

On the Waring–Goldbach Problem: Exceptional Sets for Sums of Cubes and Higher Powers

Angel V. Kumchev

Abstract. We investigate exceptional sets in the Waring–Goldbach problem. For example, in the cubic case, we show that all but $O(N^{79/84+\epsilon})$ integers subject to the necessary local conditions can be represented as the sum of five cubes of primes. Furthermore, we develop a new device that leads easily to similar estimates for exceptional sets for sums of fourth and higher powers of primes.

1 Introduction

The Waring–Goldbach problem is concerned with the solvability of the equation

$$(1.1) \quad p_1^k + \cdots + p_s^k = n,$$

where p_1, \dots, p_s are prime unknowns. It is conjectured that for any pair of integers $k, s \in \mathbb{N}$ with $s \geq k + 1$, there exist a fixed modulus $q_{k,s}$ and a collection $\mathcal{N}_{k,s}$ of congruence classes mod $q_{k,s}$ such that (1.1) is solvable for all sufficiently large $n \in \mathcal{N}_{k,s}$. While a proof of this conjecture appears to be beyond the reach of present methods, some significant progress has been made. Let $H(k)$ denote the least s for which a set of integers $\mathcal{N}_{k,s}$ as above exists. The first breakthrough came in 1937 when I. M. Vinogradov [23] developed a new method for estimating sums over primes and used it to solve the ternary Goldbach problem for sufficiently large n , that is, he proved that $H(1) \leq 3$. Shortly thereafter, Hua [8] showed that

$$(1.2) \quad H(k) \leq 2^k + 1 \quad \text{for all } k \geq 1,$$

which is the best result to date for $k \leq 3$. On the other hand, when $k \geq 4$, (1.2) has been improved on. If k is large, an approach based on I. M. Vinogradov’s mean-value theorem gives the best results. In particular, using this approach, Hua [9, Theorem 14] showed that

$$H(k) \leq 2k(2 \log k + \log \log k + O(1)).$$

For smaller $k \geq 4$, the sharpest bounds for $H(k)$ have been obtained by variants of Davenport’s iterative method. Thanigasalam [19] obtained

$$H(6) \leq 33, \quad H(7) \leq 47, \quad H(8) \leq 63, \quad H(9) \leq 83, \quad \text{and} \quad H(10) \leq 107,$$

Received by the editors April 1, 2003; revised December 14, 2003.

The author was partially supported by NSERC Grant A5123.

AMS subject classification: Primary 11P32; secondary: 11L15 11L20 11N36 11P55.

©Canadian Mathematical Society 2005.

and Kawada and Wooley [10] proved recently that

$$H(4) \leq 14 \quad \text{and} \quad H(5) \leq 21.$$

One can reduce further the number of variables needed to solve (1.1) by trying to represent *almost all* $n \in \mathcal{N}_{k,s}$ instead of all but finitely many such n . Let $E_{k,s}(x)$ be the number of $n \in \mathcal{N}_{k,s} \cap (1, x]$ for which (1.1) cannot be solved in primes p_1, \dots, p_s . Exploiting the nature of the proofs of the above bounds for $H(k)$, one can show that along with each estimate of the form $H(k) \leq s_0(k)$ one also has

$$(1.3) \quad E_{k,s}(x) \ll x(\log x)^{-A}$$

for any fixed $A > 0$ and for any $s \geq \frac{1}{2}s_0(k)$. In this paper, we pursue improvements on the right side of (1.3) for $s < s_0(k)$. The first such improvement was obtained by Vaughan [21] who showed that

$$E_{1,2}(x) \ll x \exp(-c\sqrt{\log x})$$

for some constant $c > 0$. Shortly afterward, Montgomery and Vaughan [17] proved that there exists an absolute constant $\theta < 1$ such that

$$E_{1,2}(x) \ll x^\theta,$$

and several authors used their method to give such estimates with explicit values of θ , the most recent result being $\theta = 0.914$ due to Li [13]. The first to obtain a similar estimate for an exceptional set for sums of squares or higher powers of primes were Leung and Liu [12], who showed that

$$(1.4) \quad E_{2,3}(x) \ll x^\theta,$$

with an absolute constant $\theta < 1$. Later, Bauer, Liu and Zhan [2, 15, 16] obtained a series of refinements of this estimate, the most recent being given by Liu and Zhan [16]. They establish (1.4) for every fixed θ in the interval $11/12 < \theta < 1$ and, in fact, their work and Lemma 2.3 below suffice to extend the range for θ to $7/8 < \theta < 1$ (see Kumchev [11, Theorem 6]). Furthermore, Liu and Liu [14] and Ren [18] proved that for any fixed $\varepsilon > 0$ one has

$$(1.5) \quad E_{2,4}(x) \ll x^{13/15+\varepsilon} \quad \text{and} \quad E_{3,5}(x) \ll x^{152/153+\varepsilon},$$

respectively, and Ren’s method can be easily adjusted to produce the bounds

$$(1.6) \quad E_{3,s}(x) \ll x^{1-(s-4)/153+\varepsilon} \quad (s = 6, 7, 8).$$

Using a recent refinement of the treatment of exceptional sets for additive representations (see Brüdern, Kawada and Wooley [4]), Wooley [24, 25] improved further on (1.5) and (1.6). He proved that for any fixed $\varepsilon > 0$ one has

$$(1.7) \quad \begin{aligned} E_{2,4}(x) &\ll x^{13/30+\varepsilon}, \\ E_{3,5}(x) &\ll x^{35/36+\varepsilon}, \quad E_{3,6}(x) \ll x^{17/18+\varepsilon}, \\ E_{3,7}(x) &\ll x^{23/36+\varepsilon}, \quad E_{3,8}(x) \ll x^{11/36+\varepsilon}. \end{aligned}$$

Our first theorem improves on (1.7). Let us define the sets $\mathcal{N}_{3,s}$, $5 \leq s \leq 8$, by

$$\mathcal{N}_{3,5} = \{n \in \mathbb{N} : n \equiv 1 \pmod{2}, n \not\equiv 0, \pm 2 \pmod{9}, n \not\equiv 0 \pmod{7}\},$$

$$\mathcal{N}_{3,6} = \{n \in \mathbb{N} : n \equiv 0 \pmod{2}, n \not\equiv \pm 1 \pmod{9}\},$$

$$\mathcal{N}_{3,7} = \{n \in \mathbb{N} : n \equiv 1 \pmod{2}, n \not\equiv 0 \pmod{9}\},$$

$$\mathcal{N}_{3,8} = \{n \in \mathbb{N} : n \equiv 0 \pmod{2}\}.$$

We can state our result as follows.

Theorem 1 *Let $5 \leq s \leq 8$ be an integer, let $E_{3,s}(x)$ be defined as above, and define θ_s by*

$$\theta_5 = 79/84, \quad \theta_6 = 31/35, \quad \theta_7 = 17/28, \quad \theta_8 = 23/84.$$

Then

$$E_{3,s}(x) \ll x^{\theta_s}.$$

Remark 1.1 The method of proof produces a bound of the form

$$E_{3,s}(x) \ll x^{\theta'_s + \varepsilon},$$

where θ'_s depends on the lowest positive zero of a function defined in terms of certain multiple integrals (see (7.28)). Since finding the exact value of θ'_s seems to be an impossible task, we have replaced θ'_s by a reasonably close upper bound. Thus, one can easily improve the value of θ_5 in Theorem 1 to, say, $\theta_5 = 79/84 - 10^{-7}$, but one needs a new idea in order to obtain $\theta_5 = 79/84 - 10^{-4}$.

We establish Theorem 1 in §6 and §7, using the following tools.

- (1) New exponential sum estimates from [11].
- (2) A larger than usual set of major arcs. Using the standard treatment of the major arcs in the Waring–Goldbach problem, one can only obtain estimates of the form (1.3) for exceptional sets. Earlier estimates for $E_{3,s}(x)$ by Ren [18] and Wooley [24] rely on the more refined treatment in [18, Theorem 2], but that result is insufficient for our purposes. In §3, we obtain new results concerning the major arcs in the Waring–Goldbach problem, which let us take the set of major arcs in the proof of Theorem 1 larger than allowed by Ren’s theorem. Our approach is somewhat different from that used in [18] or in related work on sums of squares [14, 15] and also enables us to treat fourth and higher powers as well as quasi-Waring–Goldbach problems (*i.e.*, problems in which some of the unknowns are almost prime instead of prime).
- (3) Sieve ideas. Using only (1) and (2), we can already obtain a considerable improvement on (1.7), but the result would be somewhat weaker than Theorem 1. For example, we would only have

$$E_{3,5}(x) \ll x^{20/21 + \varepsilon}.$$

In order to establish Theorem 1, we employ sieve techniques. We use the sieve method in Harman [6, 7], and for the result on sums of six cubes we also need a variant of the vector sieve of Brüdern and Fouvry [3]. These matters are discussed in §7.

The methods we develop for the proof of Theorem 1 can be generalized quite naturally to yield estimates for exceptional sets for sums of fourth and higher powers of primes. Next we state a few such estimates. Theorem 2, which we establish in §4 and §5, contains the results on biquadrates of primes that one can deduce from the estimates in [11] and §3. For the sake of brevity we have avoided the use of sieve methods, although such use would undoubtedly lead to a slightly stronger result. We have also excluded $E_{4,7}(x)$, $E_{4,8}(x)$ and $E_{4,9}(x)$ from consideration, since the treatment of those cases—while possible—would complicate further the treatment of the major arcs. Theorem 3, whose proof we omit, lists the results we can prove for fifth and higher powers as well as the results on seven, eight and nine biquadrates (and also some results on ten and more biquadrates that are superseded by Theorem 2).

Theorem 2 *Let $10 \leq s \leq 13$ be an integer, let*

$$\mathcal{N}_{4,s} = \{n \in \mathbb{N} : n \equiv s \pmod{240}\},$$

and let $E_{4,s}(x)$ be defined as above. Also, define θ_s by

$$\theta_{10} = 15/16, \quad \theta_{11} = 3/4, \quad \theta_{12} = 35/48, \quad \theta_{13} = 1/2.$$

Then for any fixed $\varepsilon > 0$ one has

$$E_{4,s}(x) \ll x^{\theta_s - \delta + \varepsilon},$$

where $\delta = 335/56832$ and the implied constant depends at most on ε .

