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Accuracy of Kepler approximation for fly-by orbits near an attracting centre

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Car, disois-je en moi-même, cette masse étant moindre que la nôtre, il faut que la sphère de son activité ait aussi moins d'etendue, et que, par conséquent, j'aie senti plus tard la force de son centre.

Cyrano de Bergerac. *Histoire Comique des Etats et Empires de la Lune*. 1657

1. *The statement of the problem*

In the study of the motion of a particle $\mathcal{P}$ with negligible mass in the gravitational field created by other bodies (for example, the motion of the comet within the Solar system) it is natural to decompose its trajectory into regular and singular parts.

For the regular parts, when the distances between $\mathcal{P}$ and the other bodies are sufficiently large (we will not specify this for a while), the equation of the motion can be written in the form

$$\frac{d^2x}{dt^2} = F(x, t) \tag{1.1}$$

where F is a 'good' function (i.e. continuous, analytical, etc.). The gravitational nature of the force field manifests itself in the fact that the mass of the particle $\mathcal{P}$ does not enter into (1.1). If, for example, the other bodies of the system can be regarded as the mass points whose motion is known, then

$$F(x, t) = -\frac{\partial}{\partial x}\left\{\sum_{j=1}^{N} \frac{\gamma m_j}{|x - x_j(t)|}\right\} \tag{1.2}$$

where m_j is the mass and $x_j(t)$ is the position vector of the j'th body.

On the singular parts of the trajectory the particle $\mathcal{P}$ approaches a body of the system. This is the case we are interested in. Having in mind the 'comet' interpretation, we shall call the chosen body $\mathcal{J}$ 'Jupiter' and assume that it can be regarded as a point with mass $m_{\mathcal{J}}$ and position vector $x_{\mathcal{J}}(t)$. The function $x_{\mathcal{J}}(t)$ also satisfies an equation of the form (1.1), where F describes the influence of all bodies including $\mathcal{P}$ and $\mathcal{J}$. As for the equation of motion of $\mathcal{P}$, it contains an additional singular term corresponding to Jupiter's attraction. Thus

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\frac{\mu(x - x_{\mathcal{J}}(t))}{|x - x_{\mathcal{J}}(t)|^3} + F(x, t) \\ \frac{d^2x_{\mathcal{J}}}{dt^2} &= F(x_{\mathcal{J}}, t). \end{aligned} \tag{1.3}$$

From now on we will assume that the trajectory of Jupiter is known. Passing on to the Jove-centred coordinate system

$$q = x - x_{\mathcal{J}}(t) \tag{1.4}$$

we have the equation

$$\frac{d^2q}{dt^2} = -\frac{\mu q}{|q|^3} + \varphi(q, t), \tag{1.5}$$

where

$$\varphi(q, t) = F(x_{\mathcal{J}}(t) + q, t) - F(x_{\mathcal{J}}(t), t). \tag{1.6}$$

LEMMA 1. *If the inequalities*

$$\left\|\frac{\partial F}{\partial x}\right\| \le C_1 \quad \left\|\frac{\partial^2 F}{\partial x\, \partial x}\right\| \le C_{11} \quad \left\|\frac{\partial^2 F}{\partial x\, \partial t}\right\| \le C_{12} \quad \left|\frac{dx_{\mathcal{J}}}{dt}\right| \le v \tag{1.7}$$

are satisfied in the domain

$$\{(x, t) \mid |x - x_{\mathcal{J}}(t)| < r, t \in I\},$$

then for $|q| \le r$, $t \in I$ *the perturbation* (1.6) *satisfies*

$$|\varphi(q, t)| \le C_0|q|, \quad \left\|\frac{\partial \varphi}{\partial q}\right\| \le C_1, \quad \left|\frac{\partial \varphi}{\partial t}\right| \le C_2|q| \tag{1.8}$$

where $C_2 = C_{11}v + C_{12}$ *and* $C_0 = C_1$.

The straightforward proof is omitted. We shall henceforth consider equations of the form (1.5) with φ satisfying (1.8).

