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Soluble linear groups

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The least upper bound for the nilpotent lengths of soluble linear groups of degree n is calculated. For each n it is

$$4 + 2r(n) + \left[(2n-1)/8.3^{r(n)}\right] ,$$

where $r(n) = \left[\log_3(2n-1)/4\right]$ and $[x]$ is the integral part of x .

1. Introduction

Mal'cev [4] proved that a soluble linear group G has normal subgroups N, A with $N \leq A$ such that N is nilpotent, A/N is abelian and G/A is finite. Moreover, if G has degree n, then the nilpotency class of N is at most $n - 1$ and there is a bound on the order of G/A depending only on n. This implies that there are bounds on the soluble length and the nilpotent length of a soluble linear group of degree n which depend only on n. An explicit bound for the soluble length has been obtained by Huppert and by Dixon [1] (this latter contains an error, see [5]). It is also of interest to obtain an explicit bound for the nilpotent length. For instance Makan [3] used such a bound in his work on finite soluble groups with a given number of conjugacy classes of maximal nilpotent subgroups.

Before stating the main result we recall the definition of nilpotent length and introduce some notation which is used extensively.

A chain $G = N_0 \geq N_1 \geq \ldots \geq N_u = E$ (the identity subgroup) of normal subgroups of a group G such that each section N_{i-1}/N_i $(i \in \{1, \ldots, u\})$

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is nilpotent is a *nilpotent chain* of G of length u ; the *first* and *last* sections of the chain are N_0/N_1 and N_{u-1}/N_u respectively. A group G which has a nilpotent chain has a shortest nilpotent chain. The length of a shortest nilpotent chain of G is the *nilpotent length* of G ; it will be denoted $\nu(G)$.

THEOREM A.

(i) A soluble linear group of degree n *has nilpotent length at most*

$$\alpha(n) = 4 + 2r(n) + [(2n-1)/8.3^{r(n)}]$$

where $r(n) = [\log_3(2n-1)/4]$.

(ii) There is a soluble linear group of degree n *with nilpotent length* $\alpha(n)$.

Note that $\alpha(1) = 1$, $\alpha(2) = 3$ and, for $n \geq 2$, $\alpha(n+1) = \alpha(n)$ unless $n = 2, 2.3^{r(n)}$ or $4.3^{r(n)}$ when $\alpha(n+1) = \alpha(n) + 1$; this hinges on the observation that $[(2n-1)/8.3^{r(n)}]$ is 0 or 1 according as $n \leq 4.3^{r(n)}$ or $n > 4.3^{r(n)}$ for $2n - 1 < 4.3^{r(n)+1} < 16.3^{r(n)}$.

The two parts of the theorem are proved in Sections 2 and 3 respectively.

2. An upper bound

The first part of Theorem A will be proved as a consequence of similar results on soluble permutation groups (Theorem B) and on completely reducible soluble linear groups (Theorem C). We begin with two lemmas on nilpotent chains.

LEMMA 1. *If a soluble group* G *has a nilpotent chain* C *of length* u *, then* G *has a nilpotent chain* $\mathcal{D}$ *consisting of characteristic subgroups of* G *which is similar to* C *in the sense that* $\mathcal{D}$ *also has length* u *, and if the first or last section of* C *is a finite p-group, so is the corresponding section of* $\mathcal{D}$.

Proof. Let C be the nilpotent chain

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$$G = N_0 \geq N_1 \geq \ldots \geq N_u = E .$$

Put $N_i^* = \cap\left\{N_i^\theta : \theta \in \operatorname{aut}G\right\}$ for each $i \in \{0, \ldots, u\}$, and let $\mathcal{D}$ be the chain

$$G = N_0^* \geq N_1^* \geq \ldots \geq N_u^* = E .$$

Clearly each N_i^* is characteristic in G . For each i , N_i^*/N_{i+1}^* is isomorphic to a subgroup of the direct product of the $N_i^*/N_i^* \cap N_{i+1}^\theta$ taken over all θ in $\operatorname{aut}G$, and $N_i^*/N_i^* \cap N_{i+1}^\theta$ is isomorphic to $N_{i+1}^\theta N_i^*/N_{i+1}^\theta$ which is a subgroup of $N_i^\theta/N_{i+1}^\theta$ which is isomorphic to N_i/N_{i+1} . Hence each N_i^*/N_{i+1}^* is nilpotent and so $\mathcal{D}$ is a nilpotent chain of length u . If the last section of $\mathcal{C}$ is a finite p-group, then clearly so is that of $\mathcal{D}$. If the first section of $\mathcal{C}$ is a finite p-group, then the intersection defining N_1^* needs only finitely many automorphisms and it follows, as above, that G/N_1^* is a finite p-group.

