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A NOTE ON A SPACE $H^{p,a}$ OF HOLOMORPHIC FUNCTIONS

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For $0 < p < \infty$ and $0 \le a \le 1$, we define a space $H^{p,a}$ of holomorphic functions on the unit disc of the complex plane, for which $H^{p,0} = H^\infty$, the space of all bounded holomorphic functions, and $H^{p,1} = H^p$, the usual Hardy space. We introduce a weak type operator whose boundedness extends the well-known Hardy-Littlewood embedding theorem to $H^{p,a}$, give some results on the Taylor coefficients of the functions of $H^{p,a}$ and show by an example that the inner factor cannot be divisible in $H^{p,a}$.

1. Introduction.

Let U be the unit disc in the complex plane. For a function f holomorphic in U, we write

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$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p}, \quad 0 < p < \infty ,$$

$$M_\infty(r,f) = \max_{|z|\le r} |f(z)| .$$

It is well-known that $M_p(r,f)\,(0 < p \le \infty)$ is an increasing function of $r\ (0 \le r < 1)$.

The Hardy space $H^p\ (0 < p \le \infty)$ is the class of all functions f holomorphic in U for which

$$\|f\|_p = \sup_{0\le r<1} M_p(r,f) < \infty .$$

See [1] for the general theory of H^p spaces.

For $0 < p < \infty$ and $0 \le a \le 1$, we define $H^{p,a}$ as the class of all functions f in H^p for which

$$\sup_{z\in U} (1 - |z|)^{a/p}|f(z)| < \infty .$$

Since it is well-known [1, Theorem 5.9] that if $f \in H^p$ then

$$\sup(1 - |z|)^{1/p}|f(z)| < \infty ,$$

we have $H^{p,1} = H^p$. Also, clearly $H^{p,0} = H^\infty$.

For $f \in H^{p,a}$, we define

$$\|f\|_{p,a} = \max\left(\|f\|_p , \sup_{z\in U} (1 - |z|)^{a/p}|f(z)|\right) .$$

It is routine to check that $H^{p,a}$ is a Banach space in the norm $\|\cdot\|_{p,a}$ if $1 \le p < \infty$, and a Frechét space in the invariant metric $d(f,g) = \|f-g\|^p_{p,a}$ if $0 < p < 1$.

Previous results on the class $H^{p,a}$ are in [3]. For example, Theorem B and Theorem 3.1 in [3] can now be stated respectively as follows:

THEOREM A. *If* $f \in H^{p,a}$ *and* $0 < p < q < \infty$, *then*

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$$\int_0^1 (1-r)^{qa(\frac{1}{p}-\frac{1}{q})-1} M_q(r,f)^q dr < \infty .$$

THEOREM B. *If* $f \in H^{p,a}$, *then* $I^\beta f \in H^{q,a}$, *where* $I^\beta f$ *is the fractional integral of* f *of order* β, $0 < \beta < \frac{1}{p}$ *and* $q = ap/(a - \beta p)$.

In view of Theorem A and the Hardy-Littlewood embedding theorem [1, Theorem 5.11], the following seems to be a reasonable conjecture.

Conjecture 1. *If* $f \in H^{p,a}$, $0 < p < q < \infty$ *and* $\lambda \geq p$, *then*

$$\int_0^1 (1-r)^{\lambda a(\frac{1}{p}-\frac{1}{q})-1} M_q(r,f)^\lambda \, dr \leq C_{p,a,q} \|f\|_{p,a}^\lambda ,$$

where $c_{p,a,q}$ *is a positive constant which does not depend on* f .

In section 2, we introduce an operator whose boundedness would prove conjecture 1, but we prove only that the operator is of weak-type. In section 3, we give some results on the Taylor coefficients of functions of the class $H^{p,a}$. In section 4, we show by an example that the inner factor is not divisible in the class $H^{p,a}$.

2. Weak-type inequality.

We follow the ideas of Jawerth and Torchinsky [2] in introducing our operator.

