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On a Theorem of Bers, with Applications to the Study of Automorphism Groups of Domains

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Abstract. We study and generalize a classical theorem of L. Bers that classifies domains up to biholomorphic equivalence in terms of the algebras of holomorphic functions on those domains. Then we develop applications of these results to the study of domains with noncompact automorphism group.

1 Introduction

For us a *domain* in complex space is a connected open set. If Ω is a domain, then let $\mathcal{O}(\Omega)$ denote the algebra of holomorphic functions on Ω.

In 1948, Lipman Bers [1] proved the following elegant result.

Theorem 1.1 *Let $\Omega, \widehat{\Omega}$ be domains in $\mathbb{C}$. If $\mathcal{O}(\Omega)$ is isomorphic to $\mathcal{O}(\widehat{\Omega})$ as an algebra, then the domain Ω is conformally equivalent to the domain $\widehat{\Omega}$.*

Since that time, this result has been generalized to domains in $\mathbb{C}^n$ and even to domains in Stein manifolds; see, for instance, the work of Zame [14,15]. The approach that we present below is different from, and more elementary than Zame's.

In this paper we offer some other variants of Bers's theorem and develop applications of these results to the study of the automorphism groups of domains in complex space.

2 Variants of Bers's Theorem

In this section we formulate several variants of Bers's theorem. They all have the same proof. For completeness, we provide the proof of Bers's original theorem stated in the previous section.

Proof of Theorem 1.1 In fact we will prove the result in $\mathbb{C}^n$. As we shall see below, it will be necessary to assume that the domains in question are pseudoconvex.

Let $\Omega \subseteq \mathbb{C}^n$ be a pseudoconvex domain. Let $\mathcal{O}(\Omega)$ denote the algebra of holomorphic functions from Ω to $\mathbb{C}$. Bers's theorem states, in effect, that the algebraic structure of $\mathcal{O}(\Omega)$ characterizes Ω. We begin our study by introducing a little terminology.

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Definition 2.1 Let $\Omega \subseteq \mathbb{C}^n$ be a domain. A $\mathbb{C}$-algebra homomorphism $\phi: \mathcal{O}(\Omega) \to \mathbb{C}$ is called a *character* of $\mathcal{O}(\Omega)$. If $c \in \mathbb{C}$, then the mapping

$$\begin{aligned} e_c: \mathcal{O}(\Omega) &\longrightarrow \mathbb{C}, \\ f &\longmapsto f(c), \end{aligned}$$

is called a *point evaluation*. Every point evaluation is a character.

It should be noted (and we use this fact frequently below) that if $\phi: \mathcal{O}(\Omega) \to \mathcal{O}(\widehat{\Omega})$ is not the trivial zero homomorphism, then $\phi(1) = 1$. This follows because $\phi(1) = \phi(1 \cdot 1) = \phi(1) \cdot \phi(1)$. On any open set where the holomorphic function $\phi(1)$ does not vanish, we find that $\phi(1) \equiv 1$. The result follows by analytic continuation.

It turns out that, for pseudoconvex domains, every character of $\mathcal{O}(\Omega)$ is a point evaluation. That is the content of the next lemma.

Lemma 2.2 *Assume that $\Omega \subseteq \mathbb{C}^n$ is pseudoconvex (see [8] for this concept) and bounded. Let ϕ be a character on $\mathcal{O}(\Omega)$. Then $\phi = e_c$ for some $c \in \Omega$. Indeed, if we let $\tau_j(z) = z_j$ for $j = 1, \ldots, n$ and $c_j = \phi(\tau_j)$, then $c = (c_1, c_2, \ldots, c_n) \in \Omega$.*