The following is an easy consequence of this.

LEMMA 2. *Let G be an extension of a soluble group A by a soluble group B . If A has a nilpotent chain of length u and B has a nilpotent chain of length v , then G has a nilpotent chain of length $u + v$. If, moreover, the first section of A and the last section of B are finite p-groups for the same prime p , then G has a nilpotent chain of length $u + v - 1$.*

THEOREM B. *A soluble permutation group of degree n has a nilpotent chain of length $\beta(n) = 2s(n) + \left[n/4.3^{s(n)-1}\right]$ where $s(n) = \left[\log_3 n\right]$ whose first section is a 2-group for $n < 7.3^{s(n)-1}$ and whose last section is a 2-group for $n \geq 4.3^{s(n)-1}$.*

REMARK. More explicitly this says:

for $3.3^{s(n)-1} \leq n < 4.3^{s(n)-1}$, there is a nilpotent chain of length $2s(n)$ whose first section is a 2-group;

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for $4.3^{s(n)-1} \le n < 7.3^{s(n)-1}$, there is a nilpotent chain of length $2s(n) + 1$ whose first and last sections are 2-groups;

for $7.3^{s(n)-1} \le n < 9.3^{s(n)-1}$, there is a nilpotent chain of length $2s(n) + 1$ whose last section is a 2-group.

THEOREM C. *A completely reducible soluble linear group of degree* n *has a nilpotent chain of length* $\gamma(n) = 3 + 2t(n) + [n/4.3^{t(n)}]$ *where* $t(n) = [\log_3 n/2]$ *whose first section is a finite* 2-*group except possibly when* n *is* 1 *or* 3 .

REMARK. Note that $\gamma(1) = 1$, $\gamma(2) = 3$ and, for $n \ge 3$, $\gamma(n) = \gamma(n-1)$ unless $n = 2.3^{t(n)}$ or $4.3^{t(n)}$ when $\gamma(n) = \gamma(n-1) + 1$. In particular $\gamma(n) \le n + 1$ and $\gamma(n) + 1 \le \gamma(2n)$ for all n.

At one point in the following proof we need Theorem A *(i)*, so we now prove that, if Theorem C holds up to degree n, then so does Theorem A *(i)*.

Proof of Theorem A (i) from Theorem C. Let G be a soluble linear group of degree n. If G is completely reducible, then

$$\begin{aligned}\nu(G) &\le \gamma(n) ,\\ &= 3 + 2t(n) + [n/4.3^{t(n)}] ,\\ &\le 4 + 2r(n) + [(2n-1)/8.3^{r(n)}]\end{aligned}$$

because *either* $t(n) = r(n)$ when the result is obvious *or* $t(n) = r(n) + 1$ when $12.3^{r(n)} = 4.3^{t(n)} \le 2n$, so $8.3^{r(n)} \le 2n - 1$ and the result follows; hence $\nu(G) \le \alpha(n)$.

If G is not completely reducible, then $n \ge 2$ and G contains a nilpotent normal subgroup N such that G/N is isomorphic to a completely reducible but not irreducible group of degree n ([*1*], Lemma 1). Hence

$$\begin{aligned}\nu(G) &\le 1 + \nu(G/N) ,\\ &\le 1 + \gamma(n-1) , \text{ because } G/N \text{ is not irreducible,}\\ &\le 4 + 2t(n-1) + [(n-1)/4.3^{t(n-1)}] ,\\ &\le 4 + 2r(n) + [(2n-1)/8.3^{r(n)}] , \text{ because } t(n-1) \le r(n) ,\\ &\le \alpha(n) .\end{aligned}$$

Theorems B and C are proved by (a somewhat indirect) induction on the degree n. Theorem B is clearly true for $n \in \{1, 2, 3, 4\}$ and Theorem C is true for $n = 1$. Two inductive statements are proved:

I. For $n \geq 5$, Theorem B is true provided it is true for all integers less than n and Theorem C is true for all integers less than $n - 3$;

II. For $n \geq 2$, Theorem C is true provided it, and therefore also Theorem A *(i)*, is true for all integers less than n and Theorem B is true for all integers less than $n + 3$.

