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PARTIAL ORDERS ON THE 2-CELL

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1. **Introduction.** A *partially ordered space* is an ordered pair $(X, \leq)$ where X is a compact metric space and $\leq$ is a partial ordering on X such that $\leq$ is a closed subset of the Cartesian product $X \times X$. $\leq$ is said to be a *closed partial order* on X.

If $(X, \leq)$ is a partially ordered space let $\text{Min}(X)$ (resp. $\text{Max}(X)$) denote the set of minimal (resp. maximal) elements of X. For $x \in X$ let

$$L(x) = \{y \in X \mid y \leq x\} \quad \text{and} \quad M(x) = \{y \in X \mid x \leq y\}.$$

Ward used partial orders to characterize dendrites in [7]. In [1], [2] and [3] Tymchatyn used partial orders to obtain characterizations of the two and three dimensional cells.

In this paper we use the methods developed in [1] to study a wide class of partial orders on the 2-cell. We let $(X, \leq)$ be a partially ordered space where X is a 2-cell, $\text{Min}(X)$ and $\text{Max}(X)$ are closed arcs on the boundary of X and for each $x \in X$ $L(x) \cup M(x)$ is a connected set. We let $\leq'$ be the vertical partial order on the unit square $[0, 1] \times [0, 1]$ in the euclidean plane, i.e. we set $(a, b) \leq' (c, d)$ in $[0, 1] \times [0, 1]$ if and only if $a = c$ and $d - b$ is non-negative. We show that there is a continuous order preserving function of the partially ordered space $([0, 1] \times [0, 1], \leq')$ onto $(X, \leq)$ such that the inverse image of a point of X is either a point or a horizontal line segment. It follows that $\leq$ contains a partial order that has the same properties as $\leq$ and that is obtainable in a natural way from a very simple decomposition of $([0, 1] \times [0, 1], \leq')$.

2. **Preliminaries.** We shall gather here some necessary definitions and theorems from [1] and [4].

A *chain* is a totally ordered set. An *order arc* is a compact connected chain. It is known [6] that a separable order arc is homeomorphic under an order preserving function to the closed unit interval $[0, 1]$ with its usual order (which we also denote by $\leq$) and with its usual topology. The reader should have no difficulty in determining in a particular instance whether $\leq$ represents the partial order on X or on $[0, 1]$.

If $(X, \leq)$ is a partially ordered space we let 2^X denote the space of closed subsets of X with the Hausdorff metric topology. We let $\mathscr{M}(X)$ denote the family of order arcs in X which meet both $\text{Min}(X)$ and $\text{Max}(X)$.

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THEOREM A (TYMCHATYN and WARD [4]). *Let* $(X, \leq)$ *be a partially ordered space such that* Min(X) *and* Max(X) *are closed sets and for each* $x \in X$, $L(x) \cup M(x)$ *is connected. Then* $\mathscr{M}(X)$ *is a compact subset of* 2^X *and* $\mathscr{M}(X)$ *covers* X.

An *antichain* is a set which contains no non-degenerate chain. We let $\mathscr{A}(X)$ denote the family of compact maximal antichains of X. We let $\mathscr{A}(X)$ have its relative topology as a subset of 2^X. It is known [6] that Min(X) and Max(X) are in $\mathscr{A}(X)$ if and only if they are closed subsets of X.

The following two results appear in [1]:

THEOREM B. *Let* $(X, \leq)$ *be a partially ordered space such that* $Min(X)$ *and* $Max(X)$ *are closed and* $\mathscr{M}(X)$ *covers* X. *For* $A, B \in \mathscr{A}(X)$ *define*

$$A \wedge B = \{x \in A \cup B \mid L(x) \cap (A \cup B) = \{x\}\}$$

and

$$A \vee B = \{x \in A \cup B \mid M(x) \cap (A \cup B) = \{x\}\}.$$

Then $\mathscr{A}(X)$ *with operations* $\wedge$ *and* $\vee$ *is an arcwise connected topological lattice. Furthermore,* $\mathscr{A}(X)$ *covers* X.

THEOREM C. *Let* $(X, \leq)$ *be a partially ordered space such that* Max(X) *and* Min(X) *are closed, disjoint sets and* $\mathscr{M}(X)$ *covers* X. *Then there exists a continuous order preserving function* f *of* $(X, \leq)$ *onto* $[0, 1]$ *with its usual order such that*

(i) $f^{-1}(0)=\text{Min}(X)$ *and* $f^{-1}(1)=\text{Max}(X)$ *and*
(ii) *for each* $a \in [0, 1]$ $f^{-1}(a) \in \mathscr{A}(X)$.

The partial order $\leq$ on a space X is said to be *order dense* if for each $x<y$ there exists z such that $x<z<y$. It is known [6] that if $(X, \leq)$ is a compact, order dense, partially ordered space then every chain in X is contained in a member of $\mathscr{M}(X)$.