Theorem 3 *Let k and s be integers with $4 \leq k \leq 10$ and $\frac{1}{2}s_0(k) \leq s < s_0(k)$, where $s_0(k)$ is given by the following table:*

k	4	5	6	7	8	9	10
$s_0(k)$	14	21	33	47	63	83	107

Also, let $\rho(k) = \frac{2}{3} \times (k2^k)^{-1}$ and define

$$\theta_{k,s} = \begin{cases} 1 - (s - 7)/48 & \text{if } k = 4, \\ 1 - (2s - 21)/240 & \text{if } k = 5, 11 \leq s \leq 18, \\ 4/5 - (2s - 37)/240 & \text{if } k = 5, s = 19, 20, \\ 1 - (2s - s_0(k)) \times \rho(k) & \text{if } 6 \leq k \leq 10. \end{cases}$$

Then there exists an absolute constant $\delta > 0$ such that

$$E_{k,s}(x) \ll x^{\theta_{k,s} - \delta}.$$

Notation Throughout the paper, the letter ε denotes a sufficiently small positive real number. Any statement in which ε occurs holds for each positive ε , and any implied constant in such a statement is allowed to depend on ε . Implicit constants are also allowed to depend on k (when it appears in a statement). Any additional dependence will be mentioned explicitly. The letter p , with or without indices, is reserved for prime numbers; c denotes an absolute constant, not necessarily the same in all occurrences. Also, we often use P to denote the “main parameter”; in such situations, we write $L = \log P$.

As usual in number theory, $\mu(n)$, $\phi(n)$ and $\tau(n)$ denote, respectively, the Möbius function, the Euler totient function and the number of divisors function. Also, if $z \geq 2$, we define

$$(1.8) \quad \psi(n, z) = \begin{cases} 1, & \text{if } (n, \mathcal{P}(z)) = 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{where } \mathcal{P}(z) = \prod_{p < z} p.$$

We write $e(x) = \exp(2\pi ix)$ and $(a, b) = \gcd(a, b)$ and use $m \sim M$ as an abbreviation for the condition $M \leq m < 2M$.

Throughout the paper, we use decompositions of the unit interval into major and minor arcs. If $1 \leq Y \leq X$, we define the set of major arcs $\mathfrak{M}(Y, X)$ as the union of the intervals

$$\mathfrak{M}(q, a; Y, X) = \{ \alpha \in [0, 1] : |q\alpha - a| \leq YX^{-1} \}$$

with $0 \leq a \leq q \leq Y$ and $(a, q) = 1$. The corresponding set of minor arcs is denoted by $\mathfrak{m}(Y, X) = [0, 1] \setminus \mathfrak{M}(Y, X)$.

2 Exponential Sum Estimates

In this section, we record several exponential sum estimates from [11].

Lemma 2.1 *Let $0 < \rho < 1/10$. Suppose that $\alpha \in \mathbb{R}$ and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that*

$$(2.1) \quad 1 \leq q \leq Q, \quad (a, q) = 1, \quad |q\alpha - a| \leq Q^{-1},$$

with

$$(2.2) \quad Q = P^{(9-6\rho)/5}.$$

Let $M \geq N \geq 2$, $|\xi_m| \leq 1$, $|\eta_n| \leq 1$. Then

$$(2.3) \quad \sum_{\substack{m \sim M \\ mn \sim P}} \sum_{n \sim N} \xi_m \eta_n e(\alpha(mn)^3) \ll P^{1-\rho+\varepsilon} + \frac{q^{-1/6} P^{1+\varepsilon}}{(1 + P^3 |\alpha - a/q|)^{1/2}},$$

provided that

$$(2.4) \quad \max(P^{8\rho}, P^{(2+12\rho)/5}) \leq M \leq P^{1-2\rho}.$$

Furthermore, if $\psi(n, z)$ is defined by (1.8),

$$(2.5) \quad \sum_{\substack{m \sim M \\ mn \sim P}} \sum_{n \sim N} \xi_m \psi(n, z) e(\alpha(mn)^3) \ll P^{1-\rho+\varepsilon} + \frac{q^{-1/6} P^{1+\varepsilon}}{(1 + P^3 |\alpha - a/q|)^{1/2}},$$

provided that

$$(2.6) \quad z \leq \min(P^{(3-22\rho)/5}, P^{1-10\rho})$$

and

$$(2.7) \quad M \leq \min(P^{(3-7\rho)/5}, P^{1-6\rho}).$$

Proof These estimates are [11, Lemmas 3.1 and 3.3] with $k = 3$. ■

Lemma 2.2 Let $k \geq 4$ and define $\rho(k) = \frac{2}{3} \times 2^{-k}$. Suppose that $\alpha \in \mathbb{R}$ and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying (2.1) with

$$Q = P^{(k^2 - 2k\rho(k))/(2k-1)}.$$

Then for any fixed $\varepsilon > 0$ one has

$$(2.8) \quad \sum_{p \sim P} e(\alpha p^k) \ll P^{1-\rho(k)+\varepsilon} + \frac{P^{1+\varepsilon}}{(q + P^k |q\alpha - a|)^{1/2}}.$$

Proof This is the case $k \geq 4$ of [11, Theorem 3]. ■

Lemma 2.3 Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, and suppose that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying

$$(2.9) \quad 1 \leq q \leq Q, \quad (a, q) = 1, \quad |q\alpha - a| < QP^{-k}$$

with $Q \leq P$. Then for any fixed $\varepsilon > 0$ one has

$$(2.10) \quad \sum_{p \sim P} e(\alpha p^k) \ll Q^{1/2} P^{11/20+\varepsilon} + \frac{q^\varepsilon P L^\varepsilon}{(q + P^k |q\alpha - a|)^{1/2}}.$$

Proof This is [11, Theorem 2]. ■

Lemma 2.4 Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, and suppose that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying (2.9) with $Q \leq P$. Let ξ_m, η_n be complex numbers with $|\xi_m| \leq \tau(m)^\varepsilon$, $|\eta_n| \leq \tau(n)^\varepsilon$, and let z_m be defined as $z_m = m$ or $z_m = Zm^{-1}$ with $Z \in \mathbb{R}$. Let $z \geq 2$ be such that $z_m \geq z$ for all $m \sim M$, and define

$$g(\alpha) = \sum_{\substack{m \sim M \\ mn \sim P}} \sum_{n \sim N} \sum_r \xi_m \eta_n \psi(mn, z) \psi(r, z_m) e(\alpha(mnr)^k),$$

where $\psi(n, z)$ is given by (1.8). Suppose that

$$MN \leq P^{11/20} \quad \text{and} \quad \max_{m \sim M} z_m \leq \sqrt{2P/M}.$$

Then

$$(2.11) \quad g(\alpha) \ll q^\varepsilon L^c (P\Psi(\alpha)^{-1/2} + \Psi(\alpha)^{1/2}P^{11/20} + Pz^{-1}),$$

where $\Psi(\alpha) = q + P^k|q\alpha - a|$.

Proof This result combines [11, Lemma 5.6] with the remark following its statement. ■

Occasionally, we also need the following simple tool that reduces the estimation of a bilinear sum to the estimation of a similar sum subject to “nicer” summation conditions. The proof can be found in [11, Lemma 2.7].

Lemma 2.5 Let $M, N, X \geq 2$, let $\Phi: \mathbb{N} \rightarrow \mathbb{C}$ satisfy $|\Phi(x)| \leq X$, and define the bilinear form

$$\mathcal{B}(M, N) = \sum_{m \sim M} \sum_{n \sim N} \xi_m \eta_n \Phi(mn),$$

where $|\xi_m| \leq 1, |\eta_n| \leq 1$, and m and n are subject to a joint condition of one of the forms

$$m < n \quad \text{or} \quad U \leq mn < U'.$$

Then

$$(2.12) \quad \mathcal{B}(M, N) \ll L \left| \sum_{m \sim M} \sum_{n \sim N} \xi'_m \eta'_n \Phi(mn) \right| + 1,$$

where $|\xi'_m| \leq |\xi_m|, |\eta'_n| \leq |\eta_n|$ and $L = \log(2MNX)$.

3 The Major Arcs in the Waring–Goldbach Problem

In this section, we establish two general results concerning estimates for mean values of Weyl sums over major arcs. Let P_1, \dots, P_s be real numbers with

$$(3.1) \quad P^\varepsilon \leq P_s \leq \dots \leq P_1 \leq P,$$

and define

$$S^*(q, a) = \sum_{\substack{x=1 \\ (x,q)=1}}^q e\left(\frac{ax^k}{q}\right), \quad B(n, q) = \sum_{\substack{a=1 \\ (a,q)=1}}^q \frac{S^*(q, a)^s}{\phi(q)^s} e\left(-\frac{an}{q}\right),$$

$$f_i(\alpha) = \sum_{p \sim P_i} e(\alpha p^k), \quad v_i(\beta) = \int_{P_i}^{2P_i} \frac{e(\beta y^k)}{\log y} dy.$$

We obtain the following proposition.

Proposition 1 Let $k \geq 2, s \geq 5$, and $P^k \ll N \ll P^k$. Let $N \leq n < 2N$ and define

$$(3.2) \quad \mathfrak{S}_{k,s}(n) = \sum_{q=1}^{\infty} B(n, q), \quad J_{k,s}(n) = \int_{\mathbb{R}} v_1(\beta) \cdots v_s(\beta) e(-n\beta) d\beta.$$

Suppose that P_1, \dots, P_s satisfy (3.1), that $P_s \gg P$, and that $Q \leq P^{1/2-\varepsilon}$. Then for any $A > 0$, we have

$$(3.3) \quad \int_{\mathfrak{M}(Q,N)} f_1(\alpha) \cdots f_s(\alpha) e(-n\alpha) d\alpha = \mathfrak{S}_{k,s}(n) J_{k,s}(n) + O(P_1 \cdots P_s N^{-1} L^{-A}),$$

where the implied constant depends at most on A, k, s , and ε .

We should point out that (3.3) is not necessarily an asymptotic formula. The hypothesis $s \geq 5$ implies the absolute convergence of the singular series $\mathfrak{S}_{k,s}(n)$ and the singular integral $J_{k,s}(n)$, but we need to make additional assumptions to ensure that the “main term” in (3.3) dominates the error term. We have left the related analysis of $\mathfrak{S}_{k,s}(n)$ and $J_{k,s}(n)$ out of Proposition 1, since it is quite standard and can be carried quickly (and often more efficiently) in any particular case, in which one may want to refer to the proposition.

We derive Proposition 1 from a more general result, Proposition 2 below, in which $f_1(\alpha), \dots, f_s(\alpha)$ are replaced by the exponential sums

$$(3.4) \quad g_i(\alpha) = \sum_{m \sim P_i} \lambda_i(m) e(\alpha m^k) \quad (1 \leq i \leq s),$$

where $\lambda_1(m), \dots, \lambda_s(m)$ are arithmetic functions having the following properties:

- (A₁) $\lambda_i(m) \ll 1$;
- (A₂) $\lambda_i(m) = 0$ unless $(m, \mathcal{P}(P^\varepsilon)) = 1$;
- (A₃) $\lambda_i(m)$ is well distributed in arithmetic progressions to small moduli, that is, for any $y \sim P_i, q \geq 1, (a, q) = 1$ and $A > 0$, we have

$$\sum_{\substack{m \leq y \\ m \equiv a \pmod{q}}} \lambda_i(m) - \frac{1}{\phi(q)} \sum_{\substack{m \leq y \\ (m,q)=1}} \lambda_i(m) \ll y L^{-A},$$

where the implied constant depends at most on A .