Proof of I. Let G be a soluble permutation group of degree n, $n \geq 5$. It is sufficient to consider the case when G is transitive, and then there are two cases.

(I.1). *If G is imprimitive* and G has k sets of imprimitivity $(1 < k < n)$ of degree $m = n/k$, then G is isomorphic to a subgroup of the permutational wreath product M wr K, where M and K are soluble permutation groups of degrees m and k respectively ([2], Satz II.1.2). Hence G is an extension of a direct product of copies of M by K. Therefore, by Lemma 2, G has a nilpotent chain of length $\beta(k) + \beta(m)$ and one of length $\beta(k) + \beta(m) - 1$ when $k \geq 4.3^{s(k)-1}$ and $m < 7.3^{s(m)-1}$. Moreover there is such a chain whose first section is a 2-group whenever $k < 7.3^{s(k)-1}$ and whose last section is a 2-group whenever $m \geq 4.3^{s(m)-1}$.

There are nine cases which may be summed up in the table on page 36.

If a group has a nilpotent chain of length u, then it also has a nilpotent chain of length $u + 1$ whose first (*or* last) section is a (trivial) 2-group. This observation is used without further comment below.

In Case 1 of the table, $n \geq 3^{s(k)+s(m)}$ so $s(n) \geq s(k) + s(m)$ and the result follows. In Cases 2, 4, 5, $n \geq 4.3^{s(k)+s(m)-1}$ and G has a nilpotent chain of length $2s(k) + 2s(m) + 1$ whose first and last sections are 2-groups as required. The result follows similarly in Cases 3, 7, because $n \geq 7.3^{s(k)+s(m)-1}$, in Cases 6, 8, because $n \geq 3^{s(k)+s(m)+1}$, and in Case 9 because $n \geq 4.3^{s(k)+s(m)+1}$.

	k	m	G has a nilpotent chain of length	whose first section is a	whose last section is a
1	[3, 4)	[3, 4)	$2s(k)+2s(m)$	2-group	
2		[4, 7)	$2s(k)+2s(m)+1$	2-group	2-group
3		[7, 9)	$2s(k)+2s(m)+1$	2-group	2-group
4	[4, 7)	[3, 4)	$2s(k)+2s(m)$	2-group	
5		[4, 7)	$2s(k)+2s(m)+1$	2-group	2-group
6		[7, 9)	$2s(k)+2s(m)+2$	2-group	2-group
7	[7, 9)	[3, 4)	$2s(k)+2s(m)$		
8		[4, 7)	$2s(k)+2s(m)+1$		2-group
9		[7, 9)	$2s(k)+2s(m)+2$		2-group

Here $[a, b)$ in the column headed k indicates $a.3^{s(k)-1} \leq k < b.3^{s(k)-1}$, and similarly for the column headed m .

(I.2). *If G is primitive*, then $n = p^k$ for some prime p and positive integer k and G has a self-centralizing minimal normal subgroup A of order p^k . Thus G/A can be regarded as a subgroup of aut$A$ in the usual way, that is, G/A is isomorphic to an irreducible subgroup of $GL(k, p)$. (See [2], Satz II.3.2 or [1], p. 154.)

Since $k \leq n - 4$ (because $n \geq 5$) , it follows by the induction hypothesis that G has a nilpotent chain of length $\gamma(k) + 1$ whose first section is a 2-group when $k \neq 1, 3$.