In case there is more than one partial order on a space X we shall write Min$(X, \leq)$, Max$(X, \leq)$, $L(x, \leq)$, $M(x, \leq)$, $\mathscr{M}(X, \leq)$ and $\mathscr{A}(X, \leq)$ for Min(X), Max(X), $L(x)$, $M(x)$, $\mathscr{M}(X)$ and $\mathscr{A}(X)$ respectively.

3. Orders on the 2-cell.

THEOREM 1. *Let* $(X, \leq)$ *be a partially ordered space where* X *is a closed* 2-*cell. Suppose* Min(X) *and* Max(X) *are arcs in the boundary* S^1 *of* X. *If* $\mathscr{M}(X)$ *covers* X *then* $\mathscr{M}(X)$ *admits the structure of a compact, connected, topological lattice.*

Proof. Case 1. Suppose Min(X) is disjoint from Max(X).

Let F and E be the closures of the two components of $S^1-(\text{Min}(X) \cup \text{Max}(X))$. We wish to prove that F and E are in $\mathscr{M}(X)$. Notice that F and E are arcs which meet both Min(X) and Max(X). We need only prove that F and E are chains. Let $x, y \in F$. We may suppose that x separates y from Min$(X) \cap F$ in F. By hypothesis there exists $T \in \mathscr{M}(X)$ such that $x \in T$. If $y \in T$ we are done. If $y \notin T$ then

$T \cap M(x)$ is an arc in the 2-cell X which separates y from $\mathrm{Min}(X) \cap L(y)$. By hypothesis there exists $S \in \mathscr{M}(X)$ such that $y \in S$. Now $S \cap L(y)$ is an order arc which meets both y and $\mathrm{Min}(X) \cap L(y)$. Hence, there exists

$$z \in S \cap L(y) \cap T \cap M(x).$$

Since $x \leq z \leq y$ it follows that $x \leq y$ and F is a chain. Similarly, E is a chain.

By Theorem C there is a continuous function $f:X \to [0, 1]$ such that $f^{-1}(0) = \mathrm{Min}(X)$, $f^{-1}(1) = \mathrm{Max}(X)$ and for each $a \in [0, 1]$, $f^{-1}(a) \in \mathscr{A}(X)$. It follows that for each $a \in [0, 1]$, $f^{-1}(a)$ meets each member of $\mathscr{M}(X)$ in precisely one point.

If $a \in]0, 1[$, it is clear that $f^{-1}(a)$ separates X into precisely two components $f^{-1}([0, a[)$ and $f^{-1}(]a, 1])$. Furthermore, $f^{-1}(a)$ is arcwise accessible from each of these two components. By Theorem II.5.38 in Wilder [8], $f^{-1}(a)$ is an arc. Since $f^{-1}(a)$ is irreducible with respect to separating X it follows that one endpoint of $f^{-1}(a)$ is in F and the other is in E.

For each $a \in [0, 1]$ give the arc $f^{-1}(a)$ its natural total order with minimal point in $F \cap f^{-1}(a)$. Denote the resulting partial order on X by $\leq''$. Then $x \leq'' y$ if and only if $f(x) = f(y)$ and x separates y from $f^{-1}(f(x)) \cap F$ in the arc $f^{-1}(f(x))$.

CLAIM. *$(X, \leq'')$ is a partially ordered space.*

Proof of claim. Let $x_i)$ and $y_i)$ be sequences in X which converge to x and y respectively such that for each i, $x_i \leq'' y_i$. We must prove that $x \leq'' y$. For each i, $f(x_i) = f(y_i)$ and f is continuous so $f(x) = f(y)$. Either $x \leq'' y$ or $y \leq'' x$ since $f^{-1}(f(x))$ is linearly ordered with respect to the partial order $\leq''$. Just suppose $y <'' x$. Let $t \in X$ such that $y <'' t <'' x$. By hypothesis there exists $T \in \mathscr{M}(X, \leq)$ such that $t \in T$. The arc T separates x and y in X. The sequence $x_i)$ (resp. $y_i)$) is eventually in the same component of $X - T$ as is x (resp. y) since X is locally connected. Also, $F \cap f^{-1}(f(x_i))$ is eventually in the same component of $X - T$ as $F \cap f^{-1}(f(x))$ and y. Since for each i, $f^{-1}(f(x_i))$ meets T in precisely one point, it follows that eventually $y_i <'' x_i$. This is a contradiction. Thus $x \leq'' y$ and the claim is proved.