LEMMA 1. *If* $f \in H^{p,a}$ *and* $0 < p < q < \infty$, *then*

$$M_q(r,f) \leq \|f\|_{p,a}(1-r)^{-a(\frac{1}{p}-\frac{1}{q})} .$$

Proof. Since $|f(z)| \leq \|f\|_{p,a} (1 - |z|)^{-a/p}$ from the definition, we have

$$M_q(r,f) = (\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, |f(re^{i\theta})|^{q-p} \, d\theta)^{1/q}$$

$$\leq [\, \|f\|_{p,a} (1-r)^{-a/p}]^{(q-p)/q} M_p(r,f)^{p/q}$$

$$\leq \|f\|_{p,a} (1-r)^{-a(\frac{1}{p}-\frac{1}{q})} . \qquad \square$$

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From Lemma 1, Conjecture 1 is equivalent to the following.

Conjecture 1'. *If* $f \in H^{p,a}$ *and* $0 < p < q < \infty$ *, then*

$$\int_0^1 (1-r)^{pa(\frac{1}{p}-\frac{1}{q})-1} M_q(r,f)^p\, dr \le C_{p,a,q} \|f\|_{p,a}^p$$

where $C_{p,a,q}$ *is a positive constant which does not depend on* f *.*

In fact, if conjecture 1' is true and $\lambda > p$, then we have

$$\int_0^1 (1-r)^{\lambda a(\frac{1}{p}-\frac{1}{q})-1} M_q(r,f)^\lambda\, dr$$

$$\le \int_0^1 (1-r)^{\lambda a(\frac{1}{p}-\frac{1}{q})-1} M_q(r,f)^p \left[\|f\|_{p,a} (1-r)^{-a(\frac{1}{p}-\frac{1}{q})} \right]^{\lambda-p} dr$$

$$= \|f\|_{p,a}^{\lambda-p} \int_0^1 (1-r)^{pa(\frac{1}{p}-\frac{1}{q})-1} M_q(r,f)^p dr$$

$$\le \|f\|_{p,a}^{\lambda-p} C_{p,a,q} \|f\|_{p,a}^p = C_{p,a,q} \|f\|_{p,a}^\lambda$$

We now define an operator T_q on $H^{p,a}$ by

$$(T_q f)(r) = (1-r)^{-a/q} M_q(r,f) .$$

Since

$$\int_0^1 T_q f(r)^p (1-r)^{a-1}\, dr = \int_0^1 (1-r)^{pa(\frac{1}{p}-\frac{1}{q})-1} M_q(r,f)^p dr ,$$

we see that Conjecture 1' is now equivalent to the following.

Conjecture 1". T_q *is a bounded operator from* $H^{p,a}$ *into* $L^p((1-r)^{a-1}\, dr)$.

The following theorem supports the truth of Conjecture 1".

THEOREM 2. T_q *is of weak-type* (p,p) *from* $H^{p,a}$ *into* $L^p((1-r)^{a-1} dr)$.

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Proof. We note that by Lemma 1

$$(T_q f)(r) \le \|f\|_{p,a} (1-r)^{-a/p} .$$

Hence

$$\int_{(T_q f)(r) > \mu} (1-r)^{a-1}\, dr \le \int_{\|f\|_{p,a}(1-r)^{-a/p} > \mu} (1-r)^{a-1}\, dr$$

$$= \int_{r=1-(\|f\|_{p,a}/\mu)^{p/a}}^{r=1} (1-r)^{a-1}\, dr$$

$$= \frac{1}{a} \left(\frac{\|f\|_{p,a}}{\mu}\right)^p . \qquad \square$$

We remark that the Marcinkiewicz interpolation theorem does not seem to apply immediately because of the nature of the norm $\|\cdot\|_{p,a}$.

The following theorem also supports the truth of Conjecture 1.