Using (A₁)–(A₃) and partial summation, we find that if $\alpha \in \mathfrak{M}(q, a; L^A, N)$,

$$(3.5) \quad g_1(\alpha) \cdots g_s(\alpha) = \frac{S^*(q, a)^s}{\phi(q)^s} g_1(\beta) \cdots g_s(\beta) + O(P_1 \cdots P_s L^{-3A}),$$

where $\beta = \alpha - a/q$. Integrating both sides of (3.5) over $\mathfrak{M}(L^A, N)$, we get

$$\int_{\mathfrak{M}(L^A, N)} g_1(\alpha) \cdots g_s(\alpha) e(-n\alpha) d\alpha = \sum_{q \leq L^A} B(n, q) J(n, L^A / (qN)) + O(P_1 \cdots P_s N^{-1} L^{-A}),$$

where

$$J(n, Z) = \int_{-Z}^Z g_1(\beta) \cdots g_s(\beta) e(-n\beta) d\beta.$$

We now record the bounds

$$(3.6) \quad B(n, q) \ll q^{1-s/2+\varepsilon} \quad \text{and} \quad J(n, Z) \ll Z P_1 \cdots P_s.$$

The latter is trivial and the former is a direct consequence from the estimate (see [9, Lemma 8.5])

$$(3.7) \quad S^*(q, a) \ll q^{1/2+\varepsilon}.$$

Using (3.6), we conclude that if $s \geq 5$,

$$\sum_{q \leq L^A} B(n, q) J(n, L^A / (qN)) = \mathfrak{R}(n, L^A) + O(P_1 \cdots P_s N^{-1} L^{-A/3}),$$

where

$$(3.8) \quad \mathfrak{R}(n, X) = \sum_{q=1}^{\infty} B(n, q) J(n, X / (qN)).$$

Therefore, if $s \geq 5$ and $A > 0$, we have

$$(3.9) \quad \int_{\mathfrak{M}(L^{3A}, N)} g_1(\alpha) \cdots g_s(\alpha) e(-n\alpha) d\alpha = \mathfrak{R}(n, L^{3A}) + O(P_1 \cdots P_s N^{-1} L^{-A}),$$

where the implied constant in the O -term depends at most on A and s .

Suppose now that $L^{3A} \leq Q \leq P$, $P \ll P_t \ll P$ ($t \leq s$), and $\lambda_1(m), \dots, \lambda_t(m)$ satisfy the following additional hypothesis:

(A₄) if $\alpha \in \mathfrak{M}(q, a; Q, N)$, with $(a, q) = 1$ and $q \leq Q$, we have

$$g_i(\alpha) \ll q^\varepsilon L^\varepsilon (P \Psi(\alpha)^{-1/2} + P^{1-\rho_1} Q^{1/2} + P^{1-\rho_2}),$$

where $\rho_1, \rho_2 > 0$ and $\Psi(\alpha) = q + N|q\alpha - a|$.

Then for $\alpha \in \mathfrak{M}(q, a; Q, N) \subset \mathfrak{M}(Q, N)$, we have

$$g_1(\alpha) \cdots g_s(\alpha) \ll q^\varepsilon L^\varepsilon P_1 \cdots P_s (\Psi(\alpha)^{-t/2} + P^{-t\rho_1} Q^{t/2} + P^{-t\rho_2}).$$

If $t \geq 5$, we deduce that

$$(3.10) \quad \int_{\mathfrak{R}} g_1(\alpha) \cdots g_s(\alpha) e(-n\alpha) d\alpha \ll L^\varepsilon P_1 \cdots P_s N^{-1} E,$$

where $\mathfrak{R} = \mathfrak{M}(Q, N) \setminus \mathfrak{M}(L^{3A}, N)$ and

$$E = L^{-A} + Q^{(t/2)+2+\varepsilon} P^{-t\rho_1} + Q^{2+\varepsilon} P^{-t\rho_2}.$$

Since the condition

$$Q \leq \min(P^{2t\rho_1/(t+4)-\varepsilon}, P^{t\rho_2/2-\varepsilon})$$

implies that $E \ll L^{-A}$, combining (3.9) and (3.10), we obtain the following result.

Proposition 2 *Let $k \geq 2$, $s \geq 5$, and $P^k \ll N \ll P^k$. Let $g_i(\alpha)$, $1 \leq i \leq s$, be defined by (3.4) with $\lambda_i(m)$ satisfying (A₁)–(A₃) and P_i satisfying (3.1). Suppose that $P_5 \gg P$, that $\lambda_1(m), \dots, \lambda_5(m)$ satisfy (A₄), and that*

$$(3.11) \quad Q \leq \min(P^{10\rho_1/9-\varepsilon}, P^{5\rho_2/2-\varepsilon}),$$

where ρ_1 and ρ_2 are the numbers appearing in (A₄). Then for any $A > 0$ we have

$$\int_{\mathfrak{M}(Q, N)} g_1(\alpha) \cdots g_s(\alpha) e(-n\alpha) d\alpha = \mathfrak{R}(n, L^{3A}) + O(P_1 \cdots P_s N^{-1} L^{-A}),$$

where $\mathfrak{R}(n, X)$ is defined by (3.8) and the implied constant depends at most on A, k, s , and ε .

Proof of Proposition 1 We apply Proposition 2, choosing each function $\lambda_i(m)$ equal to the characteristic function of the set of primes. Axioms (A₁) and (A₂) are then immediate, and (A₃) is the Siegel–Walfisz theorem. Furthermore, Lemma 2.3 yields (A₄) with $\rho_1 = 9/20$ and $\rho_2 = \infty$, so (3.11) follows from the assumption $Q \leq P^{1/2-\varepsilon}$ of Proposition 1. Thus, it remains to show that

$$\mathfrak{R}(n, L^{3A}) = \mathfrak{S}(n)J(n) + O(P_1 \cdots P_s N^{-1} L^{-A}).$$

This asymptotic formula follows easily from (3.6) and the estimates

$$(3.12) \quad g_i(\beta) = v_i(\beta) + O(P_i L^{-4A}) \quad \text{for } |\beta| \leq L^{3A} N^{-1},$$

and

$$(3.13) \quad v_i(\beta) \ll P_i L^{-1} (1 + P_i^k |\beta|)^{-1} \quad \text{for all } \beta \in \mathbb{R}.$$

The approximation (3.12) follows from the Prime Number Theorem by partial summation, while (3.13) can be obtained by partial integration (see [22, Lemma 6.2]). ■

4 Minor Arc Estimates, I: Biquadrates

The results of this section will be used in the proof of Theorem 2 to estimate the contribution from the minor arcs. We start by fixing some notation. We put

$$\nu_1 = 1, \quad \nu_2 = \frac{13}{16}, \quad \nu_3 = \left(\frac{13}{16}\right)^2, \quad \nu_4 = \left(\frac{13}{16}\right)^2 \frac{91}{111}, \quad \nu_5 = \left(\frac{13}{16}\right)^2 \frac{78}{111},$$

$$P_0 = \frac{1}{4}P, \quad P_j = P^{\nu_j} \quad (1 \leq j \leq 5), \quad f_j(\alpha) = \sum_{p \sim P_j} e(\alpha p^4) \quad (0 \leq j \leq 5).$$

Also, we write

$$F(\alpha) = f_1(\alpha) \cdots f_4(\alpha) f_5(\alpha)^2.$$

The exponents ν_j were suggested by Thanigasalam [20, Theorem 3], which we state in the following form.

Lemma 4.1 *Define*

$$g_j(\alpha) = \sum_{x \sim P_j} e(\alpha x^4) \quad \text{and} \quad G(\alpha) = g_1(\alpha) \cdots g_4(\alpha) g_5(\alpha)^2.$$

Then

$$\int_0^1 |G(\alpha)|^2 d\alpha \ll P^\varepsilon G(0).$$

Lemma 4.2 *Let $Q \geq P^{1/12}$, let $P^4 \ll N \ll P^4$, and write $m = m(Q, N)$. We have*

$$(4.1) \quad \int_m^1 |f_0(\alpha)F(\alpha)|^2 d\alpha \ll F(0)^2 P^{-2-\delta+\varepsilon},$$

where $\delta = 355/14208 = 0.0249 \dots$.

Proof We have

$$(4.2) \quad \int_m^1 |f_0(\alpha)F(\alpha)|^2 d\alpha \leq \sup_{\alpha \in m} |f_0(\alpha)|^2 \int_0^1 |F(\alpha)|^2 d\alpha.$$

Comparing the underlying diophantine equations, we deduce

$$(4.3) \quad \int_0^1 |F(\alpha)|^2 d\alpha \leq \int_0^1 |G(\alpha)|^2 d\alpha.$$

Since $G(0) \ll F(0)L^6$, (4.1) follows from (4.2), (4.3), Lemma 4.1 and the estimate

$$(4.4) \quad \sup_{\alpha \in m} |f_0(\alpha)| \ll P^{23/24+\varepsilon}.$$

By Dirichlet’s theorem on diophantine approximation, every real α has a rational approximation a/q subject to

$$1 \leq q \leq P^{47/21}, \quad (a, q) = 1, \quad |q\alpha - a| \leq P^{-47/21}.$$

Since for $\alpha \in \mathfrak{m}$ we also have

$$q + P^4|q\alpha - a| > Q \geq P^{1/12},$$

(4.4) follows from Lemma 2.2 with $k = 4$. ■

The next lemma is the main result of this section.

