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TOTALLY UMBILICAL SUBMANIFOLDS OF QUATERNION-SPACE-FORMS

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Abstract

Totally umbilical submanifolds of dimension greater than four in quaternion-space-forms are completely classified.

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1. Introduction

A quaternion manifold is defined as a Riemannian manifold whose holonomy group is a subgroup of $\mathrm{Sp}(m)\cdot\mathrm{Sp}(1)=\mathrm{Sp}(m)\times\mathrm{Sp}(1)/\{\pm \text{ identity}\}$. The irreducible symmetric spaces $\mathrm{Sp}(1+m)/\mathrm{Sp}(1)\times\mathrm{Sp}(m)$ and $\mathrm{Sp}(1,m)/\mathrm{Sp}(1)\times\mathrm{Sp}(m)$ are the two most important examples of quaternion manifolds. It is well known that these two spaces have constant quaternion sectional curvature for m greater than or equal to 2. We simply call quaternion manifolds with constant quaternion sectional curvature the *quaternion-space-forms*.

In this paper, we shall completely classify totally umbilical submanifolds of dimension greater than 4 in quaternion-space-forms. The dimension of a manifold will always indicate its real dimension. We shall prove the following

THEOREM. *Let N be an n-dimensional ($n>4$) totally umbilical submanifold in a $4m$-dimensional quaternion-space-form M of quaternion sectional curvature $c\neq 0$. Then N is one of the following submanifolds:*

(a) *a quaternion-space-form immersed in M as a totally geodesic, quaternion submanifold, or*

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(b) *a complex-space-form immersed in M as a totally geodesic, half-quaternion submanifold, or*

(c) *a real-space-form immersed in M as a totally geodesic, totally real submanifold, or*

(d) *a real-space-form immersed in M as a totally real extrinsic sphere.*

Case (b) (*respectively*, (c) *and* (d)) *occurs only when* $n \leqslant 2m$ (*respectively*, $n \leqslant m$ *and* $n < m$).

Since every $4m$-dimensional quaternion-space-form of zero quaternion sectional curvature is a flat space, classification of totally umbilical submanifolds in such a space follows from a well-known result of Cartan (1946). For the classifications of totally umbilical submanifolds in a complex-space-form and Cayley plane, see Chen-Ogiue (1974) and Chen (1977), respectively.

2. Preliminaries

Let M be a $4m$-dimensional quaternion manifold with metric g. There exists a 3-dimensional vector space V of tensors of type $(1,1)$ with local basis of almost Hermitian structures I, J, K such that (i) $IJ = -JI = K$, and (ii) for any local cross-section ψ of V, $\nabla_X \psi$ is also a cross -section of V, where X is an arbitrary vector field in M and ∇ the Levi-Civita connection on M. It is well known that the existence of such vector bundle V on a Riemannian manifold implies that it is a quaternion manifold (Ishihara, 1974).

Let X be a unit vector on the quaternion manifold M. Then X, IX, JX and KX form an orthonormal frame in M. We denote by $Q(X)$ the 4-plane spanned by them, and call it the *quaternion 4-plane* determined by X. Every 2-plane (or simply plane) in a quaternion 4-plane is called a *quaternion plane*. The sectional curvature for a quaternion plane is called a quaternion sectional curvature. A quaternion manifold is called a quaternion-space-form if its quaternion sectional curvatures are equal to a constant c. It is known that a quaternion manifold is a quaternion-space-form if and only if its curvature tensor R is of the following form:

$$(2.1) \qquad \begin{aligned} R(X, Y)Z = \tfrac{1}{4}c\{&g(Y,Z)X - g(X,Z)Y + g(IY,Z)IX - g(IX,Z)IY \\ &+ 2g(X,IY)IZ + g(JY,Z)JX - g(JX,Z)JY \\ &+ 2g(X,JY)JZ + g(KY,Z)KX - g(KX,Z)KY \\ &+ 2g(X,KY)KZ\} \end{aligned}$$

for some constant c. Moreover, it is known that quaternion-space-forms are locally symmetric (Ishihara, 1974).

For any two vectors X, Y in M, if $Q(X)$ and $Q(Y)$ are orthogonal, the plane $\pi(X, Y)$ spanned by X, Y is said to be *totally real*. An isometric immersion $x\colon N \to M$

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from a Riemannian manifold N into M is said to be *totally real* if each tangent plane of N is carried into a totally real plane by x in M.

Let N' be an almost Hermitian manifold and $y: N' \to M$ an isometric immersion form N' into M. Then N' is called a *half-quaternion submanifold* of M if each holomorphic plane in N' is carried into a quaternion plane in M. A Riemannian manifold is called a *real-space-form* if it has constant sectional curvature and a Kaehler manifold is called a *complex-space-form* if it has constant holomorphic sectional curvature.

