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On the Continuity of the Eigenvalues of a Sublaplacian

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Abstract. We study the behavior of the eigenvalues of a sublaplacian Δ_b on a compact strictly pseudoconvex CR manifold M, as functions on the set $\mathcal{P}_+$ of positively oriented contact forms on M by endowing $\mathcal{P}_+$ with a natural metric topology.

1 Introduction

Let M be a compact strictly pseudoconvex CR manifold, of CR dimension n, without boundary. Let $\mathcal{P}$ be the set of all C^∞ pseudohermitian structures on M. Every $\theta \in \mathcal{P}$ is a contact form on M, *i.e.*, $\theta \wedge (d\theta)^n$ is a volume form. Let $\mathcal{P}_\pm$ be the sets of $\theta \in \mathcal{P}$ such that the Levi form G_θ is positive definite (respectively, negative definite). For $\theta \in \mathcal{P}_+$, let Δ_b be the sublaplacian

$$\Delta_b u = -\operatorname{div}(\nabla^H u) \tag{1}$$

of (M, θ) acting on smooth real valued functions $u \in C^\infty(M, \mathbb{R})$. As Δ_b is a subelliptic operator (of order $1/2$) it has a discrete spectrum

$$0 = \lambda_0(\theta) < \lambda_1(\theta) \leq \lambda_2(\theta) \leq \cdots \uparrow +\infty$$

(the eigenvalues of Δ_b are counted with their multiplicities). Each eigenvalue $\lambda_\nu(\theta)$, $\nu = 0, 1, 2, \dots$, is thought of as a function of $\theta \in \mathcal{P}_+$. We shall deal mainly with the following problem: *Is there a natural topology on $\mathcal{P}_+$ such that each eigenvalue function $\lambda_\nu \colon \mathcal{P}_+ \to \mathbb{R}$ is continuous*? The analogous problem for the spectrum of the Laplace–Beltrami operator on a compact Riemannian manifold was solved by S. Bando and H. Urakawa [2], and our main result is imitative of their Theorem 2.2 (*cf.* [2, p. 155]). We shall establish the following.

Corollary 1 *For every compact strictly pseudoconvex CR manifold M, the space of positively oriented contact forms $\mathcal{P}_+$ admits a natural complete distance function $d \colon \mathcal{P}_+ \times \mathcal{P}_+ \to [0, +\infty)$ such that each eigenvalue function $\lambda_k \colon \mathcal{P}_+ \to \mathbb{R}$ is continuous relative to the d-topology.*

By a result of J. M. Lee [8], for every $\theta \in \mathcal{P}_+$ there is a Lorentzian metric $F_\theta \in \operatorname{Lor}\big(C(M)\big)$ (the Fefferman metric) on the total space $C(M)$ of the canonical circle

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bundle $S^1 \to C(M) \xrightarrow{\pi} M$. Also, if $\Box$ is the Laplace–Beltrami operator of F_θ (the wave operator), then $\sigma(\Delta_b) \subset \sigma(\Box)$. Therefore the eigenvalues λ_k may be thought of as functions $\lambda_k^{\uparrow} \colon \mathcal{C} \to \mathbb{R}$ on the set $\mathcal{C} = \{F_\theta \in \operatorname{Lor}(C(M)) : \theta \in \mathcal{P}_+\}$ of all Fefferman metrics on $C(M)$. On the other hand, $\operatorname{Lor}(C(M))$ may be endowed with the distance function d_g^∞ considered by P. Mounoud [10] (associated to a fixed Riemannian metric g on $C(M)$), and hence $(\mathcal{C}, d_g^\infty)$ is itself a metric space. It is then a natural question whether $\lambda_k^{\uparrow}$ are continuous functions relative to the d_g^∞-topology.

The paper is organized as follows. In Section 2, we recall the needed material on CR and pseudohermitian geometry. The distance function d (in Corollary 1) is built in Section 3. In Section 4, we establish a Max-Mini principle (*cf.* Proposition 2) for the eigenvalues of a sublaplacian. Then Corollary 1 follows from Theorem 1 in Section 5. In Section 6, we prove the continuity of the eigenvalues with respect to the Fefferman metric (*cf.* Corollary 2), though only as functions on $\mathcal{C}_+ = \{e^{u\circ\pi}F_{\theta_0} : u \in C^\infty(M,\mathbb{R}), u > 0\}$.

2 Review of CR and Pseudohermitian Geometry

Let $(M, T_{1,0}(M))$ be a CR manifold, of CR dimension n, where $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$ is its CR structure, *cf.*, *e.g.*, [5, pp. 3–4]. The *Levi distribution* is

$$H(M) = \Re\{T_{1,0}(M) \oplus \overline{T_{1,0}(M)}\}.$$

The Levi distribution carries the complex structure $J \colon H(M) \to H(M)$ given by $J(Z - \overline{Z}) = i(Z - \overline{Z})$ for any $Z \in T_{1,0}(M)$ (here $i = \sqrt{-1}$). A *pseudohermitian structure* is a globally defined nowhere zero section $\theta \in C^\infty(H(M)^\perp)$ in the conormal bundle $H(M)^\perp \subset T^*(M)$. Pseudohermitian structures do exist by the mere assumption that M be orientable. Let $\mathcal{P}$ be the set of all pseudohermitian structures on M. As $H(M)^\perp \to M$ is a real line bundle for any θ, $\theta_0 \in \mathcal{P}$ there is a C^∞ function $\lambda \colon M \to \mathbb{R} \setminus \{0\}$ such that $\theta = \lambda\theta_0$. Given $\theta \in \mathcal{P}$ the *Levi form* is $G_\theta(X,Y) = (d\theta)(X, JY)$ for every $X, Y \in \mathfrak{X}(M)$. Then $G_{\lambda\theta_0} = \lambda G_{\theta_0}$. The CR manifold M is *strictly pseudoconvex* if G_θ is positive definite (write $G_\theta > 0$) for some $\theta \in \mathcal{P}$. If M is strictly pseudoconvex then each $\theta \in \mathcal{P}$ is a contact form, *i.e.*, $\Psi_\theta = \theta \wedge (d\theta)^n$ is a volume form on M. Clearly, if G_θ is positive definite then $G_{-\theta}$ is negative definite. Hence $\mathcal{P}$ admits a natural orientation $\mathcal{P}_+$ ($G_\theta > 0$ for each $\theta \in \mathcal{P}_+$). Let M be a strictly pseudoconvex CR manifold and $\theta \in \mathcal{P}_+$. The *Reeb vector* field is the globally defined, nowhere zero, tangent vector field $T \in \mathfrak{X}(M)$, transverse to $H(M)$, determined by $\theta(T) = 1$ and $(d\theta)(T, X) = 0$ for any $X \in \mathfrak{X}(M)$ (*cf.* [5, Proposition 1.2, p. 8]). The *Webster metric* is the Riemannian metric g_θ on M given by