Clearly, $\mathscr{M}(X, \leq) \subset \mathscr{A}(X, \leq'')$. By Theorem A $\mathscr{M}(X, \leq)$ is compact. By Theorem B $\mathscr{A}(X, \leq'')$ is a topological lattice. We must prove that $\mathscr{M}(X, \leq)$ is actually a sublattice of $\mathscr{A}(X, \leq'')$. If $S, T \in \mathscr{M}(X, \leq)$ then $T \wedge S$ and $T \vee S$ are in $\mathscr{A}(X, \leq'')$ since $\mathscr{A}(X, \leq'')$ is a lattice. In particular $T \vee S$ and $T \wedge S$ are compact subsets of X. The continuous function f takes each of the compact sets $T \wedge S$ and $T \vee S$ by a one-to-one correspondence onto $[0, 1]$. Hence, $T \wedge S$ and $T \vee S$ are arcs. We check that $T \wedge S$ and $T \vee S$ are chains with respect to $\leq$. Let $x, y \in T \wedge S$. We may suppose that $f(x) < f(y)$. If $x, y \in T$ then $x \leq y$ since T is a chain. Suppose, therefore, that $x \in S - T$ and $y \in T - S$. Let z be the maximal element of the compact chain

$$(T \wedge S) \cap S \cap f^{-1}([0, f(y)]).$$

Then $z \in T \cap S$ since $f^{-1}(f(z)) \cap (T \wedge S)=\{z\}$, $T \wedge S$ is compact and

$$(T \wedge S) \cap f^{-1}([f(z), f(y)]) \subset T.$$

Hence, $x \leq z \leq y$ and $T \wedge S$ is a chain with respect to $\leq$. Since $T \wedge S$ is a compact, connected chain which meets both $\text{Min}(X, \leq)$ and $\text{Max}(X, \leq)$ it follows that $T \wedge S \in \mathscr{M}(X, \leq)$. Similarly, $T \vee S \in \mathscr{M}(X, \leq)$.

For $T, S \in \mathscr{M}(X, \leq)$ define $T \leq^* S$ if and only if $T \wedge S = T$. Then $(\mathscr{M}(X, \leq), \leq^*)$ is a partially ordered space. We shall prove that $\leq^*$ is order dense. Let $S, T \in \mathscr{M}(X, \leq)$ such that $T \wedge S = S$ and $S \neq T$. Let $a \in [0, 1]$ such that $f^{-1}(a) \cap S \neq f^{-1}(a) \cap T$ and let $b \in f^{-1}(a)$ such that

$$f^{-1}(a) \cap S <'' b <'' f^{-1}(a) \cap T.$$

By hypothesis there exists $P \in \mathscr{M}(X, \leq)$ such that $b \in P$. Let $R=(S \vee P) \wedge T$. Then $b \in R \in \mathscr{M}(X, \leq)$. Notice that $T \wedge R = R$ and $R \wedge S = S$. Hence, $S <^* P <^* T$. Thus, $\leq^*$ is order dense. Since F is the unique minimal element in the partially ordered space $(\mathscr{M}(X, \leq), \leq^*)$, $\mathscr{M}(X, \leq^*)$ is connected by the remarks following Theorem C.

Case 2. Suppose $\text{Min}(X, \leq) \cap \text{Max}(X, \leq)$ is non-void. Make the disjoint union of X and $\text{Max}(X, \leq) \times [0, 1]$ into a partially ordered space by setting $x \leq' y$ in $X \cup (\text{Max}(X, \leq) \times [0, 1])$, if:

(a) $x, y \in X$ and $x \leq y$
(b) $x \in X$ and $y=(a, b) \in \text{Max}(X, \leq) \times [0, 1]$ where $x \leq a$ in X or
(c) $x=(a, b)$ and $y=(c, d)$ are in $\text{Max}(X, \leq) \times [0, 1]$, $a=c$ and $b \leq d$ in $[0, 1]$.

Form the adjunction space X' of X and $\text{Max}(X, \leq) \times [0, 1]$ by identifying (m, o) and m for each $m \in \text{Max}(X, \leq)$. The partial order $\leq'$ on $X \cup (\text{Max}(X, \leq) \times [0, 1]$ induces a partial order $\leq^\circ$ on X' such that $(X', \leq^\circ)$ satisfies the hypotheses of the theorem. Notice that $\text{Min}(X', \leq^\circ)=\text{Min}(X, \leq)$ and $\text{Max}(X', \leq^\circ)=\text{Max}(X, \leq) \times \{1\}$. By Case 1 $\mathscr{M}(X', \leq^\circ)$ is a compact, connected topological lattice. It is easy to see that $\mathscr{M}(X, \leq)$ is homeomorphic and isomorphic to $\mathscr{M}(X', \leq^\circ)$ under the correspondence that takes a member A of $\mathscr{M}(X', \leq^\circ)$ to the unique member B of $\mathscr{M}(X, \leq)$ such that $B \subset A$.

The following theorem was proved in [1].