A special case of these equations is the classical restricted three-body problem. If the mass and length units are chosen so that the sum of the masses of the sun and Jupiter as well as the gravitational constant are equal to 1, then the motion of the 'comet' is described by the equation

$$\frac{d^2x}{dt^2} = -\frac{\mu(x - x_{\mathcal{J}}(t))}{|x - x_{\mathcal{J}}(t)|^3} - \frac{(1-\mu)(x - x_{\odot}(t))}{|x - x_{\odot}(t)|^3},$$

or, in the jove-centred system, by

$$\frac{d^2q}{dt^2} = -\frac{\mu q}{|q|^3} + (1-\mu)\left\{\frac{\rho(t)}{|\rho(t)|^3} - \frac{q + \rho(t)}{|q + \rho(t)|^3}\right\} \tag{1.9}$$

where $\rho(t) = x_{\mathcal{J}}(t) - x_{\odot}(t)$ is the position vector of Jupiter relative to the sun.

If Jupiter has an elliptic orbit, then $\rho(t)$ is periodic and, in this important special case, the estimates (1.8) for the perturbation (1.9) can be specified exactly.

LEMMA 2. *Let*

$$\rho_0 \le |\rho(t)|, \quad |q| \le \varepsilon |\rho(t)|, \quad \left|\frac{d\rho}{dt}\right| \le v, \quad -\infty < t < +\infty.$$

Then the function

$$\varphi(q, t) = (1-\mu)\left\{\frac{\rho(t)}{|\rho(t)|^3} - \frac{q + \rho(t)}{|q + \rho(t)|^3}\right\}$$

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obeys the estimates (1.8) *with*

$$C_0=\frac{1}{\rho_0^3}\frac{2-\varepsilon}{(1-\varepsilon)^2},\quad C_1=\frac{2}{\rho_0^3(1-\varepsilon)^3},\quad C_2=\frac{v}{\rho_0^4}\frac{6-6\varepsilon+2\varepsilon^2}{(1-\varepsilon)^3}.$$

This assertion will not be used in the following so we omit its proof.

2. *Informal description of the result. Contiguous topics*

The aim of this paper is to determine the size of a neighbourhood of Jupiter within which the fly-by orbits of the comet can be approximated by Kepler osculating hyperbolic orbits with an appropriate accuracy. Such a neighbourhood deserves the name 'sphere of Kepler's asymptotics'. Its radius depends not only on Jupiter's mass μ and on the properties of the external perturbating field but also on the relative energy of the comet. We have found conditions which allow us to show that the external (with respect to the Jupiter–comet system) perturbing field causes uniformly small perturbations in the osculating elements of the comet as well as in the derivatives of the current elements with respect to the initial ones. For perturbations obeying the estimates (1.8), the above variables uniformly approach zero provided that $r\mu^{\frac{1}{3}}\to 0$ and $\mu^{\frac{2}{3}}/\varepsilon\to 0$ where r denotes the radius of the sphere and $\varepsilon\to 0$ is the estimate from below for the initial values of the relative energy of the comet. If we permit $\varepsilon\to 0$, then the relative energy h and the angular momentum c should be normalized being replaced by $\log h$ and $\sqrt{2hc}$.

A similar problem was solved by Perko [**8**] and, later and in a rather general form, by Guillaume [**7**] in the framework of the so-called Breakwell–Perko's matching theory [**6**]. In short, this consists of the following. Let the planetoid's trajectory in the restricted three-body problem pass the vicinity of Jupiter. One constructs the asymptotic approximations for two of its parts: the inner one, which lies within some δ-neighbourhood of Jupiter, and the outer one, which lies outside of a somewhat smaller $O(\delta)$-neighbourhood. As approximations, one takes several terms of the asymptotic expansion with respect to μ, where the first term corresponds in the case of the outer expansion to the Kepler orbit focused at the Sun, and in the case of the inner one to the Kepler hyperbolic orbit focused at Jupiter. One picks the value of δ to make the degree of approximation for both expansions on the common part. For the inner part which is of interest to us, Guillaume has constructed an asymptotic approximation which is valid in a δ-neighbourhood of Jupiter where

$$O(\mu^{\frac{1}{2}})\le\delta\le O(\mu^{\frac{1}{3}}).$$

It follows from his arguments that the true orbit is asymptotically close to the hyperbolic one which osculates at the perijove provided

$$\delta=O(\mu^{\frac{1}{3}}).$$

This agrees with our result mentioned above. However Guillaume's estimates hold only for trajectories whose 'collision parameter' d (in other words, the distance between the focus and the asymptote of hyperbola) occurs in the interval

$$O(\delta^2)\le d\le O(\delta^4/\mu).$$

Such a restriction does not allow us to determine the *sphere* of Kepler's asymptotics properly, i.e. to decide which neighbourhood of Jupiter in the configuration space is such that *everywhere* within it the perturbed trajectories (having a fixed Jacobi constant for instance) can be treated as Keplerian orbits with a uniformly small error. Breakwell–Perko's theory deals in general with individual trajectories. Therefore, in particular, it does not provide any possibility of estimating the equations of variation of the perturbation of their solutions or, equivalently, the perturbation of the derivative of the current elements with respect to the initial ones.