If $k = 1$, then G/A is abelian, so G has a nilpotent chain of length 2 . Since $\beta(7) = 3$ and $\beta(n) \geq 4$ for $n > 7$, G has a nilpotent chain of length $\beta(n)$ whose last section is a 2-group when $n \geq 7$ and whose first and last sections are 2-groups when $n > 7$, so the

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result is proved for $n \geq 7$. If $n = 5$, then G/A is isomorphic to a subgroup of $\text{aut}C_5$, so G/A is a 2-group; hence G has a nilpotent chain of length 3 whose first and last sections are 2-groups, as required.

If $k \geq 2$ *and* $p \geq 3$, then $3^k \leq n < 3^{s(n)+1}$, so $k \leq s(n)$. Therefore $\gamma(k) + 1 \leq k + 2 \leq 2k \leq 2s(n)$, and the result follows.

If $p = 2$, then A is a 2-group, so the result will follow whenever $\gamma(k) + 1 \leq \beta(n)$. Now $2^k = n < 3^{s(n+1)}$, hence $k < (s(n)+1) \log_2 3 < 5(s(n)+1)/3$, therefore $k \leq 2s(n)$ when $s(n) \geq 2$, that is, when $k \geq 4$. When $k > 4$, then $\gamma(k) \leq k - 1$ and hence $\gamma(k) + 1 \leq k \leq 2s(n) \leq \beta(n)$, so the result follows when $k > 4$. When $k = 4$, however, the result also follows, since $\gamma(4) + 1 = 5 = \beta(2^4)$.

The only case, then, that remains to be considered is the case $n = 2^3$. In this case A is of order 8 and G/A acts as a permutation group on the seven non-identity elements of A . Hence, by the induction hypothesis, G/A has a nilpotent chain of length 3 whose last section is a 2-group. Since A is also a 2-group, it follows by Lemma 2 that G has a nilpotent chain of length 3 whose last section is a 2-group, as required.

Proof of II. Let G be a completely reducible linear group of degree n , $n > 1$. We may assume that the underlying field is algebraically closed (remark after Lemma 2 of [*1*]). It is enough to consider the case when G is irreducible, and then there are two cases.

(II.1). *If* G *is imprimitive* with a system of imprimitivity consisting of k $(1 < k \leq n)$ subspaces of dimension $m = n/k$, then G has a normal subgroup N which is isomorphic to a subgroup of a direct product of completely reducible soluble linear groups of degree m and G/N is a soluble permutation group of degree k (see [*1*], p. 155). Since $k \leq n$ and $m < n$, Theorems B and C may now be invoked to give information about the nilpotent chains of N and G/N . The result follows by a routine checking of cases, in essentially the same manner as in (I.1).

(II.2). *If G is primitive,* let $n = p_1^{k_1} \ldots p_h^{k_h}$ be the canonical prime factorization of n. By a result of Suprunenko ([6], Theorems 9 and 11), G has a nilpotent normal subgroup A such that

(i) G/A is isomorphic to a subgroup of the direct product of the symplectic groups $\mathrm{Sp}(2k_i, p_i)$, $i \in \{1, \ldots, h\}$;

(ii) $A/Z(G)$ is a direct product of abelian groups of order $p_i^{2l_i}$, $i \in \{1, \ldots, h\}$; $0 \le l_i \le k_i$.

If $h \ge 2$, set $k = \max_i k_i$. Then $n \ge 2^k.3$ and hence $2k < n$, so it follows by the induction hypothesis that G has a nilpotent chain of length $\alpha(2k) + 1$. The result in this case follows by observing that $\alpha(2k) + 2 \le \gamma(n)$; because $\alpha(2) + 2 \le \gamma(2.3)$ and, for $k > 1$,

$$\alpha(2k) + 2 \le \alpha(2k-2) + 3 \le \gamma(2^{k-1}.3) + 1 \le \gamma(2^k.3) .$$

If $h = 1$, then $n = p^k$, say. In this case $A/Z(G)$ is a p-group and G/A is isomorphic to a subgroup of $\mathrm{Sp}(2k, p)$. Therefore G/A contains a normal p-group B/A such that G/B is isomorphic to a completely reducible subgroup of $\mathrm{GL}(2k, p)$ ([1], Lemma 1). Since $B/Z(G)$ is a p-group, B is nilpotent. Consequently, if $n \ne 2, 2^2$ (so that $2k < n$), it follows that G has a nilpotent chain of length $\gamma(2k) + 1$ whose first section is a 2-group.