Lemma 4.3 *Let s be an integer with $10 \leq s \leq 13$. Let $Q \geq P^{1/12}$, $P^4 \ll N \ll P^4$, and write $\mathfrak{m} = \mathfrak{m}(Q, N)$. Also, let the set $\mathcal{Z} \subset \mathbb{N}$ have cardinality Z , and write*

$$K(\alpha) = \sum_{n \in \mathcal{Z}} e(n\alpha).$$

Then

$$(4.5) \quad \int_{\mathfrak{m}} |f_0(\alpha)^{s-6} F(\alpha) K(\alpha)| \, d\alpha \ll F(0) Z^{1/2} P^{\sigma_s - \delta/2 + \varepsilon} + F(0) Z P^{s-10-\delta/3},$$

where $\delta = 335/14208$ and σ_s is defined as follows:

$$\sigma_{10} = 15/8, \quad \sigma_{11} = 5/2, \quad \sigma_{12} = 83/24, \quad \sigma_{13} = 4.$$

Proof We write

$$I_s(P) = \int_{\mathfrak{m}} |f_0(\alpha)^{s-7} K(\alpha)|^2 \, d\alpha.$$

By Cauchy’s inequality and Lemma 4.2, the integral on the left side of (4.5) is bounded above by

$$I_s(P)^{1/2} \left(\int_{\mathfrak{m}} |f_0(\alpha) F(\alpha)|^2 \, d\alpha \right)^{1/2} \ll F(0) P^{-1-\delta/2+\varepsilon} I_s(P)^{1/2}.$$

Therefore, it suffices to show that

$$(4.6) \quad I_s(P) \ll P^{2\sigma_s+2+\varepsilon} Z + P^{2s-18+\varepsilon} Z^2.$$

Case 1 ($s = 10$) By orthogonality,

$$(4.7) \quad \int_0^1 |K(\alpha)|^2 \, d\alpha = Z.$$

Hence, recalling (4.4), we get

$$I_{10}(P) \leq \sup_{\alpha \in \mathfrak{m}} |f_0(\alpha)|^6 \int_0^1 |K(\alpha)|^2 \, d\alpha \ll P^{23/4+\varepsilon} Z,$$

which establishes (4.6) for $s = 10$.

Case 2 ($s = 11$) We have

$$(4.8) \quad I_{11}(P) \leq \int_0^1 |g(\alpha)^8 K(\alpha)^2| d\alpha,$$

where

$$g(\alpha) = \sum_{x \sim P_0} e(\alpha x^4).$$

By Weyl’s differencing lemma (see [22, Lemma 2.3]),

$$(4.9) \quad |g(\alpha)|^8 \ll P^4 \sum_{|h_1| < P_0} \sum_{|h_2| < P_0} \sum_{|h_3| < P_0} \sum_{x \in \mathcal{J}} e(\alpha \Delta(x^4; \mathbf{h})),$$

where $\mathcal{J} = \mathcal{J}(\mathbf{h})$ is a subinterval of $[P_0, 2P_0]$ and $\Delta(x^4; \mathbf{h})$ is the third-order forward difference of the function $x \mapsto x^4$ with steps h_1, h_2, h_3 , that is,

$$\Delta(x^4; \mathbf{h}) = 12h_1h_2h_3(2x + h_1 + h_2 + h_3).$$

Thus, we deduce from (4.8) that

$$(4.10) \quad I_{11}(P) \ll P^4 J(P),$$

where $J(P)$ is the number of solutions of the diophantine equation

$$(4.11) \quad \Delta(x^4; \mathbf{h}) = n_1 - n_2$$

subject to

$$(4.12) \quad P_0 \leq x \leq 2P_0, \quad |h_i| < P_0, \quad n_j \in \mathbb{Z}.$$

The number of solutions of (4.11), (4.12) with $n_1 = n_2$ is bounded by P^3Z . Also, for each pair (n_1, n_2) with $n_1 \neq n_2$ there are at most P^ε choices for x, \mathbf{h} . Hence,

$$J(P) \ll P^3Z + P^\varepsilon Z^2.$$

In conjunction with (4.10), this estimate establishes (4.6) for $s = 11$.

Case 3 ($s = 12$) We have

$$I_{12}(P) \leq \sup_{\alpha \in \mathfrak{m}} |f_0(\alpha)|^2 I_{11}(P) \ll P^{107/12+\varepsilon} Z + P^{6+\varepsilon} Z^2,$$

on combining (4.4) and Case 2.

Case 4 ($s = 13$) We have

$$I_{13}(P) \leq \int_0^1 |g(\alpha)^{12} K(\alpha)^2| d\alpha.$$

Hence, by (4.9) and Parseval’s identity,

$$(4.13) \quad I_{13}(P) \ll P^4 J(P),$$

where $J(P)$ is the number of solutions of the diophantine equation

$$(4.14) \quad \Delta(x^4; \mathbf{h}) = n_1 - n_2 + x_1^4 + x_2^4 - x_3^4 - x_4^4,$$

subject to

$$(4.15) \quad P_0 \leq x, x_1, \dots, x_4 \leq 2P_0, \quad |h_i| < P_0, \quad n_j \in \mathcal{Z}.$$

Let $J_0(P)$ denote the number of solutions of (4.14), (4.15) subject to

$$(4.16) \quad n_2 - n_1 = x_1^4 + x_2^4 - x_3^4 - x_4^4,$$

and let $J_1(P)$ be the number of solutions of (4.14), (4.15) for which (4.16) fails. A standard divisor function argument reveals that for each choice of $n_1, n_2, x_1, \dots, x_4$, $J_1(P)$ counts at most P^ε solutions of (4.14). On the other hand, every solution counted by $J_0(P)$ must have $h_1 h_2 h_3 (2x + h_1 + h_2 + h_3) = 0$, and therefore, each solution of (4.16) gives rise of $O(P^3)$ solutions of (4.14), (4.15). We conclude that

$$(4.17) \quad J(P) = J_0(P) + J_1(P) \ll P^3 J_2(P) + P^{4+\varepsilon} Z^2,$$

where $J_2(P)$ denotes the number of solutions of (4.16) in n_i, x_j subject to (4.15).

We now estimate $J_2(P)$. We have

$$J_2(P) = \int_0^1 |g(\alpha)^4 K(\alpha)^2| d\alpha.$$

By Weyl’s differencing lemma,

$$|g(\alpha)|^4 \ll P \sum_{|h_1| < P_0} \sum_{|h_2| < P_0} \sum_{x \in \mathcal{J}} e(\alpha \Delta(x^4; \mathbf{h})),$$

where $\mathcal{J} = \mathcal{J}(\mathbf{h})$ is a subinterval of $[P_0, 2P_0)$ and $\Delta(x^4; \mathbf{h})$ is the second-order forward difference of the function $x \mapsto x^4$ with steps h_1, h_2 ,

$$\Delta(x^4; \mathbf{h}) = 2h_1 h_2 (6x^2 + 6(h_1 + h_2)x + 2h_1^2 + 3h_1 h_2 + 2h_2^2).$$

Thus,

$$J_2(P) \ll P J_3(P),$$

where $J_3(P)$ is the number of solutions of

$$\Delta(x^4; \mathbf{h}) = n_1 - n_2$$

subject to

$$P_0 \leq x \leq 2P_0, \quad |h_i| < P_0, \quad n_j \in \mathcal{Z}.$$

Estimating $J_3(P)$ similarly to the quantity $J(P)$ from Case 2, we obtain

$$(4.18) \quad J_2(P) \ll P(P^2Z + P^\varepsilon Z^2).$$

Combining (4.13), (4.17) and (4.18), we get (4.6) with $s = 13$ and complete the proof of the lemma. ■

5 Proof of Theorem 2

Let $10 \leq s \leq 13$ and let $\mathcal{E}_s = \mathcal{E}_{4,s}(N)$ denote the set of those $n \in \mathcal{N}_{4,s} \cap [N, 2N)$ for which (1.1) with $k = 4$ has no solution in prime numbers p_1, \dots, p_s . It suffices to show that

$$(5.1) \quad |\mathcal{E}_s(N)| \ll N^{\theta_s + \varepsilon}$$

for all $N \geq N_0(\varepsilon)$.

We define P by $2P^4 = N$ and adopt the notation set in §4. Also, we write $R_s(n)$ for the number of solutions of (1.1) with $k = 4$ in primes p_1, \dots, p_s subject to

$$P_j \leq p_j < 2P_j \quad (1 \leq j \leq 5), \quad P_5 \leq p_6 < 2P_5, \quad P_0 \leq p_7, \dots, p_s < 2P_0,$$

and if \mathfrak{B} is a measurable subset of $[0, 1]$, we define

$$R_s(n; \mathfrak{B}) = \int_{\mathfrak{B}} f_0(\alpha)^{s-6} F(\alpha) e(-n\alpha) d\alpha.$$

Let us partition the unit interval into sets of major and minor arcs, $\mathfrak{M} = \mathfrak{M}(Q, N)$ and $\mathfrak{m} = \mathfrak{m}(Q, N)$, where $Q = P^{1/2-\varepsilon}$. By orthogonality,

$$(5.2) \quad R_s(n) = R_s(n; [0, 1]) = R_s(n; \mathfrak{M}) + R_s(n; \mathfrak{m}).$$

By Proposition 1,

$$(5.3) \quad R_s(n; \mathfrak{M}) = \mathfrak{S}_{4,s}(n) J_{4,s}(n) + O(F(0)P^{s-10}L^{-s-1}).$$

Here, $\mathfrak{S}_{4,s}(n)$ is defined by (3.2) and

$$J_{4,s}(n) = \int_{\mathbb{R}} v_1(\beta) \cdots v_4(\beta) v_5(\beta)^2 v_0(\beta)^{s-6} e(-n\beta) d\beta,$$

where

$$v_j(\beta) = \int_{P_j}^{2P_j} \frac{e(\beta t^4)}{\log t} dt = \int_{P_j^4}^{16P_j^4} \frac{u^{-3/4} e(\beta u)}{\log u} du.$$

A simple argument using Fourier transforms reveals that

$$(5.4) \quad F(0)P^{s-10}L^{-s} \ll J_{4,s}(n) \ll F(0)P^{s-10}L^{-s}$$

for all $n \in [N, 2N)$ (the reader can find detailed expositions of the treatments of similar singular integrals in [10, §5] or in §7.2 of the present paper). Furthermore, if $n \in \mathcal{N}_{4,s}$, we have

$$(5.5) \quad 1 \ll \mathfrak{S}_{4,s}(n) \ll 1.$$

The upper bound follows immediately from [9, Lemma 8.10], and the argument of [9, Lemma 8.12] gives

$$(5.6) \quad \mathfrak{S}_{4,s}(n) \gg \left(1 + \sum_{j=1}^4 B(n, 2^j t)\right) \prod_{p \in \{3,5\}} (1 + B(n, p))$$

whenever $s \geq 8$. A direct computation then reveals that the right side of (5.6) is non-zero if and only if $n \in \mathcal{N}_{4,s}$. Combining (5.3)–(5.5), we conclude that

$$(5.7) \quad R_s(n; \mathfrak{M}) \gg F(0)P^{s-10}L^{-s}$$

for all $n \in \mathcal{N}_{4,s} \cap [N, 2N)$.