$$g_\theta(X,Y) = G_\theta(X,Y), \quad g_\theta(X,T) = 0, \; g_\theta(T,T) = 1,$$

for every $X, Y \in H(M)$. Let $S^1 \to C(M) \xrightarrow{\pi} M$ be the canonical circle bundle (*cf.* [5, Definition 2.9, p. 119]). For every $\theta \in \mathcal{P}_+$ there is a Lorentzian metric F_θ on $C(M)$ (the *Fefferman metric*, *cf.* [5, Definition 2.15, p. 128]) such that the set $\mathcal{C} = \{F_\theta : \theta \in \mathcal{P}_+\}$ of all Fefferman metrics is given by $\mathcal{C} = \{e^{u\circ\pi}F_\theta : u \in C^\infty(M,\mathbb{R})\}$ for each fixed contact form $\theta \in \mathcal{P}_+$ (by a result of Lee [8], or [5, Theorem 2.3, p. 128]). $\mathcal{C}$ is also referred to as the *restricted conformal class* of F_θ and it is a CR invariant.

If $u \in C^\infty(M,\mathbb{R})$ then the *horizontal gradient* $\nabla^H u \in C^\infty\big(H(M)\big)$ is given by $\nabla^H u = \Pi_H \nabla u$. Here $\Pi_H\colon T(M) \to H(M)$ is the projection relative to the decomposition $T(M) = H(M) \oplus \mathbb{R}T$ and ∇u is the gradient of u with respect to the Webster metric, *i.e.*, $g_\theta(\nabla u, X) = X(u)$ for any $X \in \mathfrak{X}(M)$. The divergence operator div: $\mathfrak{X}(M) \to C^\infty(M,\mathbb{R})$ is meant with respect to the volume form Ψ_θ, *i.e.*, $\mathcal{L}_X \Psi_\theta = \operatorname{div}(X)\Psi_\theta$ for any $X \in \mathfrak{X}(M)$. The *sublaplacian* Δ_b of (M,θ) is then the formally self-adjoint, second order, degenerate elliptic (in the sense of J. M. Bony [4]) operator given by $\Delta_b u = -\operatorname{div}(\nabla^H u)$ for any $u \in C^\infty(M,\mathbb{R})$. A systematic application of functional analysis methods to the study of sublaplacians (on domains in strictly pseudoconvex CR manifolds) was started in [3]. By a result following essentially from work in [9] (*cf.* also [12]), if M is compact, then Δ_b has a discrete spectrum $\sigma(\Delta_b) = \{\lambda_\nu : \nu \geq 0\}$ such that $\lambda_0 = 0$ and $\lambda_\nu \uparrow +\infty$ as $\nu \to \infty$.

3 A Topology on the Space of Oriented Contact Forms

Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a finite open covering of M such that the closure of each U_λ is contained in a larger open set V_λ which is both the domain of a local frame $\{X_a : 1 \leq a \leq 2n\} \subset C^\infty\big(V_\lambda, H(M)\big)$ with $X_{\alpha+n} = JX_\alpha$ for any $1 \leq \alpha \leq n$, and a coordinate neighborhood with the local coordinates $(x^1,\dots,x^{2n+1})$. For each point $x \in M$, let P_x (respectively S_x) be the set of all symmetric positive definite (respectively merely symmetric) bilinear forms on $T_x(M)$. Let us consider the anti-reflexive partial order relation on S_x defined by

$$\varphi < \psi \Longleftrightarrow \psi - \varphi \in P_x, \quad \varphi, \psi \in S_x.$$

Next let $\rho_x''\colon P_x \times P_x \to [0,+\infty)$ be the distance function given by

$$\rho_x''(\varphi,\psi) = \inf\{\delta > 0 : \exp(-\delta)\varphi < \psi < \exp(\delta)\varphi\}$$

for any $\varphi, \psi \in P_x$. Then (P_x, ρ_x'') is a complete metric space (by [2, Lemma 1.1 (iii), p. 158]).

Let $\mathcal{M}$ be the set of all Riemannian metrics on M, so that $g_\theta \in \mathcal{M}$ for every $\theta \in \mathcal{P}_+$. Following [2], one may endow $\mathcal{M}$ with a complete distance function ρ. Indeed, as M is compact, one may set

$$\rho''(g_1,g_2) = \sup_{x\in M} \rho_x''(g_{1,x}, g_{2,x}), \quad g_1, g_2 \in \mathcal{M}.$$

Also let $S(M)$ be the space of all C^∞ symmetric $(0,2)$-tensor fields on M, organized as a Fréchet space by the family of seminorms $\big\{\, |\cdot|_k : k \in \mathbb{N} \cup \{0\}\big\}$, where

$$|g|_k = \sum_{\lambda\in\Lambda} |g|_{\lambda,k}, \quad |g|_{\lambda,k} = \sup_{x\in\overline{U}_\lambda} \sum_{|\alpha|\leq k} |D^\alpha g_{ij}(x)|,$$

where

$$D^\alpha = \partial^{|\alpha|}/\partial(x^1)^{\alpha_1}\cdots\partial(x^{2n+1})^{\alpha_{2n+1}}, \quad g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j) \in C^\infty(V_\lambda,\mathbb{R}),$$

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for any $g \in S(M)$. The topology of $S(M)$ as a locally convex space is compatible to the distance function

$$\rho'(g_1, g_2) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|g_1 - g_2|_k}{1 + |g_1 - g_2|_k}, \quad g_1, g_2 \in S(M).$$