Proof Let c be defined as in the statement of the lemma. Let $\mu_j(z) = \tau_j(z) - c_j$. Then

$$\phi(\mu_j) = \phi(\tau_j) - \phi(c_j) = c_j - c_j = 0.$$

If it were not the case that $c \in \Omega$, then $(\tau_1(z), \tau_2(z), \ldots, \tau_n(z))$ would never take the value c on Ω. Therefore the expression

$$\big(\tau_1(z) - c_1, \tau_2(z) - c_2, \ldots, \tau_n(z) - c_n\big)$$

would never vanish on Ω. But then we can iteratively choose complex constants $\lambda_1, \lambda_2, \ldots, \lambda_n$ so that

$$\tau(z) = \lambda_1\big(\tau_1(z) - c_1\big) + \lambda_2\big(\tau_2(z) - c_2\big) + \cdots + \lambda_n\big(\tau_n(z) - c_n\big)$$

will never vanish on Ω. Indeed,

$$\sum_{j=1}^{n} |\tau_j(z) - c_j|$$

will have a minimum, nonzero value X on $\overline{\Omega}$. Let $\lambda_j = 10^j \cdot 2 \cdot \operatorname{diam} \Omega / X$.

Thus, τ will be a unit in $\mathcal{O}(\Omega)$. But then

$$1 = \phi(\tau \cdot \tau^{-1}) = \phi(\tau) \cdot \phi(\tau^{-1}) = 0 \cdot \phi(\tau^{-1}) = 0.$$

That is a contradiction, so $c \in \Omega$.

Now let $g \in \mathcal{O}(\Omega)$ be arbitrary. Then we may write

$$g(z) = g(c) + \sum_{j=1}^{n} \mu_j(z) \cdot \widetilde{g}_j(z), \tag{2.1}$$

where $\widetilde{g}_j \in \mathcal{O}(\Omega)$. Thus,

$$\phi(g) = \phi\big(g(c)\big) + \sum_{j=1}^{n} \phi(\mu_j) \cdot \phi(\widetilde{g}_j) = g(c) + 0 = g(c) = e_c(g).$$

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We conclude that $\phi = e_c$, as claimed.

It is clear (because the algebra of holomorphic functions separates points) that c is unique. ∎

Now we can prove Bers's theorem. We formulate the result in slightly greater generality than stated heretofore.

It should be noted that Bers's theorem (stated below) is false for non-pseudoconvex domains. Consider the example of $\Omega = B$ (where B is the unit ball in $\mathbb{C}^2$) and $\widehat{\Omega} = B \setminus \{(0,0)\}$. Then, by the Hartogs extension phenomenon, $\mathcal{O}(\Omega)$ and $\mathcal{O}(\widehat{\Omega})$ are identical (hence certainly isomorphic), but the two domains are not even homeomorphic, much less biholomorphic.

Theorem 2.3 *Let Ω, $\widehat{\Omega}$ be pseudoconvex domains in $\mathbb{C}^n$. Suppose that*

$$\phi\colon \mathcal{O}(\Omega) \longrightarrow \mathcal{O}(\widehat{\Omega})$$

is a $\mathbb{C}$-algebra homomorphism. Then there exists one and only one holomorphic mapping $h : \widehat{\Omega} \to \Omega$ such that

$$\phi(f) = f \circ h \quad \text{for all } f \in \mathcal{O}(\Omega).$$

In fact, using the notation of the lemma, we define $h_j = \phi(\tau_j)$ and $h = (h_1, h_2, \dots, h_n)$.

The homomorphism ϕ is bijective if and only if h is biholomorphic, that is, a one-to-one and onto holomorphic mapping from $\widehat{\Omega}$ to Ω.

Proof We define h as in the statement of the theorem. It is clear from the construction that h is a holomorphic mapping.

If $a \in \widehat{\Omega}$, then $e_a \circ \phi$ is a character of $\mathcal{O}(\Omega)$. Thus, our lemma tells us that $e_a \circ \phi$ must in fact be a point evaluation at some point c in Ω. As a result, $e_a \circ \phi = e_c$, with

$$c_j = (e_a \circ \phi)(\tau_j) = e_a(h_j) = h_j(a)$$

for $j = 1, \dots, n$. Thus, if $f \in \mathcal{O}(\Omega)$, then

$$\phi(f)(a) = e_a(\phi \circ f) = (e_a \circ \phi)(f) = e_{h(a)}(f) = f\big(h(a)\big) = (f \circ h)(a)$$

for all $a \in \widehat{\Omega}$. We conclude that $\phi(f) = f \circ h$ for all $f \in \mathcal{O}(\Omega)$.