We now turn to $R_s(n; \mathfrak{m})$. Using (5.2) and the definition of \mathcal{E}_s , we obtain

$$\sum_{n \in \mathcal{E}_s} R_s(n; \mathfrak{M}) = - \sum_{n \in \mathcal{E}_s} R_s(n; \mathfrak{m}),$$

so (5.7) implies that

$$(5.8) \quad F(0)P^{s-10-\varepsilon}|\mathcal{E}_s| \ll \left| \sum_{n \in \mathcal{E}_s} R_s(n; \mathfrak{m}) \right|.$$

Furthermore, on writing

$$K_s(\alpha) = \sum_{n \in \mathcal{E}_s} e(\alpha n),$$

Lemma 4.3 yields

$$(5.9) \quad \left| \sum_{n \in \mathcal{E}_s} R_s(n; \mathfrak{m}) \right| = \left| \int_{\mathfrak{m}} f_0(\alpha)^{s-6} F(\alpha) K_s(-\alpha) d\alpha \right| \\ \ll F(0)|\mathcal{E}_s|^{1/2} P^{\sigma_s - \delta/2 + \varepsilon} + F(0)|\mathcal{E}_s| P^{s-10-\delta/3}.$$

The desired estimate (5.1) follows immediately from (5.8) and (5.9). ■

6 Minor Arc Estimates, II: Cubes

In this section, we prove minor arc estimates to be used in the proof of Theorem 1. Throughout the section, \mathcal{Z} denotes a set of integers having cardinality Z and

$$K(\alpha) = \sum_{n \in \mathcal{Z}} e(\alpha n).$$

We also define the exponential sums

$$(6.1) \quad f_j(\alpha) = \sum_{p \sim P_j} e(\alpha p^3), \quad g_j(\alpha) = \sum_{m \sim P_2} \lambda_j(m) e(\alpha m^3) \quad (j = 1, 2),$$

where $\lambda_1(m)$ and $\lambda_2(m)$ are bounded arithmetic functions and $P \ll P_1, P_2 \ll P$.

Lemma 6.1 *Let $0 < \rho < 1/10$ and suppose that $g_1(\alpha)$ satisfies*

$$(6.2) \quad g_1(\alpha) \ll P^{1-\rho+\varepsilon} + \frac{q^{1/6} P^{1+\varepsilon}}{(1 + P^3 |\alpha - a/q|)^{1/2}},$$

whenever there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying (2.1) with $Q = P^{(9-6\rho)/5}$. Also, let

$$(6.3) \quad X \geq P^{2\rho} \quad \text{and} \quad P^3 \ll N \ll P^3,$$

and write $m = m(X, N)$. Then

$$(6.4) \quad \int_m^1 |f_1(\alpha) f_2(\alpha)^2 g_1(\alpha) g_2(\alpha) K(\alpha)| \, d\alpha \ll Z^{1/2} P^{7/2-\rho+\varepsilon}.$$

Proof By Cauchy’s inequality, the integral on the left side of (6.4) does not exceed

$$\left(\int_0^1 |K(\alpha)|^2 \, d\alpha \right)^{1/2} \left(\int_m^1 |f_1(\alpha) f_2(\alpha)^2 g_1(\alpha) g_2(\alpha)|^2 \, d\alpha \right)^{1/2}.$$

By orthogonality,

$$\int_0^1 |K(\alpha)|^2 \, d\alpha = Z,$$

so the lemma will follow if we show that

$$(6.5) \quad \int_m^1 |f_1(\alpha) f_2(\alpha)^2 g_1(\alpha) g_2(\alpha)|^2 \, d\alpha \ll P^{7-2\rho+\varepsilon}.$$

We start the proof of (6.5) by defining the sets

$$\mathfrak{R} = \mathfrak{R}(P^{6\rho}, N) \quad \text{and} \quad \mathfrak{n} = \mathfrak{n}(P^{6\rho}, N).$$

By Dirichlet’s theorem on diophantine approximation, every real α has a rational approximation a/q satisfying (2.1) with $Q = P^{(9-6\rho)/5}$. Thus, by hypothesis (6.2),

$$(6.6) \quad \sup_{\alpha \in \mathfrak{n}} |g_1(\alpha)| \ll P^{1-\rho+\varepsilon}.$$

On the other hand, if $X < P^{6\rho}$, any $\alpha \in m \cap \mathfrak{R}$ has a rational approximation a/q subject to

$$X < q + N|q\alpha - a| \leq P^{6\rho},$$

and Lemma 2.3 with $k = 3$ yields

$$(6.7) \quad \sup_{\alpha \in \mathfrak{m} \cap \mathfrak{R}} |f_1(\alpha)| \ll P^{1+\varepsilon} X^{-1/2} + P^{11/20+3\rho+\varepsilon} \ll P^{1-\rho+\varepsilon}.$$

Furthermore, using Hölder’s inequality, we deduce from Hua’s lemma [22, Lemma 2.5] that

$$(6.8) \quad \int_0^1 |G(\alpha)|^2 d\alpha \ll P^{5+\varepsilon},$$

whenever $G(\alpha)$ is a product of the form

$$G(\alpha) = f_1(\alpha)^{u_1} f_2(\alpha)^{u_2} g_1(\alpha)^{v_1} g_2(\alpha)^{v_2},$$

with $u_1, u_2, v_1, v_2 \geq 0$ and $u_1 + u_2 + v_1 + v_2 = 4$. The desired estimate (6.5) follows readily from (6.6)–(6.8). ■

Lemma 6.2 *Let $0 < \rho < 1/10$ and suppose that $g_1(\alpha)$ satisfies (6.2), whenever there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying (2.1) with $Q = P^{(9-6\rho)/5}$. Also, assume (6.3) and write $\mathfrak{m} = \mathfrak{m}(X, N)$. Then*

$$\int_{\mathfrak{m}} |f_1(\alpha) f_2(\alpha) g_1(\alpha)^2 g_2(\alpha)^2 K(\alpha)| d\alpha \ll Z^{1/2} P^{9/2-2\rho+\varepsilon}.$$

Proof The proof is similar to the proof of Lemma 6.1, using the bounds

$$\sup_{\alpha \in \mathfrak{n}} |g_1(\alpha)|^2 \ll P^{2-2\rho+\varepsilon} \quad \text{and} \quad \sup_{\alpha \in \mathfrak{m} \cap \mathfrak{R}} |f_1(\alpha) f_2(\alpha)| \ll P^{2-2\rho+\varepsilon}$$

instead of (6.6) and (6.7). ■

Lemma 6.3 *Let $0 < \rho < 1/10$ and suppose that $g_1(\alpha)$ satisfies (6.2), whenever there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying (2.1) with $Q = P^{(9-6\rho)/5}$. Also, assume (6.3) and write $\mathfrak{m} = \mathfrak{m}(X, N)$. Then for $j = 4, 5$ we have*

$$(6.9) \quad \int_{\mathfrak{m}} |f_1(\alpha) f_2(\alpha)^j g_1(\alpha) g_2(\alpha) K(\alpha)| d\alpha \ll Z^{1/2} P^{3+j/2-\rho+\varepsilon} + Z P^{j-\rho+\varepsilon}.$$

Proof By Cauchy’s inequality, the left side of (6.9) does not exceed

$$\left(\int_0^1 |f_2(\alpha)^{2j-4} K(\alpha)^2| d\alpha \right)^{1/2} \left(\int_{\mathfrak{m}} |f_1(\alpha) f_2(\alpha)^2 g_1(\alpha) g_2(\alpha)|^2 d\alpha \right)^{1/2}.$$

Thus, (6.9) follows from (6.5) and the estimates

$$\int_0^1 |f_2(\alpha)^{2j-4} K(\alpha)^2| d\alpha \ll P^\varepsilon (P^{2j-7} Z^2 + P^{j-1} Z),$$

for which we refer to [24, Lemmas 5.1 and 6.2]. ■

7 Proof of Theorem 1

Let $\mathcal{E}_s = \mathcal{E}_{3,s}(N)$ denote the set of those $n \in \mathcal{N}_{3,s} \cap [N, 2N)$ for which (1.1) with $k = 3$ has no solution in prime numbers p_1, \dots, p_s . We proceed to show that

$$(7.1) \quad |\mathcal{E}_s(N)| \ll N^{\theta_s}$$

for all $N \geq N_0(\varepsilon)$, our approach being similar to that used in the proof of Theorem 2. We choose $P = N^{1/3}$ and set

$$(7.2) \quad P_1 = \frac{3}{4}P, \quad P_2 = \frac{1}{6}P.$$

Let $R_s(n)$ denote the number of solutions of (1.1) with $k = 3$ in primes p_1, \dots, p_s subject to

$$(7.3) \quad P_1 \leq p_1 < 2P_1, \quad P_2 \leq p_2, \dots, p_s < 2P_2.$$

Also, let ρ be a parameter with $0 < \rho < 1/11$ and define the sets of major and minor arcs,

$$\mathfrak{M} = \mathfrak{M}(Q, N) \quad \text{and} \quad \mathfrak{m} = \mathfrak{m}(Q, N),$$

with

$$(7.4) \quad Q = \min(P^{1/2-\varepsilon}, P^{5/2-25\rho-\varepsilon}).$$

We shall construct a generating function $F_s(\alpha) = F_s(\rho; \alpha)$ with the following properties:

- (F₁) $F_s(\alpha)$ is a trigonometric polynomial with real coefficients;
- (F₂) if $N \leq n < 2N$,

$$R_s(n) \geq \int_0^1 F_s(\alpha)e(-n\alpha) d\alpha;$$

- (F₃) if $\mathcal{Z} \subset [N, 2N)$,

$$\sum_{n \in \mathcal{Z}} \int_{\mathfrak{m}} F_s(\alpha)e(-n\alpha) d\alpha \ll |\mathcal{Z}|^{1/2} P^{\sigma_5-\rho+\varepsilon} + |\mathcal{Z}| P^{s-3-\rho+\varepsilon},$$

where $\sigma_5 = 7/2, \sigma_6 = 9/2 - \rho, \sigma_7 = 5$ and $\sigma_8 = 11/2$;

- (F₄) if $n \in \mathcal{N}_{3,s} \cap [N, 2N)$,

$$\int_{\mathfrak{M}} F_s(\alpha)e(-n\alpha) d\alpha = (C_s(\rho) + O(L^{-1})) P^{s-3} L^{-s},$$

where $C_s(\rho)$ is a non-increasing function of ρ satisfying $C_s(\rho_s + 10^{-6}) > 0$ with $\rho_6 = 3/35$ and $\rho_s = 5/56$ for $s = 5, 7, 8$.

By (F₁) and the symmetry of \mathfrak{M} mod 1, the integral in (F₄) is a real number. Thus, on choosing $\rho = \rho_s + 10^{-6}$, the estimate

$$|\mathcal{E}_s(N)| \ll N^{\theta_s-(2/3) \times 10^{-6}+\varepsilon}$$

follows from (F₂)–(F₄) in the same way as (5.1) follows from (5.2), (5.7) and (5.9). This establishes (7.1).