In particular $\big(S(M), \rho'\big)$ is a complete metric space. If

$$\rho(g_1, g_2) = \rho'(g_1, g_2) + \rho''(g_1, g_2)$$

then $(\mathcal{M}, \rho)$ is a complete metric space (*cf.* [2, Proposition 2, p. 158]). Each metric $g \in \mathcal{M}$ determines a Laplace–Beltrami operator Δ_g, hence the eigenvalues of Δ_g may be thought of as functions of g and as such the eigenvalues are (by [2, Theorem 2.2, p. 161]) continuous functions on $(\mathcal{M}, \rho)$. To deal with the similar problem for the spectrum of a sublaplacian, we start by observing that the natural counterpart of $\mathcal{M}$ in the category of strictly pseudoconvex CR manifolds is the set $\mathcal{M}_H$ of all sub-Riemannian metrics on $\big(M, H(M)\big)$. Nevertheless, only a particular sort of sub-Riemannian metric gives rise to a sublaplacian, *i.e.*, Δ_b is associated to $G_\theta \in \mathcal{M}_H$ for some positively-oriented contact form $\theta \in \mathcal{P}_+$. Of course $\mathcal{P}_+ \subset \Omega^1(M)$ and one may endow $\Omega^1(M)$ with the C^∞ topology. One may then attempt to repeat the arguments in [2] (by replacing $S(M)$ with $\Omega^1(M)$). The situation at hand is however much simpler since, once a contact form $\theta_0 \in \mathcal{P}_+$ is fixed, all others are parametrized by $C^\infty(M, \mathbb{R})$, *i.e.*, for any $\theta \in \mathcal{P}_+$ there is a unique $u \in C^\infty(M, \mathbb{R})$ such that $\theta = e^u \theta_0$. We may then use the canonical Fréchet space structure (and corresponding complete distance function) of $C^\infty(M, \mathbb{R})$. Precisely, for every $u \in C^\infty(M, \mathbb{R})$, $\lambda \in \Lambda$ and $k \in \mathbb{N} \cup \{0\}$ we set

$$p_{\lambda,k}(u) = \sup_{x \in \overline{U}_k} \sum_{|\alpha| \le k} |D^\alpha u(x)|,$$

$$p_k(u) = \sum_{\lambda \in \Lambda} p_{\lambda,k}(u), \quad |u|_{C^\infty} = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{p_k(u)}{1 + p_k(u)}.$$

If $\theta_0 \in \mathcal{P}_+$ is a fixed contact form then we set

$$d'(\theta_1, \theta_2) = |u_1 - u_2|_{C^\infty}, \quad \theta_1, \theta_2 \in \mathcal{P}_+,$$

where $u_i \in C^\infty(M, \mathbb{R})$ are given by $\theta_i = e^{u_i} \theta_0$ for any $i \in \{1, 2\}$. The definition of d' doesn't depend upon the choice of $\theta_0 \in \mathcal{P}_+$.

Lemma 1 *$(\mathcal{P}_+, d')$ is a complete metric space.*

Proof Let $\{\theta_\nu\}_{\nu \ge 1}$ be a Cauchy sequence in $(\mathcal{P}_+, d')$. If $u_\nu \in C^\infty(M, \mathbb{R})$ is the function determined by $\theta_\nu = e^{u_\nu} \theta_0$ then (by the very definition of d') $\{u_\nu\}_{\nu \ge 1}$ is a Cauchy sequence in $C^\infty(M, \mathbb{R})$. Here $C^\infty(M, \mathbb{R})$ is organized as a Fréchet space by the (countable, separating) family of seminorms $\big\{ p_k : k \in \mathbb{N} \cup \{0\} \big\}$. Hence there is

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$u \in C^\infty(M, \mathbb{R})$ such that $|u_\nu - u|_{C^\infty} \to 0$ as $\nu \to \infty$. Finally if $\theta = e^u \theta_0 \in \mathcal{P}_+$ then $d'(\theta_\nu, \theta) \to 0$ as $\nu \to \infty$. ∎

Let $S(H) \subset H(M)^* \otimes H(M)^*$ be the subbundle of all bilinear symmetric forms on $H(M)$. For every $G \in C^\infty\big(S(H)\big)$, $k \in \mathbb{Z}$, $k \geq 0$, and $\lambda \in \Lambda$ we set

$$|G|_{\lambda,k} = \sup_{x \in \overline{U}_\lambda} \sum_{|\alpha| \leq k} \sum_{a,b=1}^{2n} |D^\alpha G_{ab}(x)|,$$

$$|G|_k = \sum_{\lambda \in \Lambda} |G|_{\lambda,k}, \quad |G|_{C^\infty} = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|G|_k}{1 + |G|_k},$$

where $G_{ab} = G(X_a, X_b) \in C^\infty(V_\lambda, \mathbb{R})$. Moreover we set

$$\rho'_H(G_1, G_2) = |G_1 - G_2|_{C^\infty}, \quad G_1, G_2 \in C^\infty\big(S(H)\big).$$

Lemma 2 *$\big\{ |\cdot|_k : k \in \mathbb{N} \cup \{0\} \big\}$ is a countable separating family of seminorms organizing $\mathfrak{X} = C^\infty\big(S(H)\big)$ as a Fréchet space. In particular $(\mathfrak{X}, \rho'_H)$ is a complete metric space.*

