THE DEFICIENCY AND THE MULTIPLICATOR OF FINITE NILPOTENT GROUPS

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(Received 13 June 1969, revised 7 December 1969)

Communicated by G. E. Wall

1. Introduction

I. Šafarevič [2] proves that for a pro-finite p-group G, the deficiency of G (in the topological sense) equals the negative of $d(m(G))$. This means that if G is a finite p-group, where $d(m(G)) = n$, then there exists a group K with deficiency $-n$, such that G is the maximal p factor of K.

In this paper we extend this result to finite nilpotent groups by proving

THEOREM. *Let G be a finite nilpotent group, where $d(m(G)) = n$, then there exists a group K with deficiency $-n$, such that G is the maximal nilpotent factor group of K.*

We also prove the corresponding theorem relating to finite soluble groups.

2. Notations and definitions

DEFINITION 2.1. If a finite group G is generated by n elements and defined by m relations between them, then G has a presentation

$$G = \{x_1, \cdots, x_n | R_1, \cdots, R_m\} = F/R$$

where F is the free group on $x_1, \cdots, x_n$ and R is the smallest normal subgroup of F containing $R_1, \cdots, R_m$. Clearly $m \geqq n$ and the value $n-m$ is said to be the deficiency of the given presentation. The deficiency of G, denoted $\operatorname{def}(G)$, is the maximum of the deficiencies of all the finite presentations of G.

DEFINITION 2.2. The ring of integers will be denoted by Z, and the integral group ring, ZG, comprises all finite formal sums,

$$\sum z_i g_i; \; z_i \in Z, \; g_i \in G,$$

with addition and multiplication induced in the natural way. We state without proof the following

LEMMA 2.3. *Let G be a finite group and* $u \in ZG$, *so that* $u = \sum z_i g_i \in G$. *If* $\sum z_i = 0$ *then* $(\sum_{g \in G} g)u = 0$.

DEFINITION 2.4. G-module will always mean left ZG-module and if M is a G-module then $d_G(M)$ denotes the minimal number of generators of M as a ZG-module.

The minimal number of generators of a group G will be denoted by $d(G)$.

DEFINITION 2.5. Given a presentation for G, $G = F/R$, then R/R' may be considered a G-module with the action of G given by conjugation, and the value $d_G(R/R') - n$ is said to be the sufficiency of the presentation. The sufficiency of G, denoted suf(G), is the minimum of the sufficiencies of all the finite presentations of G.

DEFINITION 2.6. The multiplicator of G, denoted $m(G)$, is the second cohomology group of G with coefficients in T where T is the additive group of rationals modulo Z.

3. The Lyndon resolution

Let G be a finite group, then we construct a sequence of matrices with elements in ZG as follows.

$$M^0 = \begin{pmatrix} y_1 - 1 \\ \cdot \quad \cdot \quad \cdot \\ y_{\alpha_1} - 1 \end{pmatrix},$$

a column matrix, where $y_1, \ldots, y_{\alpha_1}$ is a set of elements generating G.

Given M^{r-1}, let M^r be any matrix whose rows $u_1, \cdots, u_{\alpha_{r+1}}$ generate the G-module formed by all vectors v such that

$$v \cdot M^{r-1} = 0.$$

Since G is finite we may choose α_r finite for all r and the $\alpha_{r+1} \times \alpha_r$ matrix M^r is said to be the r^{th} incidence matrix for G.

If K is any left G-module, let K_r be the set of all column vectors

$$k = \begin{pmatrix} k_1 \\ \cdots \\ k_{\alpha_r} \end{pmatrix}, \qquad k_i \in K.$$

Since $M^r K$ lies in K_{r+1} we obtain a sequence of mappings due to R. Lyndon [1] called the Lyndon resolution

$$\leftarrow K_{r+1} \overset{M^r}{\leftarrow} K_r \overset{M^{r-1}}{\leftarrow} K_{r-1} \leftarrow \cdots \leftarrow K_1 \overset{M^0}{\leftarrow} K_0 = K$$

where, since $M^r M^{r-1} = 0$, Image $M^{r-1} \subset$ Kernel M^r. The r^{th} cohomology group of G with coefficients in K may be defined as

$$H^r(G, K) = \text{Kernel } M^r / \text{Image } M^{r-1}, \qquad r > 0.$$

LEMMA 3.1. (Lyndon [1]) *Let G be a finite group.*

If $\{x_1, \cdots, x_n | R_1, \cdots, R_m\}$ *is a presentation for G, we may take the first incidence matrix,* M^1, *to be the matrix*

$$M^1 = (\gamma(\partial R_i/\partial x_j\}),$$

where γ *is the homomorphism of ZF onto ZG induced by the natural homomorphism of F onto G and* $\partial R_i/\partial x_j$ *denotes the Fox derivative of* R_i *with respect to* x_j.

Let $\tau : ZG \to Z$ be the homomorphism induced by $\tau(g) = 1$, for all $g \in G$, then we have

LEMMA 3.2. *Let G be a finite group, then we may choose a presentation for G such that*

$$M^1 = (\gamma\{\partial R_i/\partial x_j\}), \qquad \tau(M^1) = \begin{pmatrix} M_n \\ 0 \end{pmatrix},$$

where M_n *is a non singular* $n \times n$ *integral matrix, and*

$$\tau(M^2) = \begin{pmatrix} 0 & D_{m-n} \\ 0 & 0 \end{pmatrix},$$

where D_{m-n} *is a non-singular diagonal* $(m-n) \times (m-n)$ *integral matrix:*

$$D_{m-n} = \mathrm{diag}\,(z_1, \cdots, z_{m-n}).$$

PROOF. Clearly we can carry out elementary row operations on M^1 and M^2. Thus M^1 may be put in the required form. With M^1 in this form then the first n columns of $\tau(M^2)$ are zero, so that column operations are then induced on the non-zero columns of $\tau(M^2)$ by carrying out row operations on the zero rows of $\tau(M^1)$.