Let M' be a quaternion manifold and $z: M' \to M$ an isometric immersion from M' into M. We call M' a *quaternion submanifold* of M if quaternion 4-planes in M' are carried into quaternion 4-planes by z.

3. Basic formulas

Let N be an n-dimensional submanifold of a quaternion manifold M with metric g and local almost Hermitian structures I, J, K. We have

$$I^2 = J^2 = K^2 = -1, \tag{3.1}$$

$$IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J, \tag{3.2}$$

$$g(IX, IY) = g(JX, JY) = g(KX, KY) = g(X, Y). \tag{3.3}$$

Moreover, for any two orthonormal vectors X, Y in M which span a totally real plane, we have

$$g(\psi X, \rho Y) = 0, \quad \psi, \rho = I, J, \text{ or } K. \tag{3.4}$$

Let ∇ and ∇' be the Levi–Civita connections on M and N, respectively. The second fundamental form h of the immersion is defined by the equation; $h(X, Y) = \nabla_X Y - \nabla'_X Y$ for vector fields X, Y tangent to N. For a vector field ξ normal to N, we write

$$\nabla_X \xi = -A_\xi X + D_X \xi, \tag{3.5}$$

where $-A_\xi X$ (respectively, $D_X \xi$) denotes the tangential component (respectively, the normal component) of $\nabla_X \xi$. A normal vector field ξ is said to be parallel if $D\xi = 0$. The submanifold N is said to be *totally umbilical* if $h(X, Y) = g(X, Y) H$, for all vector fields X, Y tangent to N, where $H = \text{trace } h/n$ is the mean curvature vector of N in M. If the second fundamental form h vanishes identically, N is called a totally geodesic submanifold of M. A totally umbilical submanifold with nonzero parallel mean curvature vector is called an *extrinsic sphere*.

Let R, R' and R^N be the curvature tensors associated with ∇, ∇' and D, respectively. For the second fundamental form h of N in M, we define the covariant derivative, denoted by $\bar{\nabla}_X h$, to be

$$(3.6) \qquad (\bar{\nabla}_X h)(Y,Z) = D_X(h(Y,Z)) - h(\nabla'_X Y, Z) - h(Y, \nabla'_X Z).$$

Then, for all vector fields X, Y, Z, W tangent to N and vector fields ξ, η normal to N, the equations of Gauss, Codazzi and Ricci take the following forms (see Chen, 1973):

$$(3.7) \qquad g(R(X, Y)Z, W) = g(R'(X, Y)Z, W) + g(h(X,Z), h(Y, W)) - g(h(X, W), h(Y,Z)),$$

$$(3.8) \qquad (R(X, Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z),$$

$$(3.9) \qquad g(R(X, Y)\xi, \eta) = g(R^N(X, Y)\xi, \eta) - g([A_\xi, A_\eta](X), Y),$$

where $\perp$ in (3.8) denotes the normal component.

We call the submanifold N an *invariant submanifold* of M if we have

$$(3.10) \qquad R(X, Y)TN \subseteq TN,$$

for all X, Y in the tangent bundle TN of N.

4. Proof of the theorem

We first prove the following lemmas.

LEMMA 1. *Under the hypothesis of the theorem, N is an invariant submanifold and H is parallel.*

PROOF. Since N is a totally umbilical submanifold of dimension greater than 4 in a quaternion-space-form M, we have

$$(4.1) \qquad h(X, Y) = g(X, Y)H,$$

for all vector fields X, Y tangent to N. By (3.6), we have

$$(4.2) \qquad (\bar{\nabla}_X h)(Y,Z) = g(Y,Z)D_X H.$$

Thus, equation (3.8) reduces to

$$(4.3) \qquad (R(X, Y)Z)^{\perp} = g(Y,Z)D_X H - g(X,Z)D_Y H.$$

Since the dimension of N is greater than 4, for each unit vector field X tangent to N, there is a unit vector field Y tangent to N orthogonal to $Q(X)$. For such Y, (4.3) gives $(R(X, Y)Y)^{\perp} = D_X H$. On the other hand, (2.1) and (3.4) imply $(R(X, Y)Y)^{\perp} = 0$. Thus, $D_X H = 0$ for all vectors X tangent to N, that is, H is parallel. Substituting this into (4.3), we see that $(R(X, Y)Z)^{\perp} = 0$, for all vector fields X, Y, Z tangent to N. Thus, N is an invariant submanifold of M.