7.1 Construction of $F_5(\alpha)$

We now build a generating function $F_5(\alpha)$ having properties (F₁)–(F₄). Given an arithmetic function λ , define

$$g(\alpha; \lambda) = \sum_{m \sim P_2} \lambda(m)e(\alpha m^3).$$

We start by expressing $R_5(n)$ as a linear combination of three types of terms:

- sums of the form

$$\sum_{m \sim P_2} \gamma(m)R_4(n - m^3),$$

where $g(\alpha; \gamma)$ satisfies (6.2);

- sums of the form

$$\sum_{m_1, m_2 \sim P_2} \delta(m_1)\gamma(m_2)R_3(n - m_1^3 - m_2^3),$$

where $g(\alpha; \gamma)$ satisfies (6.2);

- other positive terms.

We produce such a representation for $R_5(n)$ by successive decompositions using Buchstab’s identity

$$(7.5) \quad \psi(n, z_1) = \psi(n, z_2) - \sum_{\substack{z_2 \leq p < z_1 \\ n = pj}} \psi(j, p) \quad (2 \leq z_2 < z_1).$$

We navigate the decompositions as to lead to exponential sums $g(\alpha; \gamma)$ that can be estimated by Lemma 2.1 (possibly in conjunction with Lemma 2.5). Also, we try to keep the terms of the last kind (“other”) to a minimum, since the larger those terms, the sooner the function $C_5(\rho)$ in (F₄) vanishes as ρ increases.

We now proceed with the decomposition of $R_5(n)$. For the sake of brevity, we define $z = p^{1-10\rho}$, $X = \sqrt{2P_2}$, $U = p^{2\rho}$, $V = p^{1-8\rho}$ and

$$W = \min(p^{1-6\rho}, p^{(3-7\rho)/5}) = \begin{cases} p^{1-6\rho} & \text{if } \rho > 2/23, \\ p^{(3-7\rho)/5} & \text{if } \rho \leq 2/23; \end{cases}$$

notice that, with the exception of X , these quantities have been chosen in accordance with Lemma 2.1. We have

$$(7.6) \quad R_5(n) = \sum_{p \sim P_2} R_4(n - p^3) = \sum_{m \sim P_2} \psi(m, X)R_4(n - m^3).$$

We now decompose the function $\psi(m, X)$.

Applying twice Buchstab’s identity, we get

$$\begin{aligned}
 (7.7) \quad \psi(m, X) &= \psi(m, z) - \sum_{\substack{m=pj \\ z \leq p \leq W}} \psi(j, z) - \sum_{\substack{m=pj \\ W < p < X}} \psi(j, p) \\
 &\quad + \sum_{\substack{m=p_1 p_2 j \\ z \leq p_2 < p_1 \leq W}} \psi(j, p_2) \\
 &= \gamma_1(m) - \gamma_2(m) - \gamma_3(m) + \gamma_4(m), \quad \text{say.}
 \end{aligned}$$

We remark that (2.5) applies to $g(\alpha ; \gamma_1)$ and $g(\alpha ; \gamma_2)$. Furthermore, we can apply (2.3) to the exponential sums corresponding to some subsums of $\gamma_4(m)$, while other subsums of $\gamma_4(m)$ we can decompose further. Thus, we now partition $\gamma_4(m)$ into five subsums. We write

$$(7.8) \quad \gamma_4(m) = \gamma_5(m) + \dots + \gamma_9(m),$$

where $\gamma_5(m), \dots, \gamma_9(m)$ are the subsums of $\gamma_4(m)$ subject to the following constraints:

- $\gamma_5(m) : p_1, p_2$ or $p_1 p_2$ lies in $[U, V]$;
- $\gamma_6(m) : z \leq p_2 < p_1 < U < V < p_1 p_2$;
- $\gamma_7(m) : z \leq p_2 < U < V < p_1 < p_1 p_2 \leq W$;
- $\gamma_8(m) : z \leq p_2 < U < V < p_1 < W < p_1 p_2$;
- $\gamma_9(m) : V < p_2 < p_1 \leq W$.

We remark that $\gamma_5(m)$ is the portion of $\gamma_4(m)$ that induces an exponential sum which can be estimated by (2.3), while $\gamma_6(m)$ and $\gamma_7(m)$ are the portions for which we can give further decompositions.

Next, we decompose $\gamma_6(m)$. Another appeal to (7.5) gives

$$\begin{aligned}
 (7.9) \quad \gamma_6(m) &= \sum_{\substack{p_1, p_2 \\ m=p_1 p_2 j}} \psi(j, z) - \sum_{\substack{p_1, p_2 \\ m=p_1 p_2 p_3 j}} \sum_{z \leq p_3 < p_2} \psi(j, p_3) \\
 &= \gamma_{10}(m) - \gamma_{11}(m), \quad \text{say.}
 \end{aligned}$$

We can apply (2.5) to $g(\alpha ; \gamma_{10})$, and we deal with $\gamma_{11}(m)$ similarly to $\gamma_4(m)$. We have

$$(7.10) \quad \gamma_{11}(m) = \gamma_{12}(m) + \gamma_{13}(m) + \gamma_{14}(m),$$

where $\gamma_{12}(m), \gamma_{13}(m)$ and $\gamma_{14}(m)$ are the subsums of $\gamma_{11}(m)$ subject to

$$p_2 p_3 \leq V, \quad V < p_2 p_3 < p_1 p_2 p_3 \leq W \quad \text{and} \quad V < p_2 p_3 < W < p_1 p_2 p_3,$$

respectively. We now apply (7.5) to $\gamma_{13}(m)$ and obtain

$$\begin{aligned}
 (7.11) \quad \gamma_{13}(m) &= \sum_{\substack{p_1, p_2, p_3 \\ m=p_1 p_2 p_3 j}} \psi(j, z) - \sum_{\substack{p_1, p_2, p_3 \\ m=p_1 \cdots p_4 j}} \sum_{z \leq p_4 < p_3} \psi(j, p_4) \\
 &= \sum_{m=p_1 p_2 p_3 j} \psi(j, z) - \sum_{\substack{m=p_1 \cdots p_4 j \\ p_3 p_4 \leq V}} \psi(j, p_4) - \sum_{\substack{m=p_1 \cdots p_4 j \\ p_3 p_4 > V}} \psi(j, p_4) \\
 &= \gamma_{15}(m) - \gamma_{16}(m) - \gamma_{17}(m), \quad \text{say.}
 \end{aligned}$$

Combining (7.9)–(7.11), we conclude that

$$(7.12) \quad \gamma_6(m) = \gamma_{10}(m) - \gamma_{12}(m) - \gamma_{14}(m) - \gamma_{15}(m) + \gamma_{16}(m) + \gamma_{17}(m),$$

where (2.5) applies to $g(\alpha; \gamma_{10})$ and $g(\alpha; \gamma_{15})$ and, since $z^2 > U$, (2.3) applies to $g(\alpha; \gamma_{12})$ and $g(\alpha; \gamma_{16})$.

We now turn to $\gamma_7(m)$. Applying (7.5) once more, we get

$$\begin{aligned}
 (7.13) \quad \gamma_7(m) &= \sum_{\substack{p_1, p_2 \\ m=p_1 p_2 j}} \psi(j, z) - \sum_{\substack{p_1, p_2 \\ m=p_1 p_2 p_3 j}} \sum_{z \leq p_3 < p_2} \psi(j, p_3) \\
 &= \sum_{m=p_1 p_2 j} \psi(j, z) - \sum_{\substack{m=p_1 p_2 p_3 j \\ p_2 p_3 \leq V}} \psi(j, p_3) - \sum_{\substack{m=p_1 p_2 p_3 j \\ p_2 p_3 > V}} \psi(j, p_3) \\
 &= \gamma_{18}(m) - \gamma_{19}(m) - \gamma_{20}(m), \quad \text{say,}
 \end{aligned}$$

where $g(\alpha; \gamma_{18})$ and $g(\alpha; \gamma_{19})$ can be estimated by (2.5) and (2.3), respectively.

Finally, combining (7.7), (7.8), (7.12) and (7.13), we deduce that

$$\begin{aligned}
 (7.14) \quad \psi(m, X) &= \gamma_1(m) - \gamma_2(m) - \gamma_3(m) + \gamma_5(m) + \gamma_8(m) + \gamma_9(m) \\
 &\quad + \gamma_{10}(m) - \gamma_{12}(m) - \gamma_{14}(m) - \gamma_{15}(m) \\
 &\quad + \gamma_{16}(m) + \gamma_{17}(m) + \gamma_{18}(m) - \gamma_{19}(m) - \gamma_{20}(m).
 \end{aligned}$$

Substituting (7.14) into the right side of (7.6), we obtain the identity

$$(7.15) \quad R_5(n) = R_5(n; \lambda_1) - R_5(n; \lambda_2) + R_5(n; \lambda_3),$$

where

$$\begin{aligned}
 R_5(n; \lambda) &= \sum_{m \sim P_2} \lambda(m) R_4(n - m^3), \\
 \lambda_1(m) &= \gamma_1(m) - \gamma_2(m) + \gamma_5(m) + \gamma_{10}(m) - \gamma_{12}(m) \\
 &\quad - \gamma_{15}(m) + \gamma_{16}(m) + \gamma_{18}(m) - \gamma_{19}(m), \\
 \lambda_2(m) &= \gamma_3(m) + \gamma_{14}(m) + \gamma_{20}(m),
 \end{aligned}$$

and

$$\lambda_3(m) = \gamma_8(m) + \gamma_9(m) + \gamma_{17}(m).$$

We remark that $\lambda_1(m)$ is the sum of those $\gamma_j(m)$'s for which we can estimate $g(\alpha; \gamma_j)$ using one of the bounds in Lemma 2.1, while $\lambda_2(m)$ and $\lambda_3(m)$ collect the terms in the decomposition for which we cannot estimate the induced exponential sums and which contribute, respectively, negative and positive quantities to $\psi(m, X)$. Accordingly, $R_5(n; \lambda_1)$ represents the terms in the decomposition of $R_5(n)$ of the first kind mentioned above, $R_5(n; \lambda_2)$ will give rise of the terms of the second kind and some of the third kind, and $R_5(n; \lambda_3)$ contributes terms of the third kind.