COROLLARY 3.3. *With* M^2 *in the form of the lemma we have*

$$H^2(G, T) = m(G) \cong Z_1 \times Z_2 \times \cdots Z_{m-n}$$

where Z_i *is the cyclic group of order* z_i.

4. The main theorem

THEOREM 4.1. *Let G be a finite nilpotent group, where* $d(m(G)) = n$, *then there exists a group K with deficiency* $-n$ *such that G is the maximal nilpotent factor group of K.*

PROOF. Choose a presentation for G as in lemma 3.2 with $\tau(M^1)$ and $\tau(M^2)$ in the desired form. Then we have

$$G = F/R = \{x_1, \cdots, x_r | R_1, \cdots, R_r, S_1, \cdots, S_t, T_1, \cdots, T_n\}$$

where $z_1 = z_2 = \cdots = z_t = 1, z_{t+i} \neq 1$ for $i = 1, \cdots, n$.

Let $K = F/N = \{x_1, \cdots, x_r | R_1, \cdots, R_r, T_1, \cdots, T_n\}$.

Clearly K has deficiency $-n$, otherwise this would carry back to G and contradict the fact that $d(m(G)) = n$.

We may also assume $t > 0$, otherwise the theorem holds trivially.

We have in F modulo R', by lemma 3.2,

$$S_i \equiv \sum_{1 \leqq j \leqq r} u_j R_j + \sum_{1 \leqq j \leqq t} v_j S_j + \sum_{1 \leqq j \leqq n} w_j T_j, \qquad (u_j, v_j, w_j \in ZG)$$

with $\tau(u_j) = \tau(v_j) = \tau(w_j) = 0$, which gives modulo $R'N$,

$$S_i \equiv \sum_{1 \leqq j \leqq t} v_j S_j, \qquad \text{with } \tau(v_j) = 0.$$

Let M be the normal closure in F of the S_j; then $\tau(v_j) = 0$ implies $v_j S_j \in [M, F]$, whence $S_i = S_i' r$ modulo N, where $S_i' \in [M, F]$ and $r \in R'$. However, $R' \subset [M, M] \subset [M, F]$ modulo N yielding $M \subset [M, F]$ modulo N or

$$MN/N \subset [MN/N, F/N].$$

Let $MN/N = M_0 \subset K$ and we have

$$M_0 \subset [M_0, K] \subset (M_0, K, K] \subset \cdots$$

or $M_0 \subset \Gamma_k(K)$, for all k where $\Gamma_k(K)$ is the k^{th} term in the lower central series of K.

Since $K/M_0 \cong G$ we have

(i) G is a nilpotent factor group of K,

(ii) if L is a nilpotent factor group of K, of class k, then L is a factor group of $K/\Gamma_k(K)$.

However $K/\Gamma_k(K)$ is a factor group of G since $M_0 \subset \Gamma_k(K)$ and the theorem is proved.

Next we prove the corresponding theorem for soluble groups.

Theorem 4.2. *Let G be a finite soluble group with sufficiency n, then there exists a group K with deficiency $-n$ such that G is the maximal soluble factor group of K.*

Proof. Choose a presentation G, in which the sufficiency is realised,

$$G = F/R = \{X_1, \cdots, X_r | R_1, \cdots, R_m, T_1, \cdots, T_t\}$$

where $\{R_1, \cdots, R_m\}$ is a minimal generating set for R modulo R', $T_i \in R'$ for $i = 1, \cdots, t$, and $m - r = n$.

Let $K = F/N = \{x_1, \cdots, x_r | R_1, \cdots, R_m\}$.

Clearly K has deficiency $-n$, otherwise this would carry back to G and contradict the fact that the sufficiency was realized in the above presentation for G.

If M is the normal closure in F of the T_i, then modulo N we have

$R' \subset [M, M]$, *whence* $M \subset [M, M]$, *whence* $MN/N \subset [MN/N, MN/N]$.

Let $MN/N = M_0 \subset K$ and we have

$$M_0 \subset [M_0, M_0] \quad \text{or} \quad M_0 \subset K^{(k)} \quad \text{for all } k$$

where $K^{(k)}$ is the k^{th} term in the derived series of K.

Since $K/M_0 \cong G$ we have

(i) G is a soluble factor group of K,

(ii) if L is a soluble factor group of K, of class k, then L is a factor group of $K/K^{(k)}$.

However $K/K^{(k)}$ is a factor group of G since $M_0 \subset K^{(k)}$ and the theorem is proved.

References

[1] R. C. Lyndon, 'Cohomology theory of groups with a single defining relation', *Annals of Math.* (2) 52 (1950), 650–665.

[2] I. R. Šafarevič, 'Extensions with prescribed branch points', *Inst. Hautes 'Etudes Sci. Publ. Math.* No. 18 (1963), 71–95; *Translations Amer. Math. Soc.*, Ser. 2, Vol. 59, 128–149.

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