Proof For each $k \in \mathbb{N} \cup \{0\}$ and $N \in \mathbb{N}$ we set

$$V(k, N) = \{G \in \mathfrak{X} : |G|_k < 1/N\}. \tag{2}$$

Let $\mathcal{B}$ be the collection of all finite intersections of sets (2). Then $\mathcal{B}$ is (*cf.*, *e.g.*, [11, Theorem 1.37, p. 27]) a convex balanced local base for a topology τ on $\mathfrak{X}$ that makes $\mathfrak{X}$ into a locally convex space such that every seminorm $|\cdot|_k$ is continuous and a set $E \subset \mathfrak{X}$ is bounded if and only if every $|\cdot|_k$ is bounded on E. The topology τ is compatible with the distance function ρ'_H. Let $\{G_m\}_{m \geq 1} \subset \mathfrak{X}$ be a Cauchy sequence relative to ρ'_H. Thus, for every fixed $k \in \mathbb{N} \cup \{0\}$ and $N \in \mathbb{N}$ one has $G_m - G_p \in V(k, N)$ for m, p sufficiently large. Consequently

$$|D^\alpha (G_m)_{ab}(x) - D^\alpha (G_p)_{ab}(x)| < 1/N,$$
$$x \in \overline{U}_\lambda, \ \lambda \in \Lambda, \ |\alpha| \leq k, \ 1 \leq a, b \leq 2n.$$

It follows that each sequence $\{D^\alpha (G_m)_{ab}\}_{m \geq 1}$ converges uniformly on $\overline{U}_\lambda$ to a function G^α_{ab}. In particular for $\alpha = \mathbf{0}$ one has $(G_m)_{ab}(x) \to G^{\mathbf{0}}_{ab}(x)$ as $m \to \infty$, uniformly in $x \in \overline{U}_\lambda$. If $\lambda, \lambda' \in \Lambda$ are such that $U_\lambda \cap U_{\lambda'} \neq \varnothing$ and

$$X'_b = A^a_b X_a, \quad A \equiv [A^a_b] \colon U_\lambda \cap U_{\lambda'} \to \mathrm{GL}(2n, \mathbb{R}),$$

is a local transformation of the frame in $H(M)$ then

$$(G_m)'_{ab} = A^c_a A^d_b (G_m)_{cd} \quad \text{on } U_\lambda \cap U_{\lambda'}$$

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so that (for $m \to \infty$) $G'^{\mathbf{0}}_{ab} = A^c_a A^d_b G^{\mathbf{0}}_{cd}$ on $U_\lambda \cap U_{\lambda'}$. Thus $G^{\mathbf{0}}_{ab} \in C^\infty(U_\lambda)$ glue up to a (globally defined) bilinear symmetric form $G^{\mathbf{0}}$ on $H(M)$ and $G_m \to G^{\mathbf{0}}$ in $\mathfrak{X}$ as $m \to \infty$. ∎

For each point $x \in M$, let $P(H)_x$ be the set of all symmetric positive definite bilinear forms on $H(M)_x$. We endow $S(H)_x$ with the anti-reflexive partial order relation

$$\varphi < \psi \Longleftrightarrow \psi - \varphi \in P(H)_x, \quad \varphi, \psi \in S(H)_x.$$

Next let $\rho''_x \colon P(H)_x \times P(H)_x \to [0, +\infty)$ be given by

$$\rho''_x(\varphi, \psi) = \inf\{\delta > 0 : \exp(-\delta)\varphi < \psi < \exp(\delta)\varphi\}$$

for any $\varphi, \psi \in P(H)_x$.

Lemma 3 *ρ''_x is a distance function on $P(H)_x$.*

Proof As $e^{-\delta}\varphi < \psi < e^{\delta}\varphi$ is equivalent to $e^{-\delta}\psi < \varphi < e^{\delta}\psi$, it follows that ρ''_x is symmetric. To prove the triangle inequality we assume that $\rho''_x(\varphi, \psi) > \rho''_x(\varphi, \chi) + \rho''(\chi, \psi)$ for some $\varphi, \psi, \chi \in P(H)_x$. Then

$$\rho''_x(\varphi, \psi) - \rho''_x(\varphi, \chi) > \inf\{\delta > 0 : \exp(-\delta)\chi < \psi < \exp(\delta)\chi\},$$

hence there is $\delta_2 > 0$ such that $e^{-\delta_2}\chi < \psi < e^{\delta_2}\chi$ and $\rho''_x(\varphi, \psi) - \rho''_x(\varphi, \chi) > \delta_2$. Similarly,

$$\rho''_x(\varphi, \psi) - \delta_2 > \inf\{\delta > 0 : \exp(-\delta)\varphi < \chi < \exp(\delta)\varphi\}$$

yields the existence of a number $\delta_1 > 0$ such that $e^{-\delta_1}\varphi < \chi < e^{\delta_1}\varphi$ and $\rho''_x(\varphi, \psi) - \delta_2 > \delta_1$. Let us set $\delta \equiv \delta_1 + \delta_2$. The inequalities written so far show that $e^{-\delta}\varphi < \psi < e^{\delta}\varphi$ and $\rho''_x(\varphi, \psi) > \delta$, a contradiction. Finally, let us assume that $\rho''_x(\varphi, \psi) = 0$, so that for any $k \in \mathbb{N}$,

$$\inf\{\delta > 0 : \exp(-\delta)\varphi < \psi < \exp(\delta)\varphi\} < 1/k$$

i.e., there is $\delta_k > 0$ such that $e^{-\delta_k}\varphi < \psi < e^{\delta_k}\varphi$ and $\delta_k < 1/k$. Thus $\lim_{k\to\infty} \delta_k = 0$ and $\psi - e^{-\delta_k}\varphi \in P(H)_x$ shows (by passing to the limit with $k \to \infty$ in $\psi(v, v) - e^{-\delta_k}\varphi(v, v) > 0$, $v \in H(M)_x \setminus \{0\}$) that $\varphi < \psi$. Similarly $e^{\delta_k}\varphi - \psi \in P(H)_x$ yields $\psi < \varphi$ in the limit, and we may conclude that $\varphi = \psi$. Vice versa, if $\varphi \in P(H)_x$ then

$$\{\delta > 0 : (1 - e^{-\delta})\varphi, (e^{\delta} - 1)\varphi \in P(H)_x\} = (0, +\infty),$$

hence $\rho''_x(\varphi, \varphi) = 0$. ∎

Lemma 4

(i) $\big(P(H)_x, \rho''_x\big)$ *is a complete metric space.*

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(ii) *Let* $\{\varphi_j\}_{j\in\mathbb{N}} \subset P(H)_x$ *such that* $\lim_{j\to\infty}\varphi_j = \varphi \in P(H)_x$ *in the* ρ''_x*-topology. Then* $\lim_{j\to\infty}\varphi_j(v,w) = \varphi(v,w)$ *for any* $v, w \in H(M)_x$.