For the last statement of the theorem, suppose that h is a one-to-one, onto, holomorphic mapping of $\widehat{\Omega}$ to Ω. If $g \in \mathcal{O}(\Omega)$, then set $f = g \circ h^{-1}$. It follows that $\phi(f) = f \circ h = g$. Hence, ϕ is onto. Likewise, if $\phi(f_1) = \phi(f_2)$, then $f_1 \circ h = f_2 \circ h$ hence, composing with h^{-1}, $f_1 \equiv f_2$. So ϕ is one-to-one.

Conversely, suppose that ϕ is an isomorphism. Let $a \in \Omega$ be arbitrary. Then e_a is a character on $\mathcal{O}(\Omega)$; hence $e_a \circ \phi^{-1}$ is a character on $\mathcal{O}(\widehat{\Omega})$. By the lemma, there is a point $c \in \widehat{\Omega}$ such that $e_a \circ \phi^{-1} = e_c$. It follows that $e_a = e_c \circ \phi$. Applying both sides to τ_j yields $e_a(\tau_j) = (e_c \circ \phi)(\tau_j)$. Unraveling the definitions gives $a_j = e_c(\tau_j \circ h) = h_j(c)$. Thus, $h(c) = a$ and h is surjective. The argument in fact shows that the pre-image c is uniquely determined. So h is also one-to-one. ∎

Now we formulate some variants of Bers's theorem. Again we stress that each has the same proof (the proof that we just presented).

In what follows, we will be dealing with the space $L(\Omega)$ of Lipschitz functions on Ω. These are functions that satisfy a condition of the form

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|} \leq C. \tag{2.2}$$

As usual, we use the expression (2.2) to define a norm $\|\cdot\|_{L(\Omega)}$ on $L(\Omega)$.

Proposition 2.4 *If Ω is a domain in $\mathbb{C}^n$, then let $L(\Omega)$ denote the algebra of Lipschitz holomorphic functions on Ω. The smoothly bounded, strongly pseudoconvex domains Ω and $\widehat{\Omega}$ in $\mathbb{C}^n$ are biholomorphically equivalent if and only if the algebras $L(\Omega)$ and $L(\widehat{\Omega})$ are isomorphic as algebras.*

In order to prove this result, it is necessary to verify that the functions $\widetilde{g}_j$ in formula (2.1) can be chosen to be in the space $L(\Omega)$. This entails solving a Gleason problem in the Lipschitz category using the method of $\overline{\partial}$. The latter technique is explicated in [6]. The necessary estimates for $\overline{\partial}$ can be found, for instance, in [12].

Proposition 2.5 *The smoothly bounded, strongly pseudoconvex domains Ω and $\widehat{\Omega}$ in $\mathbb{C}^n$ are biholomorphically equivalent, with a biholomorphism that is bi-Lipschitz, if and only if the algebras $L(\Omega)$ and $L(\widehat{\Omega})$ are isomorphic as algebras.*

For this result, one need only observe that $h_j = \phi(\tau_j)$ is the image under ϕ of a Lipschitz function. And ϕ is assumed to be an isomorphism of Lipschitz algebras.

We remark that it is possible to formulate versions of these results for Sobolev spaces of holomorphic functions, for Besov spaces of holomorphic functions, and in other contexts as well. We leave the details to the interested reader.

3 Applications

Our intention here is to study the automorphism groups of domains in $\mathbb{C}^n$. Here, if $\Omega \subseteq \mathbb{C}^n$ is a domain, then the automorphism group of Ω (denoted $\mathrm{Aut}(\Omega)$) is the collection of biholomorphic mappings of Ω to itself. The usual topology on $\mathrm{Aut}(\Omega)$ is that of uniform convergence on compact sets (equivalently, the compact-open topology). For a bounded domain Ω, this topology turns $\mathrm{Aut}(\Omega)$ into a real Lie group. Note, however, that the automorphism group of $\Omega = \mathbb{C}^n$ with $n > 1$ is infinite dimensional hence certainly *not* a Lie group.

