sign, it follows that $\epsilon < 0$. Choose $x_0$ as any point of $F_0^{d-1}\backslash F^j$; then clearly $\langle x_0, v\rangle > 0$ and $\langle x_0, u\rangle = 0$. Choose any point $y_0 \in P\backslash F_0^{d-1}$; then $\langle y_0, u\rangle > 0$. Hence

$$\langle x_0, u_0 + \epsilon v\rangle = \epsilon\langle x_0, v\rangle < 0,$$

and

$$\langle y_0, u_0 + \epsilon v\rangle = \langle y_0, u_0\rangle + \epsilon\langle y_0, v\rangle.$$

By choosing $n_1$ sufficiently close to $U_0$ we may make $\epsilon$ arbitrarily small, and since $\langle y_0, u_0\rangle > 0$, we can ensure that $\langle y_0, u_0 + \epsilon v\rangle > 0$. Thus points $x_0$ and $y_0$ lie on opposite sides of $H$, and so $H$ does not support $P$. This completes the proof of assertion II.

Write $f_j^{(1)}(F_0^{d-1})$ for the number of $j$-faces of $F_0^{d-1}$ which satisfy condition I above, and let $f_j^{(2)}(F_0^{d-1})$ be the number of $j$-faces which satisfy exactly the same condition with $n_1$ replaced by $n_2$ throughout. Then we have established and that

$$f_j(\pi_{H_1}(P)) = f_j^{(0)}(\pi_{H_1}(P)) + f_j^{(1)}(F_0^{d-1}) + f_j^{(1)}(^*F_0^{d-1}),$$
$$f_j(\pi_{H_2}(P)) = f_j^{(0)}(\pi_{H_2}(P)) + f_j^{(2)}(F_0^{d-1}) + f_j^{(2)}(^*F_0^{d-1})$$
$$(0 \leqslant j \leqslant d-1).$$

By the assumption that $n_1$ and $n_2$ lie on opposite sides of $U_0$ and their projections onto $U_0$ coincide, we see that $\langle n_1, u_0\rangle$, $\langle n_2, u_0\rangle$ are of opposite sign, and that $\langle n_1, v\rangle = \langle n_2, v\rangle$ for all $v \in U_0$. Thus if, for example, $\langle n_1, u_0\rangle < 0$, then $f_j^{(1)}(F_0^{d-1})$ is the number of $j$-faces of $F_0^{d-1}$ for which we can choose $v$ so that $\langle n_1, v\rangle > 0$, and by central symmetry of $F_0^{d-1}$, this is exactly equal to the number of $j$-faces of $F_0^{d-1}$ for which $v$ can be chosen so that $\langle n_1, v\rangle = \langle n_2, v\rangle < 0$. Since $\langle n_2, u_0\rangle > 0$, we deduce that $f_j^{(1)}(F_0^{d-1}) = f_j^{(2)}(F_0^{d-1})$. Similarly, $f_j^{(1)}(^*F_0^{d-1}) = f_j^{(2)}(^*F_0^{d-1})$. We have already shown that $f_j^{(0)}(\pi_{H_1}(P)) = f_j^{(0)}(\pi_{H_2}(P))$ and so we deduce that $f_j(\pi_{H_1}(P)) = f_j(\pi_{H_2}(P))$. This is true for all $j$ satisfying $0 \leqslant j \leqslant d-1$; therefore $P$ is $(d-1)$-equiprojective, and the lemma is proved.

To prove the theorem for $r < d-1$, we take any two $r$-flats $R$, $R_*$ in $E^d$ such that $\pi_R(P)$ and $\pi_{R*}(P)$ are regular projections. Let $S_1, \ldots, S_t$ be any sequence of $(r+1)$-flats such that $R \subset S_1$, $R_* \subset S_t$, and $S_i \cap S_{i+1} = R_i$ is an $r$-flat. It is easy to see that such a sequence may always be constructed, and further that we may do so in such a way that $\pi_{S_i}(P)$ $(i = 1, \ldots, t)$ and $\pi_{R_i}(P)$ $(i = 1, \ldots, t-1)$ are also regular projections. (If, for example, one of the $\pi_{S_i}(P)$ is not regular, then it may be made so by an arbitrarily small displacement of $S_i$.) Write $R = R_0$ and $R_* = R_t$. Noticing that for $i = 1, \ldots, t$, $\pi_{S_i}(P)$ is an $(r+1)$-polytope with centrally symmetric $r$-faces, and that

$$\pi_{R_{i-1}}(P) = \pi_{R_{i-1}}(\pi_{S_i}(P)) \quad \text{and} \quad \pi_{R_i}(P) = \pi_{R_i}(\pi_{S_i}(P)),$$

we deduce from Lemma 3 that

$$f_j(\pi_{R_{i-1}}(P)) = f_j(\pi_{R_i}(P))$$

for $i = 1, \ldots, t$ and $j = 0, \ldots, r-1$. Thus

$$f_j(\pi_R(P)) = f_j(\pi_{R_*}(P))$$

for $j = 0, \ldots, r-1$. Since this applies to any two $r$-flats $R$ and $R_*$ for which the projections $\pi_R(P)$ and $\pi_{R_*}(P)$ are regular, we deduce that $P$ is $r$-equiprojective. This completes the proof of Theorem 3.

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