Proof (i) Let $\{\varphi_j\}_{j\in\mathbb{N}} \subset P(H)_x$ be a Cauchy sequence in the ρ''_x-topology, *i.e.*, for any $\epsilon > 0$ there is $j_\epsilon \in \mathbb{N}$ such that $\rho''_x(\varphi_{j+p}, \varphi_j) > \epsilon$ for any $j \geq j_\epsilon$ and any $p = 1, 2, \dots$. Hence there is $\delta_\epsilon > 0$ such that $e^{-\delta_\epsilon}\varphi_j < \varphi_{j+p} < e^{\delta_\epsilon}\varphi_j$ and $\delta_\epsilon < \epsilon$. Consequently

$$|\log\varphi_{j+p}(v,v) - \log\varphi_j(v,v)| < \delta_\epsilon < \epsilon$$

for any $v \in H(M)_x \setminus \{0\}$. Therefore if

$$\xi_j \equiv \big(\log\varphi_j(v,v), \dots, \log\varphi_j(v,v)\big) \in \mathbb{R}^{2n}$$

then $\{\xi_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}^{2n}$. Let then $\xi = \lim_{j\to\infty}\xi_j$ and let

$$\varphi\colon H(M)_x \times H(M)_x \to \mathbb{R}$$

be the bilinear form given by $\varphi(v,v) = \exp(\xi^a)$ for any $v \in H(M)_x \setminus \{0\}$ followed by polarization. Here $\xi = (\xi^1, \dots, \xi^{2n})$. Then $\varphi \in P(H)_x$ and $\lim_{j\to\infty}\varphi_j = \varphi$ in the ρ''_x-topology.

(ii) If $\varphi_j \to \varphi$ as $j \to \infty$ then $\log\varphi_j(v,v) \to \log\varphi(v,v)$ as $j \to \infty$, for any $v \in H(M)_x \setminus \{0\}$. Then $\lim_{j\to\infty}\varphi_j(v,v) = \varphi(v,v)$ uniformly in v and statement (ii) follows by polarization. ∎

As M is compact we may set

$$\rho''_H(G_1, G_2) = \sup_{x\in M}\rho''_x(G_{1,x}, G_{2,x}),$$

$$\rho_H(G_1, G_2) = \rho'_H(G_1, G_2) + \rho''_H(G_1, G_2), \quad G_1, G_2 \in \mathcal{M}_H.$$

Also let d be the distance function on $\mathcal{P}_+$ given by

$$d(\theta_1, \theta_2) = d'(\theta_1, \theta_2) + \rho''_H(G_{\theta_1}, G_{\theta_2}), \quad \theta_1, \theta_2 \in \mathcal{P}_+.$$

Proposition 1

(i) $(\mathcal{M}_H, \rho_H)$ *is a complete metric space.*

(ii) *The map* $\theta \in \mathcal{P}_+ \mapsto G_\theta \in \mathcal{M}_H$ *of* $(\mathcal{P}_+, d)$ *into* $(\mathcal{M}_H, \rho_H)$ *is continuous.*

(iii) $(\mathcal{P}_+, d)$ *is a complete metric space.*

Proof (i) Let $\{G_j\}_{j\geq 1}$ be a Cauchy sequence in $(\mathcal{M}_H, \rho_H)$. Then $\{G_j\}_{j\geq 1}$ is a Cauchy sequence in both $(\mathfrak{X}, \rho'_H)$ and $(\mathcal{M}_H, \rho''_H)$. Yet $(\mathfrak{X}, \rho'_H)$ is complete (by Lemma 2). Thus $\rho'_H(G_j, G) \to 0$ as $j \to \infty$ for some $G \in \mathfrak{X}$. In particular

$$\lim_{j\to\infty} G_{j,x}(v,w) = G_x(v,w)$$

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for every $x \in M$ and $v, w \in H(M)_x$. On the other hand, as $\{G_j\}_{j\geq 1}$ is Cauchy in $(\mathcal{M}_H, \rho_H'')$, for every $\epsilon > 0$ there is $N_\epsilon \geq 1$ such that

$$\rho_x''(G_{i,x}, G_{j,x}) \leq \rho_H''(G_i, G_j) < \epsilon$$

for every $i, j \geq N_\epsilon$ and $x \in M$. Thus $\{G_{j,x}\}_{j\geq 1}$ is Cauchy in the complete (by Lemma 4) metric space $\big(P(H)_x, \rho_x''\big)$ so that $\rho_x''(G_{j,x}, \varphi) \to 0$ as $j \to \infty$ for some $\varphi \in P(H)_x$. Then (by (iii) in Lemma 4) $\lim_{j\to\infty} G_{j,x}(v, w) = \varphi(v, w)$ for every $v, w \in H(M)_x$, hence $G_x = \varphi$, yielding $G \in \mathcal{M}_H$.

(ii) Let $\{\theta_\nu\}_{\nu\geq 1} \subset \mathcal{P}_+$ such that $d(\theta_\nu, \theta) \to 0$ for $\nu \to \infty$ for some $\theta \in \mathcal{P}_+$. If $\theta_\nu = e^{u_\nu}\theta_0$ and $\theta = e^{u}\theta_0$, then $|u_\nu - u|_{C^\infty} \to 0$ as $\nu \to \infty$. Then $G_{\theta_\nu} = e^{u_\nu} G_{\theta_0}$ and $G_\theta = e^u G_{\theta_0}$. Since $D^\alpha u_\nu \to D^\alpha u$ as $\nu \to \infty$, uniformly on $\overline{U}_\lambda$, for any $\lambda \in \Lambda$, $|\alpha| \leq k$, and $k \in \mathbb{N} \cup \{0\}$, it follows that $D^\alpha (G_{\theta_\nu})_{ab} \to D^\alpha (G_\theta)_{ab}$ as $\nu \to \infty$, uniformly on $\overline{U}_\lambda$ for any $1 \leq a, b \leq 2n$. Hence $G_{\theta_\nu} \to G_\theta$ in $\mathfrak{X}$ so that (by the very definition of d and ρ_H) $\rho_H(G_{\theta_\nu}, G_\theta) \to 0$.