In this paper we consider the equations of motion, which have been put into normal form by an appropriate choice of osculating elements. It gives us the possibility of estimating the perturbation both of the elements and of their derivatives for the entire trajectory within a certain sphere which has positive energy.† This estimate is concealed in the bowels of the basic theorem while its formulation involves not only estimates for the solutions of the equation of variation, but also the estimates for the derivatives of the scattering mapping (the mapping 'in–out' for the sphere of Kepler's asymptotics). Such a replacement implies the necessity of eliminating trajectories which just brush by the sphere since the derivative of mapping in question has a singularity at the points where the trajectories touch the sphere. We should note that the simple but important idea to pass from the usual coordinates to the osculating elements when considering the equations of variations has been suggested to us by the paper of V. N. Borodovski [**2**].

Several different 'gravitational spheres' have been introduced by different authors (see e.g. [**5**, p. 536]). When $\mu \to 0$ these spheres have the following radii: $\mu^{\frac{1}{2}}$ in the case of the 'gravity sphere', $\mu^{\frac{2}{5}}$ in the case of the 'sphere of action', $\mu^{\frac{1}{3}}$ in the case of 'Hill's sphere' and the 'sphere of influence'. According to what has been said the sphere of Kepler's asymptotics is close to the latter two. In connection with this it should be mentioned that M. D. Kislik [**4**] has proposed measuring the perturbation in terms of the difference between the heliocentric osculating elements of the true orbit leaving Jupiter's vicinity and the approximate one. He has shown that the 'sphere of influence' should be used in order to minimize the mean error. However, to a considerable extent his arguments are based on results of the numerical integration. Some additional remarks, relating the spheres of radii $\mu^{\frac{1}{2}}$ and $\mu^{\frac{1}{3}}$ will be given in § 10.

3. *The definitions and the formulation of basic theorem*

To formulate the basic theorem we shall pass for convenience to the reduced coordinates and time by setting

$$q = \mu Q, \quad v = V, \quad dt = \mu\, dT. \tag{3.1}$$

In these variables the unperturbed equation takes the form

$$\frac{d^2Q}{dT^2} = -\frac{Q}{|Q|^3} \tag{3.2}$$

† The same methods are applicable for the estimation of higher derivatives.

or

$$\frac{dQ}{dT}=V,\quad \frac{dV}{dT}=-\frac{Q}{|Q|^3},\quad \frac{dt}{dT}=\mu. \tag{3.3}$$

Formulae (3.3) define the vector field X in the phase space $\mathcal{M}=(\mathbb{R}^3\backslash\{0\})\times\mathbb{R}^3\times\mathbb{R}$. We concentrate our attention on the region of hyperbolic motion

$$\mathcal{M}_0=\left\{(Q,V,t)\in\mathcal{M}\left|\frac{|V|^2}{2}-\frac{1}{|Q|}>0\right.\right\} \tag{3.4}$$

which is invariant with respect to the unperturbed phase flow defined in $\mathcal{M}$ by the field X.

In the same phase space there is defined the perturbed vector field $\tilde{X}$ given by

$$\frac{dQ}{dT}=V,\quad \frac{dV}{dT}=-\frac{Q}{|Q|^3}+\Phi(Q,t,\mu),\quad \frac{dt}{dT}=\mu. \tag{3.5}$$

The perturbing term Φ is associated to the function φ in (1.5) by the equality

$$\Phi(Q,t,\mu)=\mu\varphi(\mu Q,t,\mu)=\mu\varphi(q,t,\mu). \tag{3.6}$$

If φ satisfies (1.8), then, as is easily seen

$$|\Phi|\le\mu^2C_0|Q|,\quad \left\|\frac{\partial\Phi}{\partial Q}\right\|\le\mu^2C_1,\quad \left\|\frac{\partial\Phi}{\partial t}\right\|\le\mu^2C_2|Q|. \tag{3.7}$$

Following the general plan sketched above, we shall study the behaviour of the perturbed trajectories within a certain ball $|\mathbf{Q}|\le R$ in configuration space. The theorem formulated below determines the conditions sufficient for the R-ball to be 'of Kepler's asymptotics'.