LEMMA 2. *Under the hypothesis of the theorem, N is locally symmetric.*

PROOF. Since H is parallel by Lemma 1, (4.2) gives $\bar{\nabla} h = 0$. From (3.7) and (3.8), we find

$$(4.4)\qquad g(R(X, Y)Z, W) = g(R'(X, Y)Z, W) + \alpha^2\{g(X,Z)g(Y, W) - g(X, W)g(Y,Z)\}.$$

$$(4.5)\qquad g(R(X, Y)Z, \xi) = 0,$$

for all vector fields X, Y, Z, W tangent to N and vector field ξ normal to N, where $\alpha^2 = g(H, H)$. From (4.4) and (4.5), we get

$$(4.6)\qquad R'(X, Y)Z = R(X, Y)Z + \alpha^2\{g(Y,Z)X - g(X,Z)Y\}.$$

Because H is parallel, α is a constant. Thus, by using $\nabla R = 0$, we get

$$(4.7)\qquad \begin{aligned} \nabla_U(R(X, Y)Z) &= R(\nabla_U X, Y)Z + R(X, \nabla_U Y)Z + R(X, Y)\nabla_U Z \\ &= R(\nabla'_U X, Y)Z + R(X, \nabla'_U Y)Z + R(X, Y)\nabla'_U Z \\ &\quad + R(h(X, U), Y)Z + R(X, h(U, Y))Z \\ &\quad + R(X, Y)(h(U,Z)). \end{aligned}$$

Consequently, (4.5) and (4.7) imply

$$(4.8)\qquad \begin{aligned} U(g(R(X, Y)Z, W)) &= g(R(\nabla'_U X, Y)Z, W) + g(R(X, \nabla'_U Y)Z, W) \\ &\quad + g(R(X, Y)\nabla'_U Z, W) + g(R(X, Y)Z, \nabla'_U W), \end{aligned}$$

for all vector fields X, Y, Z, W, U tangent to N. From (4.6) and (4.8), we find

$$(4.9)\qquad \begin{aligned} U(g(R(X, Y)Z, W)) &= U(g(R'(X, Y)Z, W)) - g((\nabla'_U R')(X, Y)Z, W) \\ &\quad + \alpha^2 U\{g(X,Z)g(Y, W) - g(X, W)g(Y,Z)\}. \end{aligned}$$

Therefore, (4.4) and (4.9) imply that N is locally symmetric.

LEMMA 3. *If N is an* n-*dimensional ($n \geqslant 4$) invariant submanifold of a quaternion-space-form M of quaternion sectional curvature $c \neq 0$, then N is a totally real submanifold, or a half-quaternion submanifold, or a quaternion submanifold of M.*

PROOF. Let X, Y be two vectors tangent to N. From (2.1) we find

$$(4.10)\qquad R(X, Y)X = \tfrac{1}{4}c\{g(Y, X)X - g(X, X)Y + 3g(IY, X)IX + 3g(JY, X)JX + 3g(KY, X)KX\}.$$

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Since N is an invariant submanifold of M, (4.10) implies

$$g(Y,IX)IX+g(Y,JX)JX+g(Y,KX)KX\in TN, \tag{4.11}$$

from which we see that if the component of Y in $Q(X)$ is nonzero, the component of Y in $Q(X)$ is nonzero, the component lies in TN. Therefore, we may choose an orthonormal basis for each $T_p N$, $p\in N$, in the following form:

$$e_1,\dots,e_{i_1},e_{i_1+1},\dots,e_{i_1+i_2},\dots,e_{i_1+\dots+i_{k-1}+1},\dots,e_{i_1+\dots+i_k},$$

where each $B_l=\{e_{i_1+\dots+i_{l-1}+1},\dots,e_{i_1+\dots+i_l}\}$ is a subset of a quaternion 4-plane $Q(E_l)$, $l=1,\dots,k$, for some E_l; moreover, $Q(E_1),\dots,Q(E_k)$ are mutually orthogonal. In particular, the number of elements in each B_l is fixed as p varies on N.

Case (1). If each B_l contains exactly one element, then N is a totally real submanifold of M.

Case (2). If each B_l contains exactly four elements, then each B_l spans a quaternion 4-plane. The quaternion structure on M then induces a canonical quaternion structure on N. With respect to this canonical quaternion structure, N becomes a quaternion submanifold of M.

Case (3). If neither case (1) nor case (2) holds, then there is a B_j which contains either 2 or 3 elements. Without loss of generality, we may assume that $l=1$.