(iii) If $\{\theta_\nu\}_{\nu\geq 1}$ is a Cauchy sequence in $(\mathcal{P}_+, d)$ then $\{u_\nu\}_{\nu\geq 1}$ is Cauchy in $(\mathcal{P}_+, d')$ as well. Yet (by Lemma 1) $(\mathcal{P}_+, d')$ is complete, hence $d'(\theta_\nu, \theta) \to 0$ for some $\theta \in \mathcal{P}_+$. Then, as a byproduct of the proof of statement (ii), one has $G_{\theta_\nu} \to G_\theta$ in $\mathfrak{X}$. Finally, verbatim repetition of the arguments in the proof of statement (i) yields $\rho_H''(G_{\theta_\nu}, G_\theta) \to 0$ so that $d(\theta_\nu, \theta) \to 0$. ■

4 A Max-Mini Principle

For each $k \in \mathbb{N} \cup \{0\}$ we consider a $(k + 1)$-dimensional real subspace $L_{k+1} \subset C^\infty(M, \mathbb{R})$ and set

$$\Lambda_\theta(L_{k+1}) = \sup\Big\{ \frac{\|\nabla^H f\|_{L^2}^2}{\|f\|_{L^2}^2} : f \in L_{k+1} \setminus \{0\} \Big\}.$$

Here

$$\|f\|_{L^2} = \Big(\int_M f^2 \Psi_\theta \Big)^{\frac{1}{2}}, \quad \|X\|_{L^2} = \Big(\int_M g_\theta(X, X) \Psi_\theta \Big)^{\frac{1}{2}},$$

for any $f \in C^\infty(M, \mathbb{R})$ and any $X \in \mathfrak{X}(M)$. Let $\{u_\nu\}_{\nu\geq 0} \subset C^\infty(M, \mathbb{R})$ be a complete orthonormal system relative to the L^2 inner product $(f, g)_{L^2} = \int_M fg\Psi_\theta$ such that $u_\nu \in \operatorname{Eigen}\big(\Delta_b; \lambda_\nu(\theta)\big)$ for every $\nu \geq 0$. If $f \in C^\infty(M, \mathbb{R})$ then $f = \sum_{\nu=0}^\infty a_\nu(f) u_\nu$ (L^2 convergence) for some $a_\nu(f) \in \mathbb{R}$. Let L_{k+1}^0 be the subspace of $C^\infty(M, \mathbb{R})$ spanned by $\{u_\nu : 0 \leq \nu \leq k\}$. Let $(\nabla^H)^*$ be the formal adjoint of ∇^H, *i.e.*,

$$(\nabla^H f, X)_{L^2} = \big(f, (\nabla^H)^* X \big)_{L^2}$$

for any $f \in C^\infty(M, \mathbb{R})$ and $X \in C^\infty\big(H(M)\big)$. Mere integration by parts shows that

$$(\nabla^H)^* X = -\operatorname{div}(X), \quad X \in C^\infty\big(H(M)\big),$$

implying, by (1), the useful identity

$$\|\nabla^H f\|_{L^2}^2 = (f, \Delta_b f)_{L^2}, \quad f \in C^\infty(M, \mathbb{R}). \tag{3}$$

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Let $f \in L^0_{k+1} \setminus \{0\}$ so that $f = \sum_{\nu=0}^{k} a_\nu u_\nu$ for some $a_\nu \in \mathbb{R}$. Then, by (3),

$$\|\nabla^H f\|^2_{L^2} = \sum_{\nu=0}^{k} a_\nu^2 \lambda_\nu(\theta) \le \lambda_k(\theta) \sum_{\nu=0}^{k} a_\nu^2 = \lambda_k(\theta)\|f\|^2_{L^2}$$

hence

$$\Lambda_\theta(L^0_{k+1}) \le \lambda_k(\theta). \tag{4}$$

Our purpose in this section is to establish the following.

Proposition 2 *Let M be a compact strictly pseudoconvex CR manifold and $\theta \in \mathcal{P}_+$ a positively oriented contact form. Then*

$$\lambda_k(\theta) = \inf_{L_{k+1}} \Lambda_\theta(L_{k+1})$$

where the infimum is taken over all subspaces $L_{k+1} \subset C^\infty(M,\mathbb{R})$ with $\dim_{\mathbb{R}} L_{k+1} = k+1$.

So far, by (4), $\lambda_k(\theta) \ge \Lambda_\theta(L^0_{k+1}) \ge \inf_{L_{k+1}} \Lambda_\theta(L_{k+1})$. The proof of Proposition 2 is by contradiction. We assume that $\lambda_k(\theta) > \inf_{L_{k+1}} \Lambda_\theta(L_{k+1})$, *i.e.*, there is a $(k+1)$-dimensional subspace $L_{k+1} \subset C^\infty(M,\mathbb{R})$ such that $\Lambda_\theta(L_{k+1}) < \lambda_k(\theta)$. Then $\Lambda_\theta(L_{k+1})$ is finite and

$$\|f\|^2_{L^2}\Lambda_\theta(L_{k+1}) \ge \|\nabla^H f\|^2_{L^2}, \quad f \in L_{k+1}.$$

Then, by (3),

$$\sum_{\nu=0}^{\infty} a_\nu(f)^2 \Lambda_\theta(L_{k+1}) \ge \sum_{\nu=0}^{\infty} \lambda_\nu(\theta) a_\nu(f)^2,$$

so that

$$\sum_{\Lambda_\theta(L_{k+1}) \ge \Lambda_\nu(\theta)} a_\nu(f)^2[\Lambda_\theta(L_{k+1}) - \lambda_\nu(\theta)] \ge \sum_{\Lambda_\theta(L_{k+1}) < \lambda_\nu(\theta)} a_\nu(f)^2[\lambda_\nu(\theta) - \Lambda_\theta(L_{k+1})]. \tag{5}$$