Let us consider two hypersurfaces in phase space:

$$\overset{\pm}{\underset{R}{\Sigma}}=\{(Q,V,t)\in\mathcal{M}_0|\,|Q|=R,\pm\langle Q,V\rangle>0\}. \tag{3.8}$$

The phase flows generated in $\mathcal{M}$ by the fields X and $\tilde{X}$ define Poincaré mappings from a certain subset of Σ_R^- into Σ_R^+. To investigate them we shall pass to the osculating elements.

We plan to use the traditional osculating elements such as the longitude of the ascending node, pericentre argument, etc. This leads to the appearance of singularities caused by the choice of coordinates and not by the physical nature of the problem. This may be justified by our desire to operate in the unique coordinate system which gives parallelizability of phase space. V. I. Arnol'd cautioned about making this mistake saying that 'analysts are apt to treat all bundles as Cartesian products and all manifolds are parallelizable ones'. As a phase space, it is natural to consider the completion of $\mathcal{M}_0$ with points which correspond to collision. The abstract manifold obtained, $\mathfrak{M}$, as we shall see soon, is non-parallelizable. To avoid the aforementioned mistake and at the same time not to burden ourselves with passing to local coordinates, we shall always regard the manifold $\mathfrak{M}$ as situated in Euclidean space (which may be naturally chosen as $\mathbb{R}^9$). Without other specifications, we shall extend the various objects defined on $\mathfrak{M}$ to its neighbourhood in the ambient space by means of the formulae describing them. For example, to solve the equations of variation for the vector field defined on $\mathfrak{M}$, we will extend them to a neighbourhood of $\mathfrak{M}$ and solve the usual system of linear differential equations.

We will call the following dimensional manifold the manifold of osculating elements (we should add 'hyperbolic')

$$\mathfrak{M} = (\xi, \mathscr{E}, t) = \{((\xi_1, \xi_2, \xi_3), \mathscr{E}, t) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \mid \quad |\xi_3| = 1, \langle \xi_2, \xi_3 \rangle = 0\}. \quad (3.9)$$

It is worth emphasizing that $\mathfrak{M}$ is nothing else than

$$\mathbb{R} \times T\mathbb{S}^2 \times \mathbb{R} \times \mathbb{R}.$$

Using the formulae (see any textbook on celestial mechanics, e.g. [**3**, pp. 435–438])

$$h = \frac{|V|^2}{2} - \frac{1}{|Q|}, \quad c = [Q, V], \quad f' = [V, [Q, V]] - Q/|Q| \quad (3.10)$$

for the reduced energy, angular momentum and Laplace's vector, we construct the mapping $\sigma : \mathcal{M}_0 \to \mathfrak{M}$ from the region of hyperbolic motion in phase space to the manifold of osculating elements:

$$\begin{aligned}
&\xi_1 = \log h \text{ is the 'normalized' energy;}\\
&\xi_2 = \sqrt{2h}c \text{ is the 'normalized' angular momentum;}\\
&\xi_3 = \omega = f/|f| \text{ is the direction toward the pericentre;}\\
&\mathscr{E} = \operatorname{Arsh}(-\sqrt{2h}|Q|\langle V, \omega\rangle) \text{ is the hyperbolic eccentric anomaly;}\\
&t = t \text{ is the physical unreduced time.}
\end{aligned} \quad (3.11)$$

The orbit is defined uniquely by

$$\xi = (\xi_1, \xi_2, \xi_3),$$

whereas $\mathscr{E}$ marks the position of the point on it. We shall sometimes use

$$\xi = (\xi, \mathscr{E}) = (\log h, \sqrt{2h}c, \omega, \mathscr{E})$$

for the 'geometric' coordinates of a point belonging to $\mathfrak{M}$.

The mapping σ induces a diffeomorphism of $\mathcal{M}_0$ onto

$$\mathfrak{M}' = \mathfrak{M} \backslash (\mathbb{R} \times \{0\} \times \mathbb{R}^3 \times \{0\} \times \mathbb{R}) \quad (3.12)$$