Next, we decompose $R_5(n; \lambda_2)$. Similarly to (7.6), we have

$$R_5(n; \lambda_2) = \sum_{m_1, m_2 \sim P_2} \lambda_2(m_1) \psi(m_2, X) R_3(n - m_1^3 - m_2^3).$$

We use the following identity for $\psi(m, X)$:

(7.16)

$$\begin{aligned} \psi(m, X) &= \psi(m, z) - \sum_{\substack{m=pj \\ U \leq p < V}} \psi(j, p) - \sum_{\substack{m=pj \\ V < p < X}} \psi(j, p) - \sum_{\substack{m=pj \\ z \leq p < U}} \psi(j, z) \\ &+ \sum_{\substack{m=p_1 p_2 j \\ z \leq p_2 < p_1 < U}} \psi(j, z) - \sum_{\substack{m=p_1 p_2 p_3 j \\ z \leq p_3 < p_2 < p_1 < U}} \left\{ \sum_{p_2 p_3 \leq V} + \sum_{p_2 p_3 > V} \right\} \psi(j, p_3) \\ &= \gamma_1^*(m) - \gamma_2^*(m) - \gamma_3^*(m) - \gamma_4^*(m) + \gamma_5^*(m) - \gamma_6^*(m) - \gamma_7^*(m), \text{ say.} \end{aligned}$$

The proof of this identity is similar to (and simpler than) that of (7.14), using only three Buchstab decompositions; thus, we omit it. The main point is that we only have terms for which we can estimate $g(\alpha; \gamma_i^*)$, *i.e.*, those with $i \in \{1, 2, 4, 5, 6\}$, and negative terms, *i.e.*, $\gamma_3^*(m)$ and $\gamma_7^*(m)$. We conclude that

$$(7.17) \quad R_5(n; \lambda_2) = R_5(n; \lambda_2, \lambda_1^*) - R_5(n; \lambda_2, \lambda_3^*),$$

where

$$\begin{aligned} R_5(n; \lambda, \nu) &= \sum_{m_1, m_2 \sim P_2} \lambda(m_1) \nu(m_2) R_3(n - m_1^3 - m_2^3), \\ \lambda_1^*(m) &= \gamma_1^*(m) - \gamma_2^*(m) - \gamma_4^*(m) + \gamma_5^*(m) - \gamma_6^*(m), \\ \lambda_3^*(m) &= \gamma_3^*(m) + \gamma_7^*(m). \end{aligned}$$

Combining (7.15) and (7.17), we obtain

$$R_5(n) = R_5(n; \lambda_1) - R_5(n; \lambda_2, \lambda_1^*) + R_5(n; \lambda_3) + R_5(n; \lambda_2, \lambda_3^*),$$

which is the decomposition for $R_5(n)$ we have been pursuing in this section. Let $f_1(\alpha)$ and $f_2(\alpha)$ be the exponential sums defined in (6.1). We set

$$(7.18) \quad F_5(\alpha) = f_1(\alpha)f_2(\alpha)^2(g(\alpha; \lambda_1)f_2(\alpha) - g(\alpha; \lambda_2)g(\alpha; \lambda_1^*)),$$

so that

$$(7.19) \quad \int_0^1 F_5(\alpha)e(-n\alpha) d\alpha = R_5(n; \lambda_1) - R_5(n; \lambda_2, \lambda_1^*) \leq R_5(n).$$

Then, $F_5(\alpha)$ satisfies (F₁) and (F₂). Also, $g(\alpha; \lambda_1)$ and $g(\alpha; \lambda_1^*)$ satisfy (6.2) by construction, and (7.4) implies $Q \geq P^{2\rho}$, so Lemma 6.1 yields (F₃). Therefore, in order to complete the proof of (7.1) for $s = 5$, it remains to verify that $F_5(\alpha)$ satisfies (F₄).

7.2 The Case $s = 5$ Completed

We intend to verify that the generating function (7.18) satisfies (F₄) by referring to Proposition 2. Let $\lambda(m)$ denote any of the functions $\gamma_j(m)$ or $\gamma_j^*(m)$ appearing in the definitions of λ_1, λ_2 or λ_1^* . We note that λ satisfies axioms (A₁) and (A₂) from §3 by construction and proceed to check that it satisfies axioms (A₃) and (A₄) as well.

We start our discussion with (A₃). It suffices to show that if $P \ll y \ll P, A > 0$, and χ is a non-principal character mod $q, q \leq L^{2A}$, then

$$(7.20) \quad \sum_{m \leq y} \lambda(m)\chi(m) \ll yL^{-A},$$

with an implied constant depending at most on A . To this end, we observe that the left side of (7.20) is always bounded above by an expression of the form

$$LK \left| \sum_{J \leq j < J_1} \psi(j, w)\chi(j) \right|,$$

where

$$w \geq z, \quad z \leq J < J_1 \leq 2J \quad \text{and} \quad JK \ll y.$$

Thus, (7.20) follows from the estimate

$$\sum_{J \leq j < J_1} \psi(j, w)\chi(j) \ll JL^{-A},$$

which can be derived by partial summation from [1, Lemma 5].

We now turn toward axiom (A₄), which we verify by means of Lemma 2.4. When λ is one of $\gamma_1, \gamma_2, \gamma_{10}, \gamma_{12}, \gamma_{14}, \gamma_{15}, \gamma_{18}, \gamma_1^*, \gamma_2^*, \gamma_4^*, \gamma_5^*, \gamma_6^*$ or the portion of γ_5 subject to $U \leq p_1 p_2 \leq V$, Lemma 2.4 yields (A₄) with $\rho_1 = 9/20$ and $\rho_2 = 1 - 10\rho$. In order

to deal with $\gamma_{16}, \gamma_{19}, \gamma_{20}$ and the remainder of γ_5 , we first apply Buchstab's identity in reverse. Consider, for example, γ_{20} . We have

$$\begin{aligned} \gamma_{20}(m) &= \sum_{\substack{p_1 > V \\ p_1 p_2 \leq W}} \sum_{V^{1/2} < p_2 < U} \sum_{m=p_1 p_2 j} \{ \psi(j, V p_2^{-1}) - \psi(j, p_2) \} \\ &= \gamma'_{20}(m) - \gamma''_{20}(m), \quad \text{say,} \end{aligned}$$

and we can apply Lemma 2.4 to both γ'_{20} and γ''_{20} . Finally, since γ_3 is the characteristic function of the products $p_1 p_2$ with $p_1 \geq p_2 > W$, we can estimate $g(\alpha; \gamma_3)$ by combining Lemmas 5.1 and 5.3 in [11]. In all the cases, we obtain (at least) that λ satisfies axiom (A₄) with $\rho_1 = 9/20$ and $\rho_2 = 1 - 10\rho$.

We conclude that λ_1, λ_2 and λ_1^* satisfy axioms (A₁)–(A₄) from §3. We also note that when $\rho_1 = 9/20$ and $\rho_2 = 1 - 10\rho$, (3.11) follows from (7.4). Hence, we can apply Proposition 2 to the integral appearing in (F₄). We obtain

$$(7.21) \quad \int_{\mathfrak{M}} F_5(\alpha) e(-n\alpha) d\alpha = \mathfrak{R}(n, L^{50}) + O(P^2 L^{-6}),$$

where, in accordance with the notation in §3,

$$\mathfrak{R}(n, X) = \sum_{q=1}^{\infty} B(n, q) \int_{-X/(qN)}^{X/(qN)} F_5(\beta) e(-n\beta) d\beta.$$

We now define the exponential integrals

$$v_i(\beta) = \int_{P_i}^{2P_i} \frac{e(\beta y^3)}{\log y} dy \quad (i = 1, 2)$$

and the singular integral

$$\tilde{J}(n) = \int_{\mathbb{R}} v_1(\beta) v_2(\beta)^2 (g(\beta; \lambda_1) v_2(\beta) - g(\beta; \lambda_2) g(\beta; \lambda_1^*)) e(-n\beta) d\beta.$$

Recalling the bounds (3.12) and (3.13), we obtain

$$(7.22) \quad \mathfrak{R}(n, L^{50}) = \mathfrak{S}_{3,5}(n) \tilde{J}(n) + O(P^2 L^{-6}),$$

where $\mathfrak{S}_{3,5}(n)$ is defined in (3.2).

In order to analyze the singular integral $\tilde{J}(n)$, we require approximations for the mean values of λ_1, λ_2 and λ_1^* . We use that there exist functions $\ell_1(\rho), \ell_2(\rho)$ and $\ell_1^*(\rho)$ such that if $P_2 < y \leq 2P_2$,

$$(7.23) \quad \sum_{P_2 \leq m < y} \lambda_i(m) = \ell_i \cdot (y - P_2) L^{-1} + O(PL^{-2}) \quad (i = 1, 2)$$

and

$$(7.24) \quad \sum_{P_2 \leq m < y} \lambda_1^*(m) = \ell_1^* \cdot (y - P_2)L^{-1} + O(PL^{-2}).$$

These asymptotic formulas follow by partial summation from the Prime Number Theorem and the following lemma, whose proof can be found in [5, Lemma 1].

Lemma 7.1 *Let $y^\varepsilon \leq z \leq y$, and let $\omega(u)$ be the continuous solution of the differential delay equation*

$$\begin{cases} (u\omega(u))' = \omega(u - 1) & \text{if } u > 2, \\ \omega(u) = u^{-1} & \text{if } 1 < u \leq 2. \end{cases}$$

Then

$$\sum_{m < y} \psi(m, z) = \omega\left(\frac{\log y}{\log z}\right) \frac{y}{\log z} + O(y(\log y)^{-2}),$$

with an implied constant depending at most on ε .