THEOREM D. *Let* $(X, \leq)$ *be a partially ordered space. Suppose there exists a function* $h: [0, 1] \to \mathscr{M}(X)$ *such that* $X = \cup\{h(a) \mid a \in [0, 1]\}$ *and if* $a<b<c$ *in* $[0, 1]$ *then* $h(a) \cap h(c) \subset h(b)$. *If* X *is non-degenerate and has no cutpoints then* X *is a 2-cell.*

It is shown in the proof of Theorem D that if $\leq$ is a closed partial order on the 2-cell D then in order that there exist a function $h: [0, 1] \to \mathscr{M}(D)$ as in the above theorem it is necessary that $\text{Min}(D)$ and $\text{Max}(D)$ be closed, connected sets in the boundary of D and that $\mathscr{M}(D)$ cover D. We shall show that these conditions are also sufficient.

Corollary 2. *Let $\leq$ be a closed partial order on a 2-cell D such that $\mathscr{M}(D)$ covers D and* Min(D) *and* Max(D) *are closed and connected sets in the boundary of D. Then there exists a continuous function $h:[0,1]\to\mathscr{M}(D)$ such that $D=\cup\{h(a) \mid a\in[0,1]\}$ and if $a<b<c$ in* $[0,1]$ *then $h(a)\cap h(c)\subset h(b)$.*

Proof. $\mathscr{M}(D)$ is a compact topological lattice and thus $\mathscr{M}(D)$ has a zero F and a unit E. By Theorem 1 $\mathscr{M}(D)$ is connected. By Koch's Theorem (see [5]) there is an order arc $\mathbb{C}$ in $\mathscr{M}(D)$ such that $F, E\in\mathbb{C}$. Let $h:[0,1]\to\mathbb{C}$ be a one to one continuous function such that $h(0)=F$ and $h(1)=E$. From the definition of order in $\mathscr{M}(D)$ it is clear that if $a<b<c$ in $[0,1]$ then $h(a)\cap h(c)\subset h(b)$. It remains to show only that $D=\cup\{h(a) \mid a\in[0,1]\}$.

Let $x\in D$ and let $R\in\mathscr{A}(D)$ such that $x\in R$ by Theorem B. It is easy to see that for each $a\in[0,1]$, $h(a)\cap R$ consists of exactly one point. Define $g:[0,1]\to R$ by letting $g(a)\in h(a)\cap R$ for each $a\in[0,1]$. Then g is easily seen to be a continuous function. By the proof of Theorem 1 R is an arc with endpoints in F and E. Hence $g(0)\in F$ and $g(1)\in E$. Thus, g maps $[0,1]$ onto R and $x\in\cup\{h(a) \mid a\in[0,1]\}$.

Let $(D', \leq')$ be the unit square $[0,1]\times[0,1]$ in the plane with the partial order $(a,b)\leq'(c,d)$ if and only if $a=c$ and $b\leq d$ in $[1,1]$.

Corollary 3. *Let $(D,\leq)$ be a partially ordered space such that D is a 2-cell, $\mathscr{M}(D,\leq)$ covers D and* Min$(D,\leq)$ *and* Max$(D,\leq)$ *are closed disjoint arcs in the boundary of D. There is an order preserving continuous function g of $(D',\leq')$ onto $(D,\leq)$ such that*

(i) $g^{-1}(\mathrm{Max}(D,\leq))=\mathrm{Max}(D',\leq')$

(ii) $g^{-1}(\mathrm{Min}(D,\leq))=\mathrm{Min}(D',\leq')$

(iii) *g takes every member of $\mathscr{M}(D',\leq')$ homeomorphically onto a member of $\mathscr{M}(D,\leq)$.*

(iv) *for $x\in D$ $g^{-1}(x)$ is either a point or a horizontal line segment in D'.*

Proof. Let $f:D\to[0,1]$ be a function satisfying the conditions of Theorem C. Let $h:[0,1]\to\mathscr{M}(D,\leq)$ be a function satisfying the conditions of Corollary 2. Define $g:(D',\leq')\to(D,\leq)$ by letting $g(a,b)$ be the unique point in $h(a)\cap f^{-1}(b)$ for each $(a,b)\in D'=[0,1]\times[0,1]$.

If $(D,\leq)$ and g are as in Corollary 3, there is a smallest partial order $\leq^*$ on D such that g is order preserving with respect to $\leq^*$ and $\leq^*$ has a closed graph. Clearly $\leq^*\subset\leq$, $\mathscr{M}(D,\leq^*)$ covers D, $\mathrm{Min}(D,\leq^*)=\mathrm{Min}(D,\leq)$ and $\mathrm{Max}(D,\leq^*)=\mathrm{Max}(D,\leq)$. Thus we have extracted from $\leq$ a partial order $\leq^*$ which is moderately large and which is well understood since it is completely determined by the function g.

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