Case (3.1). If $B_1=\{e_1,e_2\}$, then, for any $e\in B_l$, $l\neq 1$, we have

$$R(e_1,e_2)e=\tfrac{1}{2}c\{g(e_1,Ie_2)Ie+g(e_1,Je_2)Je+g(e_1,Ke_2)Ke\}$$

by virtue of (2.1). Since $g(e_1,e_2)=0$ and e_1,e_2 lie in the same quaternion 4-plane, at least one of $g(e_1,Ie_2)$, $g(e_1,Je_2)$ and $g(e_1,Ke_2)$ is nonzero. Thus, from (3.10), B_l contains at least two elements. This shows that if B_1 contains two elements, all other B_l contain at least two elements. If each B_l contains exactly two elements, then we have

$$B_1=\{e_1,e_2\},\quad B_2=\{e_3,e_4\},\dots,B_k=\{e_{2k-1},e_{2k}\},\quad 2k=n.$$

Since e_{2i-1} and e_{2i} are two orthonormal vectors lying in the same quaternion 4-plane, e_{2i} is a linear combination of Ie_{2i-1}, Je_{2i-1}, Ke_{2i-1}, namely,

$$e_{2i}=aIe_{2i-1}+bJe_{2i-1}+cKe_{2i-1},$$

with $a^2+b^2+c^2=1$. Now, we define a (1, 1)-tensor j on TN by

$$=aI+bJ+cK,$$

then $j^2=-1$, that is, j defines an almost complex structure on N. It is clear that N with this canonical almost complex structure forms a half-quaternion submanifold of M. Moreover, j with the induced metric is almost Hermitian.

Case (3.2). If $B_1 = \{e_1, e_2, e_3\}$, then, for any $e \in B_l$, $l \neq 1$, we have

$$R(e_\alpha, e_\beta)e = \tfrac{1}{2}c\{g(e_\alpha, Ie_\beta)Ie + g(e_\alpha, Je_\beta)Je + g(e_\alpha, Ke_\beta)Ke\} \tag{4.12}$$

for all $\alpha, \beta = 1, 2, 3$. Since N is an invariant submanifold, $R(e_\alpha, e_\beta)e \in TN$. Because the following matrix

$$\begin{pmatrix} g(e_1, Ie_2) & g(e_1, Je_2) & g(e_1, Ke_2) \\ g(e_1, Ie_3) & g(e_1, Je_3) & g(e_1, Ke_3) \\ g(e_2, Ie_3) & g(e_2, Je_3) & g(e_2, Ke_3) \end{pmatrix}$$

is of rank 3, (4.12) implies that Ie, Je and Ke lie in TN. Thus, B_l contains 4 elements. Applying the same argument to B_1, we obtain a contradiction.

LEMMA 4. *If N is a quaternion submanifold of a quaternion manifold M, then N is totally geodesic.*

PROOF. From the definition of the second fundamental form, we have

$$h(X, IY) = \nabla_X(IY) - \nabla'_X(IY) = I\nabla_X Y - \nabla'_X(IY) + (\nabla_X I)Y. \tag{4.13}$$

Since $\nabla_X I$ is a linear combination of I, J, K, $(\nabla_X I)Y$ is tangent to N. Comparing the normal components of both sides of (4.13), we find

$$h(X, IY) = Ih(X, Y). \tag{4.14}$$

Hence, we get

$$h(IX, IY) = -h(X, Y). \tag{4.15}$$

Similarly, we have

$$h(JX, JY) = h(KX, KY) = -h(X, Y). \tag{4.16}$$

On the other hand, by using $K = IJ$, we find $h(KX, KY) = -h(JX, JY) = h(X, Y)$. Comparing this with (4.16), we get $h(X, Y) = 0$. Thus, N is totally geodesic in M.

LEMMA 5. *Let N be a totally umbilical submanifold of dimension n $(n > 4)$ in a quaternion-space-form of quaternion sectional curvature $c \neq 0$. If N is a half-quaternion submanifold, then N is a complex-space-form immersed in M as a totally geodesic submanifold in M.*

PROOF. If N is a half-quaternion submanifold of M, N has a canonical almost Hermitian structure (j, g). From Lemma 2, N is locally symmetric with respect to the induced metric, thus, M is Kaehlerian. From the equation of Gauss, for any plane section π in N, the sectional curvature σ and σ' of π in M and N satisfy $\sigma'(\pi) = \sigma(\pi)+\alpha^2$, where $\alpha^2 = g(H,H)$. Since, for any holomorphic plane section $\pi(X,jX)$, the sectional curvature of the quaternion-space-form M satisfies $\sigma(\pi(X,jX)) = c$, the holomorphic sectional curvature of N is equal to the constant $c+\alpha^2$. Thus, N is a complex-space-form. Therefore, N as well as M are $\frac{1}{4}$-pinched. Consequently, the ratio of $c+\alpha^2 : \frac{1}{4}c+\alpha^2$ is $4:1$. But this is impossible unless $\alpha = 0$, that is, N is minimal. Since N is totally umbilical, N is totally geodesic.