Let $\Phi\colon L_{k+1} \to C^\infty(M,\mathbb{R})$ be the linear map given by

$$\Phi(f) = \sum_{\nu=0}^{m} a_\nu(f) u_\nu, \quad f \in L_{k+1},$$

where $m = \max\{\nu \ge 0 : \lambda_\nu(\theta) \le \Lambda_\theta(L_{k+1})\}$. Note that $0 \le m \le k-1$ (by the contradiction assumption). We claim that

$$\operatorname{Ker}(\Phi) \ne (0). \tag{6}$$

Of course (6) is only true within the contradiction loop. The statement follows from $\dim_{\mathbb{R}} \Phi(L_{k+1}) \le m+1 \le k < k+1$ (hence Φ cannot be injective). Using (6), let $f_0 \in L_{k+1}$ such that $\Phi(f_0) = 0$ and $f_0 \ne 0$. Then $a_\nu(f_0) = 0$ for any $0 \le \nu \le m$, *i.e.*, whenever $\Lambda_\theta(L_{k+1}) \ge \lambda_\nu(\theta)$. Applying (5) to $f = f_0$ yields $a_\nu(f_0) = 0$ whenever $\Lambda_\theta(L_{k+1}) < \lambda_\nu(\theta)$. Thus $f_0 = 0$, a contradiction.

5 Continuity of Eigenvalues

The scope of this section is to establish the following.

Theorem 1 *Let M be a compact strictly pseudoconvex CR manifold. If $\delta > 0$ and $\theta, \hat{\theta} \in \mathcal{P}_+$ are two contact forms on M such that $d(\theta, \hat{\theta}) < \delta$ then $e^{-\delta}\lambda_k(\theta) \le \lambda_k(\hat{\theta}) \le e^{\delta}\lambda_k(\theta)$ for any $k \ge 0$.*

Proof For any $x \in M$

$$\delta > \inf\{\epsilon > 0 : e^{-\epsilon}G_{\theta,x} < G_{\hat{\theta},x} < e^{\epsilon}G_{\theta,x}\}$$

i.e., there is $0 < \epsilon < \delta$ such that $G_{\hat{\theta},x} - e^{-\epsilon}G_{\theta,x} \in P(H)_x$ and $e^{\epsilon}G_{\theta,x} - G_{\hat{\theta},x} \in P(H)_x$. There is a unique $u \in C^\infty(M, \mathbb{R})$ such that $\hat{\theta} = e^u\theta$. Consequently

$$\hat{\theta} \wedge (d\hat{\theta})^n = e^{(n+1)u}\theta \wedge (d\theta)^n. \tag{7}$$

On the other hand $e^{-\delta}G_{\theta,x}(v,v) < G_{\hat{\theta},x}(v,v) < e^{\delta}G_{\theta,x}(v,v)$ for any $v \in H(M)_x \setminus \{0\}$ implies $|u| < \delta$. Then for every $f \in C^\infty(M)$, by (7),

$$e^{-(n+1)\delta}\int_M f^2\Psi_\theta \le \int_M f^2\Psi_{\hat{\theta}} \le e^{(n+1)\delta}\int_M f^2\Psi_\theta. \tag{8}$$

Moreover,

$$\hat{\nabla}^H f = e^{-u}\nabla^H f, \tag{9}$$

where $\hat{\nabla}^H f$ is the horizontal gradient of f with respect to $\hat{\theta}$. Thus, by (9), $\|\hat{\nabla}^H f\|^2_{\hat{\theta}} = e^{-u}\|\nabla^H f\|^2_\theta < e^{\delta}\|\nabla^H f\|^2_\theta$ so that, by (7),

$$e^{-(n+2)\delta}\int_M \|\nabla^H f\|^2_\theta\Psi_\theta \le \int_M \|\hat{\nabla}^H f\|^2_{\hat{\theta}}\Psi_{\hat{\theta}} \le e^{(n+2)\delta}\int_M \|\nabla^H f\|^2_\theta\Psi_\theta.$$

Finally, by (8)–(9),

$$e^{-\delta}\frac{\|\nabla^H f\|^2_{L^2}}{\|f\|^2_{L^2}} \le \frac{\int_M \|\hat{\nabla}^h f\|^2_{\hat{\theta}}\Psi_{\hat{\theta}}}{\int_M f^2\Psi_{\hat{\theta}}} \le e^{\delta}\frac{\|\nabla^H f\|^2_{L^2}}{\|f\|^2_{L^2}},$$

so that (by the Max-Mini principle)

$$e^{-\delta}\lambda_k(\theta) \le \lambda_k(\hat{\theta}) \le e^{\delta}\lambda_k(\theta). \tag{10}$$

Theorem 1 is proved. Corollary 1 follows from (10).

6 Spectra of Δ_b and $\Box$

Let F_θ be the Fefferman metric of (M,θ) and $\Box$ the corresponding wave operator (the Laplace–Beltrami operator of $\big(C(M),F_\theta\big)$). We set $\mathfrak{M}=C(M)$ for simplicity. Let g be a fixed Riemannian metric on $\mathfrak{M}$. The space $S(\mathfrak{M})$ of all symmetric tensor fields may be identified with the space of all fields of endomorphisms of $T(\mathfrak{M})$ which are symmetric with respect to g, *i.e.*, for each $h\in S(\mathfrak{M})$ let $\tilde h\in C^\infty\big(\mathrm{End}\big(T(\mathfrak{M})\big)\big)$ be given by

$$g(\tilde hX,Y)=h(X,Y),\quad X,Y\in\mathfrak{X}(\mathfrak{M}).$$

From now on we assume that M is compact. Then $\mathfrak{M}$ is compact as well (as $\mathfrak{M}$ is the total space of a principal bundle with compact base and compact fibres) and we endow $S(\mathfrak{M})$ with the distance function

$$d_g^\infty(h_1,h_2)=\sup_{z\in\mathfrak{M}}[\mathrm{trace}(\varphi_z^2)]^{1/2},\quad h_1,h_2\in S(\mathfrak{M}),$$