(the points of $\mathfrak{M}$ where $c = 0$ and $\mathscr{E} = 0$ correspond to the attracting centre $Q = 0$). The inverse mapping $\sigma^{-1} : \mathfrak{M}' \to \mathcal{M}_0$ is described by the formulae ([**3**, p. 510])

$$\begin{aligned}
Q &= \frac{1}{2h}(e - \operatorname{ch}\mathscr{E})\omega + \frac{1}{\sqrt{2h}} \operatorname{sh}\mathscr{E}[c, \omega],\\
V &= -\frac{1}{\sqrt{2h}} \frac{\operatorname{sh}\mathscr{E}}{|Q|}\omega + \frac{\operatorname{ch}\mathscr{E}}{|Q|}[c, \omega],\\
t &= t
\end{aligned} \quad (3.13)$$

where

$$|Q| = \frac{1}{2h}(e \operatorname{ch}\mathscr{E} - 1), \quad e = +\sqrt{1 + 2h|c|^2} = |f|. \quad (3.14)$$

The tangent mapping $T\sigma : T\mathcal{M}_0 \to T\mathfrak{M}'$ takes the vector fields $X, \tilde{X}$ into vector fields on $\mathfrak{M}'$. ξ is the first integral for the unperturbed motion, whereas the eccentric

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anomaly varies according to

$$t - t_{\text{per}} = \frac{\mu}{(2h)^{\frac{3}{2}}}(e \operatorname{sh} \mathscr{E} - \mathscr{E}) \tag{3.15}$$

(see [**3**, p. 510]) which implies

$$dt = \mu \, dT = \frac{\mu |Q|}{\sqrt{2h}} \, d\mathscr{E}. \tag{3.16}$$

Hence,

$$\frac{d\xi}{dT} = 0, \quad \frac{d\mathscr{E}}{dT} = \frac{\sqrt{2h}}{|Q|}, \quad \frac{dt}{dT} = \mu \tag{3.17}$$

so that

$$T_\sigma(X) = \frac{\sqrt{2h}}{|Q|}\frac{\partial}{\partial \mathscr{E}} + \mu \frac{\partial}{\partial t}.$$

Multiplying this vector field by the function $|Q|/\sqrt{2h}$, we obtain the field

$$\mathfrak{X} = \frac{\partial}{\partial \mathscr{E}} + \frac{\mu |Q|}{\sqrt{2h}} \frac{\partial}{\partial t},$$

which may be continuously extended to all of $\mathfrak{M}$. This multiplication is equivalent to the time scaling

$$dT = \frac{|Q|}{\sqrt{2h}} d\tau. \tag{3.18}$$

By comparing (3.18) and (3.17) we see that, in the unperturbed motion, τ and $\mathscr{E}$ coincide up to a choice of reference time.

It will be shown in the next section (lemma 3) that the perturbed vector field $(|Q|/\sqrt{2h})T_\sigma(X)$ may also be extended to the field $\tilde{\mathfrak{X}}$ which is defined on all of $\mathfrak{M}$. In the perturbed problem, τ and $\mathscr{E}$ cease to coincide. One may express this fact by saying that the eccentric anomaly splits into a dynamical (τ) and a geometrical ($\mathscr{E}$) anomaly.

As follows from (3.14), the hypersurfaces $\sum_R^\pm$ are taken by σ into

$$\mathfrak{S}_R^\pm = \left\{ (\xi, \mathscr{E}, t) \in \mathfrak{M} \,\middle|\, \mathscr{E} = \pm \operatorname{Arch} \frac{1 + 2hR}{\sqrt{1 + 2hc^2}} = \mathscr{E}_R^\pm(\xi) \right\}. \tag{3.19}$$

The Poincaré mappings from $\sum_R^-$ into $\sum_R^+$ along the trajectories of X, $\tilde{X}$ are taken into the Poincaré mappings S, $\tilde{S}$ along the trajectories of $\mathfrak{X}$, $\tilde{\mathfrak{X}}$. The extension of the vector fields from $\mathfrak{M}'$ to $\mathfrak{M}$ corresponds to the regularization of collisions with the attracting centre, and the τ introduced by (3.18) is nothing other than the 'regularizing' parameter of Thiele (traditionally attributed to Sundman).

The mapping S along the unperturbed trajectories is defined on all of $\mathfrak{S}_R^-$

$$S(\xi, \mathscr{E}_R^-(\xi), t) = \left(\xi, \mathscr{E}_R^+(\xi), t + \frac{\mu}{(2h)^{\frac{3}{2}}}(e \operatorname{sh} \mathscr{E} - \mathscr{E}) \Big|_{\mathscr{E}_R^-(\xi)}^{\mathscr{E}_R^+(\xi)} \right) \tag{3.20}$$

and $S(\mathfrak{S}_R^-) = \mathfrak{S}_R^+$. As for the perturbed mapping $\tilde{S}$, it may turn out that it is not well defined for the small energies h because a trajectory may pass to the negative energy region and remain within the ball $|Q| < R$ forever. For this reason we introduce the subset

$$\mathfrak{S}_R^-(\varepsilon) = \{(\xi, \mathscr{E}, t) \in \mathfrak{S}_R^- | h \geq \varepsilon > 0\}. \tag{3.21}$$

