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A LIMIT THEOREM FOR THE RELIABILITY OF A CONSECUTIVE-k-OUT-OF-n SYSTEM

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Abstract

A consecutive-k-out-of-n system consists of n identical and linearly ordered components. The system will fail if and only if at least k consecutive components fail. Let T_n be the system's lifetime. Then, under very general conditions we prove that there is a positive constant a, so that the distribution of the random variable $n^{(1/ka)}T_n$ converges to a Weibull distribution, as $n \to \infty$.

1. Introduction

Consecutive-k-out-of-n systems have recently been studied by Derman et al. (1982), Shanthikumar (1982), Hwang (1982), Tong (1985), Fu (1986), among others.

A consecutive-k-out-of-n system consists of n linearly ordered components. The system will fail if and only if at least k consecutive components fail. The components are assumed identical and their failures are stochastically independent of one another. Let T be a random variable which is the time of a component's failure and let $q(t) = \Pr\{T \leqq t\}$, for $t \geqq 0$, be a component's failure distribution. Let T_n be the system's time of failure. Then, our main result is the following.

Theorem 1. Let

$$q(t) = \lambda^a t^a + o(t^a)$$

where a, λ are positive real constants. Then

$$\Pr\{n^{(1/ka)}T_n \leqq t\} \to 1 - \exp(-(\lambda t)^{ak})$$

as $n \to \infty$, for all $t \geqq 0$.

The proof of the theorem is given in Section 2. The case $k = 1$ corresponds to a series system and the corresponding result is well known; see for example Barlow and Proschan (1975), p. 230. A related limit theorem, which refers only to the cases $k \leqq 4$, is proved by Chao and Lin (1984).

Table 1 gives a numerical example, showing the convergence, for the case of the exponential distribution, $q(t) = 1 - e^{-t}$, i.e. $\lambda = a = 1$, and $k = 2$. The columns of the table give the values of $\Pr\{n^{(1/ka)}T_n \leqq t\}$ for $n = 5, 10, \cdots, 35$. The computation of those values is based on Hwang (1982).

2. Proof of Theorem 1

Let $p(t) = 1 - q(t) = \Pr\{T > t\}$ be a component's reliability function and let $R(p(t), n) = \Pr\{T_n > t\}$ be the reliability of the system. Let $t_n = n^{-(1/ka)}t$ and let $x(t)$ be

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TABLE 1

n	$t = 0{\cdot}5$	1	1·5	2	2·5	3	3·5
5	0·1351	0·3686	0·5760	0·7281	0·8302	0·8955	0·9361
10	0·1601	0·4372	0·6682	0·8195	0·9070	0·9539	0·9777
15	0·1713	0·4691	0·7091	0·8564	0·9340	0·9713	0·9880
20	0·1779	0·4887	0·7337	0·8772	0·9482	0·9795	0·9923
25	0·1824	0·5023	0·7505	0·8909	0·9569	0·9842	0·9945
30	0·1858	0·5126	0·7630	0·9007	0·9628	0·9872	0·9959
35	0·1884	0·5207	0·7727	0·9082	0·9671	0·9893	0·9967
$1 - \exp(-t^2)$	0·2211	0·6321	0·8946	0·9816	0·9980	0·9998	0·9999

the unique positive root of the polynomial $1 - pz(1 + pz + \cdots + p^{k-1}z^{k-1})$. It is clear that Theorem 1 amounts to proving that

$$R(p(t_n), n) \to \exp(-(\lambda t)^{ak}) \tag{1}$$

as $n \to \infty$, for all $t \geqq 0$. On the other hand we recall from Feller (1967), p. 325, that

$$R(p(t_n), n) \sim \frac{1 - q(t_n)x(t_n)}{(k + 1 - kx(t_n))p(t_n)} \cdot x(t_n)^{-(n+1)}. \tag{2}$$

But as $t_n \to 0$, it follows immediately that $q(t_n) \to 0$ and $p(t_n) \to 1$, as $n \to \infty$. From Feller (1967), p. 326, (7.19), we know that

$$x(t) = 1 + p(t)q(t)^k + (k + 1)(p(t)q(t)^k)^2 + \cdots$$

if $(k + 1)p(t) > 1$. Since $q(t) = \lambda^a t^a + o(t^a)$, we have

$$x(t) = 1 + (\lambda t)^{ak} + o(t^{ak}), \quad \text{and} \quad \log x(t) = (\lambda t)^{ak} + o(t^{ak}).$$

By substitution we get

$$\log x(t_n) = (\lambda t)^{ak}/n + o(1/n)$$

for large n, and

$$-\frac{\log x(t_n)}{1/(n + 1)} = -\frac{(\lambda t)^{ak}/n + o(1/n)}{1/(n + 1)} \to -(\lambda t)^{ak}.$$

Taking these remarks into consideration, we get (1) from relation (2). This completes the proof of Theorem 1.

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