If $\gamma(2k) + 1 \le \gamma(p^k)$, then

$$\gamma(2k+2) + 1 \le \gamma(2k) + 2 \le \gamma(p^k) + 1 \le \gamma(p^{k+1}) ;$$

but $\gamma(8) + 1 \le \gamma(2^4)$, $\gamma(4) + 1 \le \gamma(3^2)$ and $\gamma(2) + 1 \le \gamma(p)$ for $p \ge 5$. So $\gamma(2k) + 1 \le \gamma(p^k)$ and the result follows except for $n = 2, 3, 2^2, 2^3$. We consider these remaining cases separately.

(i) $n = 2$; In this case G/A is isomorphic to a subgroup of $\mathrm{Sp}(2, 2)$, and the result follows since $\mathrm{Sp}(2, 2)$ is a group of order 6.

(ii) $n = 3$; The result is immediate since $\mathrm{Sp}(2, 3)$ has nilpotent length 2.

(iii) $n = 4$: Since $Sp(4, 2)$ is isomorphic to S_6 ([2], Satz II.9.21), G has a nilpotent chain of length $\beta(6) + 1 = 4$, whose first section is a 2-group, as required.

(iv) $n = 8$: In this case G/B is isomorphic to a completely reducible subgroup $\overline{G}$ of $GL(6, 2)$. It suffices to show that $\overline{G}$ has a nilpotent chain of length 4 whose first section is a 2-group. Let F be the algebraic closure of $GF(2)$.

If $\overline{G}$ *is reducible* over F , the result follows immediately since $\gamma(n) \le 4$ for $n < 6$.

If $\overline{G}$ *is irreducible and primitive*, Suprunenko's result implies that $\overline{G}$ has a nilpotent normal subgroup $\overline{A}$ such that $\overline{G}/\overline{A}$ is isomorphic to a subgroup of $Sp(2, 2) \times Sp(2, 3)$. Such a subgroup has nilpotent length 2 . The result follows.

If $\overline{G}$ *is irreducible and imprimitive*, then $\overline{G}$ has a normal subgroup $\overline{N}$ which is isomorphic to a subgroup of a direct product of completely reducible soluble linear groups of degree $m = 1, 2$ or 3 and $\overline{G}/\overline{N}$ is a soluble permutation group of degree 6, 3 or 2 respectively. If $m = 1$ or 3 , the result comes at once for $\gamma(1) + \beta(6) = \gamma(3) + \beta(2) = 4$ and the corresponding chains have first section a finite 2-group. A completely reducible soluble linear group of degree 2 over an algebraically closed field of characteristic 2 is either reducible and then abelian or irreducible and then metabelian ([1], p. 156). The result follows as before.

3. Examples

LEMMA 3. *Let* A *be a finite soluble group of nilpotent length* u *and* B *a finite soluble permutation group of nilpotent length* v . *Let* G *be the permutational wreath product of* A *by* B .

(i) $\nu(G) = u + v$ *or* $u + v - 1$.

(ii) $\nu(G) = u + v - 1$ *if and only if* A *has a nilpotent chain of length* u *whose first section is a* p*-group and* B *has a nilpotent chain of length* v *whose last section is a* p*-group for the same prime* p .

Proof. By Lemma 2, $\nu(G) \le u + v$ and $\nu(G) \le u + v - 1$ when the conditions in *(ii)* hold. So it remains to show

(a) $\nu(G) \geq u + v - 1$ and

(b) $\nu(G) = u + v - 1$ implies the conditions in *(ii)* hold.