THEOREM 3. *If* $f \in H^{p,a}$ *and* $0 < p < q \le \lambda$, *then*

$$\int_0^1 (1-r)^{\lambda a(\frac{1}{p}-\frac{1}{q})-1} M_q(r,f)^\lambda\, dr < \infty .$$

Proof. Since $M_q(r,f) \le M_\lambda(r,f)$, we have

$$\int_0^1 (1-r)^{\lambda a(\frac{1}{p}-\frac{1}{q})-1} M_q(r,f)^\lambda\, dr$$

$$\le \int_0^1 (1-r)^{\lambda a(\frac{1}{p}-\frac{1}{q})-1} M_\lambda(r,f)^\lambda\, dr < \infty$$

by Theorem A. $\square$

In the negative direction, conjecture 1 with $q = \infty$ is false as the following example shows.

EXAMPLE 4. Consider

$$f(z) = \frac{1}{1-z} \left\{\frac{1}{z} \log \frac{1}{1-z}\right\}^{-1} .$$

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We know that $f \in H^p$ for $0 < p < 1$ [4, p.96].
Since

$$M_\infty(r,f) \sim \frac{1}{1-r}\left\{\frac{1}{r}\log\frac{1}{1-r}\right\}^{-1}, \quad (\text{as } r \to 1^-),$$

we see that $f \in H^{p,p}$, $0 < p < 1$. But

$$\int_0^1 (1-r)^{\lambda p(\frac{1}{p}-\frac{1}{\infty})-1} M_\infty(r,f)^\lambda \, dr$$

$$\geq C \int_0^1 (1-r)^{-1}\left(\frac{1}{r}\log\frac{1}{1-r}\right)^{-\lambda} dr$$

$$= \infty$$

if $\lambda \leq 1$.

3. Taylor coefficients.

The following is an extension of a familiar result of Hardy and Littlewood [1, Theorem 6.4] on functions of H^p, $(0 < p < 1)$.

THEOREM 5. *If* $f(z) = \sum_0^\infty a_n z^n \in H^{p,a}$, $(0 < p < 1)$, *then*

$$|a_n| \leq C \, \|f\|_{p,a} \, n^{\frac{a}{p}(1-p)},$$

where C *is a positive constant.*

Proof. From the equality

$$a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\, e^{-in\theta}\, d\theta,$$

we have

$$|a_n r^n| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p |f(re^{i\theta})|^{1-p}\, d\theta$$

$$\leq [\, \|f\|_{p,a} (1-r)^{-\frac{a}{p}}]^{1-p} \, \|f\|_p^p$$

$$\leq \|f\|_{p,a} (1-r)^{-\frac{a}{p}(1-p)}.$$

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If we set $r = 1 - \frac{1}{n}$, we get

$$|a_n| \le C \ \|f\|_{p,a} \ n^{\frac{a}{p}(1-p)} . \qquad \square$$

Using Theorem A, we have the following which reduces to the familiar theorem of Hardy and Littlewood [1, Theorem 6.2] when $p = q$.

THEOREM 6. *If* $f(z) = \sum_0^\infty a_n z^n \in H^{p,a}$ *and* $0 < p \le q \le 2$, *then*

$$\sum_0^\infty (n+1)^{q(1 - a(\frac{1}{p} - \frac{1}{q})) - 2} |a_n|^q < \infty .$$

Proof. From Theorem A, we have

$$\int_0^1 (1-r)^{qa(\frac{1}{p} - \frac{1}{q}) - 1} M_q(r,f)^q \, dr < \infty .$$

Multiplying by $(1-r)^{qa(\frac{1}{p} - \frac{1}{q}) - 1}$ and integrating both sides of the following inequality of Hardy and Littlewood [1, Theorem 6.2]

$$\sum_0^\infty (n+1)^{q-2} |a_n|^q \, r^{nq} \le C_q M_q(r,f)^q ,$$

we have

$$\sum_0^\infty (n+1)^{q-2} |a_n|^q \int_0^1 r^{nq} (1-r)^{qa(\frac{1}{p} - \frac{1}{q}) - 1} dr < \infty .$$