The resulting expressions for the ℓ_i 's are sums of multiple integrals involving the function $\omega(\cdot)$, each of them accounting for the contribution of one of the γ_i 's or γ_i^* 's. For example, if $P_2 \leq y \leq 2P_2$, we have

$$\sum_{m \leq y} \gamma_{15}(m) = yL^{-1} (I_{15} + O(L^{-1})),$$

where

$$I_{15} = \frac{1}{1 - 10\rho} \iiint_{\mathfrak{D}_{15}} \omega\left(\frac{1 - u_1 - u_2 - u_3}{1 - 10\rho}\right) \frac{du_1 du_2 du_3}{u_1 u_2 u_3},$$

the integration being over u_1, u_2, u_3 subject to

$$\begin{cases} 1 - 10\rho < u_3 < u_2 < u_1 < 2\rho, & u_2 + u_3 > 1 - 8\rho, \\ u_1 + u_2 + u_3 < \min(1 - 6\rho, (3 - 7\rho)/5). \end{cases}$$

Now, we consider

$$\tilde{J}_1(n) = \int_{\mathbb{R}} v_1(\beta)v_2(\beta)^3 g(\beta; \lambda_1) e(-n\beta) d\beta.$$

Changing the order of integration, we obtain

$$(7.25) \quad \tilde{J}_1(n) = \iiint_{P_2 \leq y_1, y_2, y_3 \leq 2P_2} \sum_{m \sim P_2} \lambda_1(m) \Phi(m, \mathbf{y}) \frac{dy_1 dy_2 dy_3}{(\log y_1)(\log y_2)(\log y_3)},$$

where

$$\Phi(m, \mathbf{y}) = \int_{\mathbb{R}} \left\{ \int_{P_1^3}^{8P_1^3} \frac{y^{-2/3} e(\beta y)}{\log y} dy \right\} e(-\beta(n - y_1^3 - y_2^3 - y_3^3 - m^3)) d\beta.$$

If $N \leq n < 2N$ and $P_2 \leq m, y_1, y_2, y_3 \leq 2P_2$, (7.2) implies

$$P_1^3 < n - y_1^3 - y_2^3 - y_3^3 - m^3 < 8P_1^3,$$

so an appeal to Fourier's inversion formula gives

$$\Phi(m, \mathbf{y}) = \frac{(n - y_1^3 - y_2^3 - y_3^3 - m^3)^{-2/3}}{\log(n - y_1^3 - y_2^3 - y_3^3 - m^3)}.$$

Hence, (7.23) and partial summation lead to the approximate formula

$$(7.26) \quad \sum_{m \sim P_2} \lambda_1(m) \Phi(m, \mathbf{y}) = \ell_1 L^{-1} \int_{P_2}^{2P_2} \Phi(y_4, \mathbf{y}) dy_4 + O(P^{-1}L^{-3}).$$

Combining (7.25) and (7.26), we conclude that

$$(7.27) \quad \tilde{f}_1(n) = \frac{1}{3} \ell_1 L^{-5} J(n) + O(P^2 L^{-6}),$$

where

$$J(n) = \iiint_{P_2 \leq y_1, \dots, y_4 \leq 2P_2} (n - y_1^3 - \dots - y_4^3)^{-2/3} dy_1 \dots dy_4.$$

Similarly, we obtain

$$\int_{\mathbb{R}} v_1(\beta) v_2(\beta)^2 g(\beta; \lambda_2) g(\beta; \lambda_1^*) e(-n\beta) d\beta = \frac{1}{3} \ell_2 \ell_1^* L^{-5} J(n) + O(P^2 L^{-6}),$$

which in conjunction with (7.27) gives

$$(7.28) \quad \tilde{f}(n) = \frac{1}{3} (\ell_1 - \ell_2 \ell_1^*) L^{-5} J(n) + O(P^2 L^{-6}).$$

Finally, we observe that

$$P^2 \ll J(n) \ll P^2$$

for $N \leq n < 2N$, and that an argument similar to that leading to (5.5) yields

$$1 \ll \mathfrak{S}_{3,5}(n) \ll 1$$

for $n \in \mathcal{N}_{3,5}$. Therefore, (F₄) follows from (7.21), (7.22), (7.28) and the inequality

$$(\ell_1 - \ell_2 \ell_1^*)|_{\rho=5/56+10^{-6}} > 0,$$

which can be verified by numerical integration. This completes the proof of Theorem 1 in the case $s = 5$.

7.3 The Cases $s = 6, 7$ and 8

When $s = 7$ or 8 , the proof is essentially the same as in the case $s = 5$, using the generating functions

$$F_s(\alpha) = f_1(\alpha)f_2(\alpha)^{s-3}(g(\alpha; \lambda_1)f_2(\alpha) - g(\alpha; \lambda_2)g(\alpha; \lambda_1^*))$$

instead of $F_5(\alpha)$ and Lemma 6.3 instead of Lemma 6.1. The case $s = 6$ is also similar, with Lemma 6.2 in place of Lemma 6.1. However, since Lemma 6.2 requires upper bounds for two exponential sums instead for one, we need to sift two pairs of variables instead of one. This forces some changes, which we now describe.

We intend to combine the sieve construction used in §7.1 with a variant of the vector sieve of Brüdern and Fouvry [3]. Retaining the notation from §7.1, we define the function

$$\Lambda^-(m_1, m_2) = \lambda_1(m_1)\psi(m_2, X) - \lambda_2(m_1)\lambda_1^*(m_2).$$

We can now restate the sieve inequality underlying (7.19) as

$$\psi(m_1, X)\psi(m_2, X) \geq \Lambda^-(m_1, m_2).$$

Furthermore, by interchanging the roles of λ_2 and λ_3 in §7.1, we obtain

$$\psi(m_1, X)\psi(m_2, X) \leq \Lambda^+(m_1, m_2),$$

where

$$\Lambda^+(m_1, m_2) = \lambda_1(m_1)\psi(m_2, X) + \lambda_3(m_1)\lambda_1^*(m_2).$$

With the functions Λ^\pm in hand, we can state our version of the vector sieve as follows:

$$(7.29) \quad \prod_{j=1}^4 \psi(m_j, X) \geq \Lambda^+(\mathbf{m})\Lambda^-(\tilde{\mathbf{m}}) + \Lambda^-(\mathbf{m})\Lambda^+(\tilde{\mathbf{m}}) - \Lambda^+(\mathbf{m})\Lambda^+(\tilde{\mathbf{m}}),$$

where $\mathbf{m} = (m_1, m_2)$ and $\tilde{\mathbf{m}} = (m_3, m_4)$. Accordingly, we set

$$F_6 = F_6(\alpha) = f_1 f_2 [2(g_1 f_2 + g_3 g_1^*)(g_1 f_2 - g_2 g_1^*) - (g_1 f_2 + g_3 g_1^*)^2],$$

where $g_i = g(\alpha; \lambda_i)$ and $g_i^* = g(\alpha; \lambda_i^*)$. We can deal with $F_6(\alpha)$ similarly to $F_5(\alpha)$ (using Lemma 6.2 instead of Lemma 6.1). The only significant change occurs in the final step of the evaluation of the singular integral. Because of the change in the sifting functions, the function $\ell_1 - \ell_2 \ell_1^*$ appearing in (7.28) has to be replaced by

$$\ell = \ell(\rho) = 2(\ell_1 - \ell_2 \ell_1^*)(\ell_1 + \ell_3 \ell_1^*) - (\ell_1 + \ell_3 \ell_1^*)^2,$$

where ℓ_3 is defined analogously to ℓ_1 and ℓ_2 (i.e., we extend (7.23) to $i = 3$). Correspondingly, we need to change the value of the parameter ρ so that $\ell(\rho) > 0$.

Resorting to numerical integration, we find that we can choose $\rho = 3/35 + 10^{-6}$, which leads to the desired conclusion. ■

Acknowledgment This paper was written while the author enjoyed the benefits of a Postdoctoral Fellowship at the University of Toronto. He would like to take this opportunity to express his gratitude to the Department of Mathematics and especially to Prof. J. B. Friedlander for the support. He would also like to thank both Prof. D. R. Heath-Brown for a discussion from which the exposition benefited greatly and the anonymous referee for some very helpful suggestions.

References

- [1] R. C. Baker, G. Harman, and J. Pintz, *The exceptional set for Goldbach's problem in short intervals*. In: Sieve Methods, Exponential Sums and their Applications in Number Theory, London Math. Soc. Lecture Note Ser. 237, Cambridge University Press, Cambridge, 1997, pp. 1–54.
- [2] C. Bauer, M. C. Liu, and T. Zhan, *On a sum of three prime squares*. J. Number Theory **85**(2000), 336–359.
- [3] J. Brüdern and E. Fouvry, *Le crible à vecteurs*. Compositio Math. **102**(1996), 337–355.
- [4] J. Brüdern, K. Kawada, and T. D. Wooley, *Additive representations in thin sequences, I: Waring's problem for cubes*. Ann. Sci. École Norm. Sup. (4) **34**(2001), 471–501.
- [5] J. B. Friedlander, *Integers free from large and small primes*. Proc. London Math. Soc. (3) **33**(1976), 565–576.
- [6] G. Harman, *On the distribution of αp modulo one*. J. London Math. Soc. (2) **27**(1983), 9–18.
- [7] ———, *On the distribution of αp modulo one II*. Proc. London Math. Soc. (3) **72**(1996), 241–260.
- [8] L. K. Hua, *Some results in prime number theory*. Quart. J. Math. Oxford Ser. **9**(1938), 68–80.
- [9] ———, *Additive Theory of Prime Numbers*. Translations of Mathematical Monographs, Vol. 13 American Mathematical Society, Providence, RI, 1965.
- [10] K. Kawada and T. D. Wooley, *On the Waring–Goldbach problem for fourth and fifth powers*. Proc. London Math. Soc. (3) **83**(2001), 1–50.
- [11] A. Kumchev, *On Weyl sums over primes and almost primes*, preprint.
- [12] M. C. Leung and M. C. Liu, *On generalized quadratic equations in three prime variables*. Monatssh. Math. **115**(1993), 133–169.
- [13] H. Li, *The exceptional set of Goldbach numbers. II*. Acta Arith. **92**(2000), 71–88.
- [14] J. Y. Liu and M. C. Liu, *The exceptional set in the four prime squares problem*. Illinois J. Math. **44**(2000), 272–293.
- [15] J. Y. Liu and T. Zhan, *Distribution of integers that are sums of three squares of primes*. Acta Arith. **98**(2001), 207–228.
- [16] ———, *An iterative method in the Waring–Goldbach problem*, preprint.
- [17] H. L. Montgomery and R. C. Vaughan, *The exceptional set in Goldbach's problem*. Acta Arith. **27**(1975), 353–370.
- [18] X. Ren, *The Waring–Goldbach problem for cubes*. Acta Arith. **94**(2000), 287–301.
- [19] K. Thanigasalam, *Improvement on Davenport's iterative method and new results in additive number theory III*. Acta Arith. **48**(1987), 97–116.
- [20] ———, *On sums of positive integral powers and a simple proof of $G(6) \leq 31$* . Bull. Calcutta Math. Soc. **81** (1989), 279–294.
- [21] R. C. Vaughan, *On Goldbach's problem*. Acta Arith. **22**(1972), 21–48.
- [22] ———, *The Hardy–Littlewood Method*, second ed. Cambridge Tracts in Mathematics, 125, Cambridge University Press, Cambridge, 1997.
- [23] I. M. Vinogradov, *Representation of an odd number as the sum of three primes*. Dokl. Akad. Nauk SSSR **15**(1937), 291–294, in Russian.

- [24] T. D. Wooley, *Slim exceptional sets for sums of cubes*. *Canad. J. Math.* **54**(2002), 417–448.
- [25] ———, *Slim exceptional sets for sums of four squares*. *Proc. London Math. Soc.* (3) **85**(2002), 1–21.

*Department of Mathematics,
The University of Texas at Austin,
1 University Station C1200,
Austin, TX 78712,
U.S.A.
e-mail: kumchev@math.utexas.edu*