(a). The upper nilpotent series of a finite group H will be written

$$E = F_0(H) \leq F_1(H) \leq \ldots \leq F_i(H) \leq \ldots$$

where $F_{i+1}(H)/F_i(H)$ is the Fitting radical of $H/F_i(H)$. Let D be the base group of G ; then $D = A_1 \times \ldots \times A_k$ where k is the degree of B and each A_i is an isomorphic copy of A . We show first that $F_{u-1}(G) = F_{u-1}(D)$. Since $D \trianglelefteq G$, it follows that $F_{u-1}(G) \cap D = F_{u-1}(D)$. Now suppose $\sigma \in F_{u-1}(G)D \cap B$ and $\sigma \neq 1$. Without loss of generality we may assume $1\sigma = 2$. Since $\nu(A) = u$, there is an element a in $A \backslash F_{u-1}(A)$. Let a_i be the copy of a in A_i ; then $a_i \notin F_{u-1}(A_i)$ and $a_1^{\sigma} = a_2$. Take $d \in D$, so $\sigma d \in F_{u-1}(G)$. Then $a_1(\sigma d)a_1^{-1} = (\sigma d)a_2^d a_1^{-1} \in F_{u-1}(G)$. Hence $a_2^d a_1^{-1} \in F_{u-1}(G) \cap D = F_{u-1}(D)$. But

$$F_{u-1}(D) = F_{u-1}(A_1) \times \ldots \times F_{u-1}(A_k) ,$$

so $a_1 \in F_{u-1}(A_1)$. This contradiction implies $F_{u-1}(G)D \cap B = E$ and hence $F_{u-1}(G) = F_{u-1}(D)$. Since D has nilpotent length u , $D \leq F_u(G)$. Put $\nu = \nu(G)$. The chain

$$G/D = F_\nu(G)/D > F_{\nu-1}(G)/D > \ldots > F_u(G)/D \geq E$$

is a nilpotent chain of length $\nu - u + 1$. Since G/D is isomorphic to B and $\nu(B) = v$, it follows that $\nu - u + 1 \geq v$ and so $\nu \geq u + v - 1$ as required.

(b). If $\nu = u + v - 1$, then $F_u(G) \neq D$ because $\nu(B) = v$. Now $F_u(G) = D\big(F_u(G) \cap B\big)$, so the nilpotent group $F_u(G)/F_{u-1}(G)$ is isomorphic to the semidirect product $\big(D/F_{u-1}(G)\big)\big(F_u(G) \cap B\big)$ (with the action on

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$D/F_{u-1}(G)$ induced from that on D). Hence $D/F_{u-1}(D)$ and $F_u(G) \cap B$ are p-groups for the same prime p. Therefore $A/F_{u-1}(A)$ and $F_u(G) \cap B$ are p-groups for the same prime p. Thus

$$A > F_{u-1}(A) > F_{u-2}(A) > \ldots > E$$

and

$$B > F_{\nu-1}(G) \cap B > \ldots > F_u(G) \cap B > E$$

are nilpotent chains of the required kind.

We now construct examples to show the bounds in Theorems B and C are best possible and use these to prove Theorem A *(ii)*.

For every positive integer s, the iterated wreath product

$$S_3 \text{ wr } S_3 \text{ wr } \ldots \text{ wr } S_3 \quad (s \text{ factors})$$

is a soluble permutation group of degree 3^s which, by Lemma 3, has nilpotent length $2s$; and the group

$$S_4 \text{ wr } S_3 \text{ wr } \ldots \text{ wr } S_3 \quad (s \text{ factors})$$

is a soluble permutation group of degree 4.3^{s-1} with nilpotent length $2s + 1$. The length bound of Theorem B is therefore best possible. The first of these examples has no nilpotent chain of length $2s$ whose last section is a 2-group. Let D_{21} be the non-abelian group of order 21 considered as a permutation group of degree 7. The group

$$S_3 \text{ wr } \ldots \text{ wr } S_3 \text{ wr } D_{21} \quad (s \text{ factors})$$

has no nilpotent chain of length $2s + 1$ with first and last sections 2-groups. Thus the other conditions are also best possible.

If M is an irreducible linear group of degree m and K a transitive permutation group of degree k, then the permutational wreath product M wr K is an irreducible linear group of degree mk : for let W be the underlying linear space of M and put $V = W_1 \oplus \ldots \oplus W_k$ where each W_i is a copy of W ; each element σ of K can be regarded as an element of aut$V$ by setting $w_i\sigma = w_{i\sigma}$ for all w_i in W_i and all i ;

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for each m in M let m_i be the element of $\operatorname{aut}V$ such that for all w_j in W_j

$$w_j m_i = \begin{cases} w_j & \text{for } j \neq i \\ (wm)_j & \text{for } j = i \, ; \end{cases}$$

then $M \operatorname{wr} K = \langle m_i, \sigma : m \in M, i \in \{1, \ldots, k\}, \sigma \in K \rangle$; since M acts irreducibly on W and K is transitive, $M \operatorname{wr} K$ acts irreducibly on V .