If Ω is a fixed domain in $\mathbb{C}^n$ and if $f \in L(\Omega)$, then let us say that f is *noncompact* if there is a sequence $\phi_j \in \mathrm{Aut}(\Omega)$ such that $\{f \circ \phi_j\}$ is a noncompact set in $L(\Omega)$. Notice that, obversely, f is compact if $\{f \circ \phi_j\}$ is a compact set in $L(\Omega)$ for every choice of ϕ_j.

Proposition 3.1 *Let Ω be a smoothly bounded, pseudoconvex domain in $\mathbb{C}^n$. Then Ω has noncompact automorphism group if and only if there exists an $f \in L(\Omega)$ such that f is noncompact.*

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Proof If the automorphism group is noncompact, then (by a classical result of H. Cartan), there exist $\phi_j \in \operatorname{Aut}(\Omega)$, $P \in \Omega$, and $X \in \partial\Omega$ such that $\phi_j(P) \to X$. By a result of Ohsawa (see [11]), the Bergman metric is complete. Fix a nonconstant $f \in L(\Omega)$. Choose $p, q \in \Omega$, $p \neq q$, so that

$$|p - q| \approx \big(1/|f\|_{L(\Omega)}\big) \cdot \big|f(p) - f(q)\big|.$$

We may suppose without loss of generality that $|p - q| = 1$.

Now certainly $|\phi_j(p) - \phi_j(q)| \to 0$ (since, by the completeness of the metric, both $\phi_j(p)$ and $\phi_j(q)$ must both tend to X). We can now calculate that

$$\begin{aligned} C &= C|p - q| \\ &\approx \big(1/\|f\|_{L(\Omega)}\big) \cdot \big|f(p) - f(q)\big| \\ &= \big(1/\|f\|_{L(\Omega)}\big) \cdot \big|f\big(\phi_j^{-1}(\phi_j(p))\big) - f\big(\phi_j^{-1}(\phi_j(q))\big)\big|. \end{aligned}$$

Since $|\phi_j(p) - \phi_j(q)| \to 0$, we see that $\{f \circ \phi_j^{-1}\}$ has Lipschitz norm that is blowing up. So f is noncompact.

Conversely, if $\operatorname{Aut}(\Omega)$ is compact, then let $f \in L(\Omega)$ and consider $\{f \circ \phi_j\}$ for $\phi_j \in \operatorname{Aut}(\Omega)$. Examine

$$\big|f \circ \phi_j(p) - f \circ \phi_j(q)\big|. \tag{3.1}$$

Clearly, by compactness, $|\nabla\phi_j|$ is bounded above and below, uniformly in j, on any compact set $K \subset\subset \Omega$. By the Ascoli–Arzela theorem applied on compact sets, we see from (3.1) that $f \circ \phi_j$ has a convergent subsequence. ∎

The next well-known result, due to Bun Wong [13], is a cornerstone of the modern theory of automorphism groups of smoothly bounded domains. We now present some new proofs of this result.

Theorem 3.2 *Let Ω be a smoothly bounded, strongly pseudoconvex domain in $\mathbb{C}^n$. Suppose that there a point $P \in \Omega$ and a strongly pseudoconvex boundary point $X \in \partial\Omega$ and that there exist $\phi_j \in \operatorname{Aut}(\Omega)$ such that $\phi_j(P) \to X$. Then Ω is biholomorphic to the unit ball $B \subseteq \mathbb{C}^n$.*

Proof As advertised, we will sketch three proofs. We first note that, according to Cartan's theorem and our previous result, the hypotheses imply that there is an $f \in L(\Omega)$ that is noncompact.

First Proof of the Theorem: If Ω is *not* biholomorphic to the ball, then, by a celebrated result of Lu Qi-Keng [10] (see [4] for thorough discussion), there is a point Q in Ω where the holomorphic sectional curvature of the Bergman metric is not the constant holomorphic sectional curvature of the ball.