But, by Stirling's formula, we have

$$\int_0^1 r^{nq} (1-r)^{qa(\frac{1}{p} - \frac{1}{q}) - 1} dr = B(nq + 1, \ qa(\tfrac{1}{p} - \tfrac{1}{q}))$$

$$= \frac{\Gamma(nq + 1) \ \Gamma(qa(\frac{1}{p} - \frac{1}{q}))}{\Gamma(nq + 1 \ + qa(\frac{1}{p} - \frac{1}{q}))}$$

$$\sim \Gamma(qa(\tfrac{1}{p} - \tfrac{1}{q})) \ (\tfrac{e}{q})^{qa(\frac{1}{p} - \frac{1}{q})} (n+1)^{-qa(\frac{1}{p} - \frac{1}{q})}$$

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as $n \to \infty$. Hence we have

$$\Sigma\ (n+1)^{q(1-a(\frac{1}{p}-\frac{1}{q}))-2} |a_n|^q < \infty . \qquad \square$$

Now, suppose that Conjecture 1 were true and let $f(z) = \sum_0^\infty a_n z^n \in H^{p,a}$, $(0 < p < 1)$. We note that $|a_n r^n| \le M_1(r,f)$. We then have

$$\infty > \int_0^1 (1-r)^{ap(\frac{1}{p}-1)-1} M_1(r,f)^p\, dr$$

$$= \sum_{n=0}^\infty \int_{1-\frac{1}{n+1}}^{1-\frac{1}{n+2}} (1-r)^{ap(\frac{1}{p}-1)-1} M_1(r,f)^p\, dr$$

$$\ge \sum_{n=0}^\infty \frac{1}{(n+1)(n+2)} (n+1)^{1-ap(\frac{1}{p}-1)} M_1(1-\frac{1}{n+1}\, ,\, f)^p$$

$$\ge C \sum_{n=0}^\infty (n+1)^{-1-ap(\frac{1}{p}-1)} |a_n|^p\, (1-\frac{1}{n+1})^{np}$$

$$\ge C_p \sum_{n=0}^\infty (n+1)^{ap(1-\frac{1}{p})-1} |a_n|^p .$$

where C and C_p are positive constants.

Hence the truth of Conjecture 1 would imply that of the following.

Conjecture 2. *If* $f(z) = \sum_0^\infty a_n z^n \in H^{p,a}$, $(0 < p < 1)$, *then*

$$\sum_{n=0}^\infty (n+1)^{ap(1-\frac{1}{p})-1} |a_n|^p < \infty .$$

We note that Conjecture 2 reduces to the well-known theorem of Hardy and Littlewood [1, Theorem 6.2] when $a = 1$.

4. An example.

We give an example which shows that the inner factor of a function in $H^{p,a}$ is not divisible in $H^{p,a}$.

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EXAMPLE 7. Consider

$$f(z) = \frac{1}{1-z} e^{-\frac{1+z}{1-z}} .$$

It is trivial that $f \in H^p$ for $0 < p < 1$. Now

$$|f(re^{i\theta})| = \frac{1}{\sqrt{1-2r\cos\theta + r^2}} e^{-\frac{1-r^2}{1-2r\cos\theta+r^2}} .$$

By a routine calculation, we see that

$$M_\infty(r,f) = \max_\theta |f(re^{i\theta})|$$

is attained when $\cos\theta = \frac{3r^2-1}{2r}$; so

$$M_\infty(r,f) = \frac{1}{\sqrt{2e}} \frac{1}{\sqrt{1-r^2}} .$$

Hence $f \in H^{p,p/2}$ for $0 < p < 1$. But $\frac{1}{1-z} \notin H^{p,p/2}$. So the inner factor $e^{-\frac{1+z}{1-z}}$ of $f(z)$ is not divisible in $H^{p,p/2}$.

References

[1] P. L. Duren, *Theory of H^p spaces* (Academic Press, New York, NY, 1970).

[2] B. Jawerth and A. Torchinsky, "On a Hardy and Littlewood imbedding theorem". Preprint.

[3] H. O. Kim, "Derivatives of Blaschke products", *Pacific J. Math.*, 114 (1984), 175-190.

[4] J. E. Littlewood, *Lectures on the theory of functions* (Clarenden Press, Oxford, 1944).

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