Let t be a positive integer. The linear group $GL(2, 3)$ has only one nilpotent chain of length 3 and its first section is a 2-group. Hence

$$GL(2, 3) \operatorname{wr} S_3 \operatorname{wr} \ldots \operatorname{wr} S_3 \quad (t+1 \text{ factors})$$

is an irreducible linear group of degree 2.3^t with nilpotent length $2t + 3$. Such an example can be constructed over many fields because $GL(2, 3)$ has a faithful irreducible representation of degree 2 over every field in which there is a primitive fourth root of unity and a square root of 2 . Let M be any linear group of degree 1 which is not a 2-group; then

$$M \operatorname{wr} S_4 \operatorname{wr} S_3 \operatorname{wr} \ldots \operatorname{wr} S_3 \quad (t+2 \text{ factors})$$

is an irreducible linear group of degree 4.3^t with nilpotent length $2t + 4$. These examples show that the bound in Theorem C is best possible.

Theorem A *(ii)* is an immediate consequence of the above examples and the corollary to the following lemma.

LEMMA 4. *Let G be an irreducible linear group of degree n . There is a linear group H of degree $n + 1$ containing a non-trivial abelian normal subgroup A such that H/A is isomorphic to G and every nilpotent normal subgroup of H is contained in A .*

COROLLARY. *If there is an irreducible soluble linear group of degree n and nilpotent length t , then there is a soluble linear group of degree $n + 1$ and nilpotent length $t + 1$.*

Proof of Lemma 4. Let F denote the field and W the linear space

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underlying G. Put $V = F \oplus W$. For each $\sigma \in W^*$ (the dual of W) and $g \in G$ define the map $(\sigma, g) : V \to V$ by

$$(f, w)(\sigma, g) = (f+w\sigma, wg) \text{ for all } (f, w) \in V .$$

Then $H = \{(\sigma, g) : \sigma \in W^*, g \in G\}$ is a subgroup of aut$V$ and so linear of degree $n + 1$. Clearly $A = \{(\sigma, e) : \sigma \in W^*\}$ is an abelian normal subgroup of H such that H/A is isomorphic to G. Let N be a normal subgroup of H not contained in A. The space $U = \{w-wx : w \in W, (\sigma, x) \in N\}$ is a non-trivial G-invariant subspace of W. Since G is irreducible on W, $U = W$. Thus, for every non-trivial element (τ, e) of A there is a w in W and a (σ, x) in N such that $\left(wx^{-1}-w\right)\tau \neq 0$. Hence the commutator $[(\tau, e), (\sigma, x)] = \left(x^{-1}\tau-\tau, e\right) \neq (0, e)$. This can be repeated to give a sequence $\left(\sigma_0, x_0\right), \left(\sigma_1, x_1\right), \ldots$ of elements of N such that

$$\left[(\tau, e), \left(\sigma_0, x_0\right), \ldots, \left(\sigma_j, x_j\right)\right] \neq (0, e)$$

for all j. Thus NA is not nilpotent and therefore N is not nilpotent.

References

[1] John D. Dixon, "The solvable length of a solvable linear group", *Math. Z.* 107 (1968), 151-158.

[2] B. Huppert, *Endliche Gruppen I* (Die Grundlehren der mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[3] A.R. Makan, "On some aspects of finite soluble groups", (Ph.D. thesis, Australian National University, Canberra, 1971).

[4] A.I. Mal'cev, "On certain classes of infinite solvable groups", *Amer. Math. Soc. Transl.* (2) 2 (1956), 1-21.

[5] M.F. Newman, "The soluble length of soluble linear groups", Submitted to *Math. Z.*

[6] D. [A.] Suprunenko, *Soluble and nilpotent linear groups* (Transl. Math. Monographs, 9. Amer. Math. Soc., Providence, Rhode Island, 1963).

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