As noted in the proof of the preceding result, the Bergman metric is complete on Ω. So, in fact, any compact set $K \subset\subset \Omega$ has the property that $\{\phi_j\}$ converges uniformly on K to X. In particular, $\phi_j(Q) \to X$. But it can be calculated (see [4, 5, 7]) that the holomorphic sectional curvature of the Bergman metric tends to the constant curvature of the ball at points that approach a strongly pseudoconvex boundary point X. This contradicts the last sentence of the previous paragraph. We conclude that Ω is biholomorphic to the ball, as claimed. ∎

Second Proof of the Theorem It is convenient for this argument to equip $\mathcal{O}(\Omega)$ with the topology of uniform convergence on compact sets (*i.e.,* the compact-open topology). For convenience, and without any loss of generality, we restrict attention now to ambient dimension 2.

Let U be a small neighborhood of X. Since X is a peak point (see [8]), it is standard to argue that for any compact set $K \subseteq \Omega$, there is a J so large that $j > J$ implies that $\phi_j(K) \subseteq U \cap \Omega$. Let X' be a point of $U \cap \Omega$ that is very near to X. Let $\delta = \delta_j = \operatorname{dist}(X', \partial\Omega)$. After a normalization of coordinates, we may suppose that the complex normal direction at X is z_1 and the complex tangential direction at X is z_2.

Define

$$\psi(z_1, z_2) = \left(X_1' + (z_1 - X_1')/\delta, X_2' + (z_2 - X_2')/\sqrt{\delta} \right).$$

Then $\psi \circ \phi_j$, with j as above, will have Lipschitz norm that is bounded, independent of j. As a result, using a sequence of compact sets K_j that exhausts Ω, and neighborhoods U that shrink to X, we can derive a subsequence, convergent on compact sets. And it will converge to a mapping of Ω to the Siegel upper half space. (This is just the standard method of scaling, which is described in detail in [4]). So Ω is biholomorphic to the Siegel upper half space, which is in turn biholomorphic to the unit ball. ∎

Third Proof of the Theorem: For this proof we examine the Fefferman asymptotic expansion for the Bergman kernel near a strongly pseudoconvex boundary point (see [2, 4]). This says that, in suitable local coordinates,

$$K(z, \zeta) = \frac{\psi(z, \zeta)}{[-X(z, \zeta)]^{n+1}} + \widetilde{\psi}(z, \zeta) \cdot \log[-X](z, \zeta). \tag{3.2}$$

Here $\psi, \widetilde{\psi}$ are smooth functions on $\overline{\Omega} \times \overline{\Omega}$ and X is the Levi polynomial (see [8, Ch. 3]) on Ω.

An interesting feature of Fefferman's work, and subsequent work of Burns and Graham [3], is that the logarithmic term is always present near a boundary point that is not spherical.

Arguing as usual, if P and X exist, then any other point $Q \in \Omega$ has the property that $\phi_j(Q) \to X$ as $j \to \infty$. We begin with a point Q near the boundary at which the Fefferman expansion (3.2) is valid. If Ω is not the ball, then we can take Q to be very near to a boundary point that is not spherical.

Of course the Bergman kernel transforms under a biholomorphic mapping F of Ω by the standard formula ([8, Ch. 1])

$$\operatorname{Jac}_{\mathbb{C}} F(z) K\big(F(z), F(\zeta)\big) \overline{\operatorname{Jac}_{\mathbb{C}} F(\zeta)} = K(z, \zeta). \tag{3.3}$$

So, when we think of $\phi_j(Q) \to X$, then we can understand how the Bergman kernel transforms by applying the transformation formula (3.3) to the Fefferman expansion (3.2). On one hand, this should give rise to another Fefferman-type formula based at the point $\phi_j(Q)$. But the problem is that the logarithmic expression does not scale. The result, as $j \to \infty$, will not be a valid Fefferman formula. This is a contradiction, so Ω must be biholomorphic to the ball. ∎

The following corollary is a consequence of the first two results.

Corollary 3.3 *A strongly pseudoconvex domain $\Omega \subseteq \mathbb{C}^N$ is biholomorphic to the ball if and only if the algebra $L(\Omega)$ of Lipschitz functions is noncompact.*

4 Further Results

The next result is classical. See [9, Ch. 12] for a more traditional proof.