Now, we return to the proof of the theorem. By Lemma 1, N is invariant. Thus, Lemma 3 implies that N is a totally real submanifold, or a half-quaternion submanifold, or a quaternion submanifold.

If N is a quaternion submanifold, Lemma 4 implies that N is totally geodesic. Since the quaternion structure on N is obtained from the restriction of the quaternion structure on M, equation (3.7) of Gauss implies that the curvature tensor of N is also given in the form of (2.1). Thus N is also a quaternion-space-form.

If N is a half-quaternion submanifold, Lemma 5 implies that N is a complex-space-form immersed in M as a totally geodesic submanifold.

If N is a totally real submanifold, then, for any orthonormal vectors X, Y tangent to N, the quaternion 4-planes $Q(X)$ and $Q(Y)$ are orthogonal. Thus, (2.1) gives $\sigma(\pi(X,Y)) = \frac{1}{4}c$. By the equation of Gauss, N is a real-space-form of sectional curvature $\frac{1}{4}c+\alpha^2$. If N is not totally geodesic in M, $H \neq 0$. By the parallelism of H, we find from the equation of Ricci that

$$g(R(X,Y)H, IY) = 0, \tag{4.17}$$

for any orthonormal vectors X, Y tangent to N. On the other hand, for such X and Y, equation (2.1) gives

$$g(R(X,Y)H, IY) = -\tfrac{1}{4}cg(IX,H)g(IY,IY).$$

Thus, $g(IX,H) = 0$. Similarly, we have $g(JX,H) = g(KX,H) = 0$. Since this is true for all X tangent to N, H is perpendicular to $Q(TN)$. Thus, $m > n = \dim N$. This completes the proof of the theorem.

5. Remarks

REMARK 1. If $n = 3$, Lemma 3 is false. This follows from the fact that for any 3-dimensional subspace π of a quaternion 4-plane $Q(X)$ in a quaternion-space-form, there exists a totally geodesic submanifold in M which is tangent to π.

REMARK 2. Let M be the complete quaternion-space-form

$$\mathrm{Sp}(1+m)/\mathrm{Sp}(1)\times\mathrm{Sp}(m),$$

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and let N be the real-space-form $\mathrm{SO}(1+m)/\{\pm 1\}\times \mathrm{SO}(m)$. Then, by regarding $\mathrm{SO}(1+m)$ as a subgroup of $\mathrm{Sp}(1+m)$ in the natural way, it is clear that $\mathrm{SO}(1+m)\cap \mathrm{Sp}(1)\times \mathrm{Sp}(m)=\{\pm 1\}\times \mathrm{SO}(m)$. This shows that

$$\mathrm{SO}(1+m)/\{\pm 1\}\times \mathrm{SO}(m)$$

can be imbedded in $\mathrm{Sp}(1+m)/\mathrm{Sp}(1)\times \mathrm{Sp}(m)$ as a totally geodesic submanifold. Moreover, it is clear that the imbedding is totally real.

Since $\mathrm{SO}(m)/\{1\}\times \mathrm{SO}(m-1)$ can be imbedded in $\mathrm{SO}(m+1)/\{\pm 1\}\times \mathrm{SO}(m)$ as an extrinsic sphere, there is an $(m-1)$-dimensional real-space-form imbedded in a $4m$-dimensional quaternion-space-form as an extrinsic sphere. Thus, both cases (c) and (d) happen.

REMARK 3. Since $SU(1+m)/S(U_1\times U_m)$ can be imbedded in

$$\mathrm{Sp}(1+m)/\mathrm{Sp}(1)\times \mathrm{Sp}(m)$$

as a totally geodesic, half-quaternion submanifold in a natural way, case (b) of the theorem occurs.

REMARK 4. Since $\mathrm{Sp}(1+q)/\mathrm{Sp}(1)\times \mathrm{Sp}(q)$ can be imbedded in

$$\mathrm{Sp}(1+m)/\mathrm{Sp}(1)\times \mathrm{Sp}(m)$$

as a totally geodesic, quaternion submanifold in a natural way for $q<m$, case (a) of the theorem occurs.

REMARK 5. If we consider the corresponding results in Remarks 2, 3 and 4 for spaces of non-compact type, we see that cases (a)–(d) in the theorem occur for $c<0$ as well as $c>0$.

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