where $\varphi=\tilde h_1-\tilde h_2$ and $\varphi_z^2=\varphi_z\circ\varphi_z$. The set $\mathrm{Lor}(\mathfrak{M})$ of all Lorentz metrics on $\mathfrak{M}$ is an open set of $\big(S(\mathfrak{M}),d_g^\infty\big)$ and for any pair g_1,g_2 of Riemannian metrics on $\mathfrak{M}$ the distance functions d_{g_1} and d_{g_2} are uniformly equivalent (*cf.*, *e.g.*, [10, p. 49]). We shall use the topology induced by d_g^∞ on $\mathrm{Lor}(\mathfrak{M})$ (and therefore on $\mathcal{C}\subset\mathrm{Lor}(\mathfrak{M})$). By a result of [8], the sublaplacian Δ_b of (M,θ) is the pushforward of the wave operator, *i.e.*, $\pi_*\Box=\Delta_b$. In particular $\sigma(\Delta_b)\subset\sigma(\Box)$. Thus each $\lambda_k\colon\mathcal{P}_+\to\mathbb{R}$ may be thought of as a function $\lambda_k^\uparrow\colon\mathcal{C}\to\mathbb{R}$ such that $\lambda_k^\uparrow\circ F=\lambda_k$ for every $k\geq 0$, where $F\colon\mathcal{P}_+\to\mathcal{C}$ is the map given by $F(\theta)=F_\theta$ for every $\theta\in\mathcal{P}_+$. As another consequence of Theorem 1 we establish the following.

Corollary 2 *Let M be a compact strictly pseudoconvex CR manifold and let g be an arbitrary Riemannian metric on $\mathfrak{M}=C(M)$. Let $\theta_0\in\mathcal{P}_+$ be a fixed contact form and $\mathcal{P}_{++}=\{e^u\theta_0: u\in C^\infty(M,\mathbb{R}),u>0\}$. If $\mathcal{C}_+=\{F_\theta:\theta\in\mathcal{P}_{++}\}$ then for every $k\in\mathbb{N}\cup\{0\}$ the function $\lambda_k^\uparrow\colon\mathcal{C}_+\to\mathbb{R}$ is continuous relative to the d_g^∞-topology.*

Proof Let $\theta_i\in\mathcal{P}_+$, $i\in\{1,2\}$, and let us set $\varphi=\tilde F_{\theta_1}-\tilde F_{\theta_2}$. Let $\{E_p:1\leq p\leq 2n+2\}$ be a local g-orthonormal frame on $T(\mathfrak{M})$, defined on the open set $\mathcal{U}\subset\mathfrak{M}$. Then

$$\mathrm{trace}(\varphi^2)=\sum_{p=1}^{2n+2}g(\varphi^2E_p,E_p)=\sum_p\{F_{\theta_1}(\varphi E_p,E_p)-F_{\theta_2}(\varphi E_p,E_p)\}$$

on $\mathcal{U}$. On the other hand if $\varphi E_p=\varphi_p^qE_q$ then $\varphi_p^q=F(\theta_1)(E_p,E_q)-F(\theta_2)(E_p,E_q)$ hence

$$\mathrm{trace}(\varphi^2)=(e^{u_1\circ\pi}-e^{u_2\circ\pi})^2\|F_{\theta_0}\|_g^2,\tag{11}$$

where $u_i\in C^\infty(M,\mathbb{R})$ is given by $\theta_i=e^{u_i}\theta_0$ and $\|F_{\theta_0}\|_g$ is the norm of F_{θ_0} as a $(0,2)$-tensor field on $\mathfrak{M}$ with respect to g. Then, by (11),

$$d_g^\infty(F_{\theta_1},F_{\theta_2})=\sup_{\mathfrak{M}}|e^{u_1\circ\pi}-e^{u_2\circ\pi}|\,\|F_{\theta_0}\|_g.$$

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As $\mathfrak{M}$ is compact, $a = \inf_{z \in \mathfrak{M}} \|F_{\theta_0}\|_{g,z} > 0$. Indeed, by compactness, $a = \|F_{\theta_0}\|_{g,z_0}$ for some $z_0 \in \mathfrak{M}$. If $a = 0$ then $F_{\theta_0, z_0} = 0$, a contradiction (as F_{θ_0} is Lorentzian, and hence nondegenerate). Let $\epsilon > 0$ such that $d_g^\infty(F_{\theta_1}, F_{\theta_2}) < \epsilon$. Then $|e^{u_1} - e^{u_2}| < \epsilon/a$ everywhere on M. As both $u_1 > 0$ and $u_2 > 0$ it follows that $|u_1 - u_2| < \log(1+\epsilon/a)$. Indeed $e^{u_1} - e^{u_2} < \epsilon/a$ is equivalent to $e^{u_1 - u_2} < 1 + (\epsilon/a)e^{-u_2}$ hence (as $u_2 > 0$)

$$u_1 - u_2 < \log[1 + (\epsilon/a)e^{-u_2}] < \log(1 + \epsilon/a).$$

Therefore

$$(1 + \epsilon/a)^{-1} G_{\theta_1, x}(v, v) < G_{\theta_2, x}(v, v) < (1 + \epsilon/a) G_{\theta_1, x}(v, v)$$

for any $v \in H(M)_x \setminus \{0\}$ and any $x \in M$. Consequently $\rho_H''(G_{\theta_1}, G_{\theta_2}) < \log(1+\epsilon/a)$. The arguments in Section 5 then yield

$$(1 + \epsilon/a)^{-1} \lambda_k^\uparrow(F_{\theta_1}) \le \lambda_k^\uparrow(F_{\theta_2}) \le (1 + \epsilon/a) \lambda_k^\uparrow(F_{\theta_1})$$

and Corollary 2 follows. The problem of the behavior of $\lambda_k^\uparrow : \mathcal{C} \to \mathbb{R}$ is open. So does the more general problem of the behavior of the spectrum of the wave operator on $\mathfrak{M}$ with respect to a change of $F \in \mathrm{Lor}(\mathfrak{M})$. Further work (*cf.* [1]) on the behavior of $\sigma(\Delta_b)$ under analytic 1-parameter deformations $\{\theta(t)\}_{t \in \mathbb{R}}$ of a given contact form $\theta_0 \in \mathcal{P}_+$ builds on the Riemannian counterpart in [6] and the functional analysis results in [7].

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