Proposition 4.1 *Fix a bounded domain $\Omega \subseteq \mathbb{C}^n$. Let $\{\phi_j\}$ be automorphisms of Ω. Assume that the ϕ_j converge normally* (i.e., *uniformly on compact sets) to a limit f. Then either*

(i) *the mapping f is an automorphism of Ω; or*
(ii) *the mapping f maps to a complex variety in the boundary.*

Proof We adopt the point of view of Bers's theorem.

With $\phi_j \in \operatorname{Aut}(\Omega)$ as in the statement of the proposition, and $g \in L(\Omega)$, examine $\{g \circ \phi_j\}$.

Now either $g \circ \phi_j$ is compact or it is not. If $g \circ \phi_j$ is compact, then there exists a subsequence ϕ_{j_k} and a τ such that $g \circ \phi_{j_k} \to \tau$ with $\tau \in L(\Omega)$. So $g \circ f = \tau$, with $f \in \operatorname{Aut}(\Omega)$ (because it is a nondegenerate mapping, and a limit of automorphisms). Specifically, the mapping f is univalent, because it is the limit of univalent mappings. Also, f is onto, because we can apply our reasoning to ϕ_j^{-1}. That is part (i) of our conclusion (formulated in the language of this paper).

If instead $g \circ \phi_j$ is noncompact, then $\{g \circ \phi_j\}$ has no convergent subsequence. So $g \circ \phi_j$ blows up in norm. Hence, there are a point $P \in \Omega$ and a point $X \in \partial\Omega$ such that $\phi_{j_k}(P) \to X$ (for some subsequence ϕ_{j_k}). Hence, $g \circ \phi_{j_k}$ collapses to the boundary. This completes the proof of (ii). ∎

We now have the following proposition.

Proposition 4.2 *Suppose that $f: \Omega \to \Omega$ is a holomorphic mapping. Assume that, for some sequence $\{\phi_j\}$ of automorphisms of Ω, $f \circ \phi_{j_k}$ converges normally to a function $g \in \mathcal{O}(\Omega)$.*

(i) *If $g \in \operatorname{Aut}(\Omega)$, then $f \in \operatorname{Aut}(\Omega)$.*
(ii) *If g is not constant, then every convergent subsequence of $h_k \equiv f \circ \phi_{j_{k+1}} \circ \phi_{j_k}^{-1}$ has limit id_Ω.*

Proof This result is like a converse to compactness.

If $f(a) = f(b)$ for some distinct points $a, b \in \Omega$, then

$$f\big(\phi_{j_k} \circ \phi_{j_k}^{-1}(a)\big) = f\big(\phi_{j_k} \circ \phi_{j_k}^{-1}(b)\big).$$

Now, if the ϕ_{j_k} converge to some ψ, then we see that $g(\psi(a)) = g(\psi(b))$. If ψ is an automorphism then this is certainly a contradiction.

Of course, $f \circ \phi_{j_k}(\Omega) \subseteq f(\Omega)$ for all k. So $g(\Omega) \subseteq f(\Omega) \subseteq \Omega$. But $g(\Omega) = \Omega$, so $f(\Omega) = \Omega$. Thus, f is onto. It is also one-to-one. This proves (i).

For part (ii), we take g to be holomorphic and nonconstant. Let h be a subsequential limit of $f \circ \phi_{j_{k+1}} \circ \phi_{j_k}^{-1} \equiv h_k$. As a result, $f \circ \phi_{j_{k+1}} = h_k \circ \phi_{j_k}$, so $g = h \circ \psi$. But then $h = g \circ \psi^{-1}$, so h differs from g by an automorphism. Certainly then h is nonconstant. We note further that $g = h \circ \psi$ so that g is an automorphism. ∎

5 Concluding Remarks

Bers's theorem is a very classical result, originally proved in the context of one complex variable. Yet the ideas that it represents are still meaningful and of considerable interest today. This paper represents a brief exploration of Bers's theorem in a new context.

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