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Moreover, the estimates of the derivatives grow with no limit near the points where the trajectories touch the surface of section $\mathfrak{S}_R^-(\varepsilon)$. So one must exclude from consideration any trajectory with a small entrance angle into the R-ball by introducing another subset

$$\mathfrak{S}_R^-(\varepsilon, K) = \{(\xi, \mathscr{E}, t) \in \mathfrak{S}_R^-(\varepsilon) \mid |\cos(V^\wedge, Q)| > K\}.$$

Now we are ready to formulate the basic theorem.

THEOREM. *Let the function $\varphi(q, t, \mu)$ in the differential equation*

$$\frac{d^2q}{dt^2} = -\frac{\mu q}{|q|^3} + \varphi(q, t, \mu)$$

describing the perturbed Kepler problem be C^1 with respect to q and t, and also satisfy the condition

$$|\varphi(q, t, \mu)| \le C_0|q|, \quad \left\|\frac{\partial \varphi}{\partial q}(q, t, \mu)\right\| \le C_0, \quad \left\|\frac{\partial \varphi}{\partial t}(q, t, \mu)\right\| \le C_2|q| \tag{3.22}$$

in the ball $|q| \le \mu R$ for all t.

Then there exist $\delta > 0$ and $C > 0$ such that the inequality

$$\mu^2\left(\frac{1}{2\varepsilon} + R\right)^3\left(\frac{C_0}{K^2} + \frac{\mu R}{\sqrt{2\varepsilon}} C_2\right) < \delta$$

holds for some $\varepsilon > 0$, $K \in (0, 1)$ and implies the existence of a Poincaré mapping

$$\tilde{S} : \mathfrak{S}_R^-(\varepsilon) \to \mathfrak{S}_R^+((\xi^-, \mathscr{E}^-, t^-) \to (\tilde{\xi}^+, \tilde{\mathscr{E}}^+, \tilde{t}^+))$$

along the trajectories of the perturbed phase flow. This also gives an estimate of the deviation from the unperturbed Poincaré mapping.

$$\tilde{S} : \mathfrak{S}_R^-(\varepsilon) \to \mathfrak{S}_R^+((\xi^-, \mathscr{E}^-, t^-) \to (\tilde{\xi}^+, \tilde{\mathscr{E}}^+, \tilde{t}^+))$$

in the subset $\mathfrak{S}_R^-(\varepsilon, K) \subset \mathfrak{S}_R^-(\varepsilon)$ in the following way:

$$\|\tilde{\xi}^+ - \xi^+\| \le C\mu^2\left(\frac{1}{2h^-} + R\right)^2 RC_0, \quad \|\tilde{t}^+ - t^+\| \le \frac{C}{K}\frac{\mu^3}{\sqrt{2h^-}}\left(\frac{1}{2h^-} + R\right)^3 RC_0,$$

$$\left\|\frac{\partial \tilde{\xi}^+}{\partial \xi^-} - \frac{\partial \xi^+}{\partial \xi^-}\right\| = \left\|\frac{\partial \tilde{\xi}^+}{\partial \xi^-} - \mathrm{id}\right\| \le \frac{C}{K}\mu^2\left(\frac{1}{2h^-} + R\right)^3\left[\frac{C_0}{K} + \frac{\mu R}{\sqrt{2h^-}} C_2\right],$$

$$\left\|\frac{\partial \tilde{\xi}^+}{\partial t^-} - \frac{\partial \xi^+}{\partial t^-}\right\| \le C\mu^2\left(\frac{1}{2h^-} + R\right)^2 RC_2,$$

$$\left\|\frac{\partial \tilde{t}^+}{\partial \xi^-} - \frac{\partial t^+}{\partial \xi^-}\right\| \le \frac{C}{K^2}\frac{\mu^3}{\sqrt{2h^-}}\left(\frac{1}{2h^-} + R\right)^4\left[\frac{C_0}{K^2} + \frac{\mu R}{\sqrt{2h^-}} C_2\right],$$

$$\left\|\frac{\partial \tilde{t}^+}{\partial t^-} - \frac{\partial t^+}{\partial t^-}\right\| \le \frac{C}{K}\frac{\mu^3}{\sqrt{2h^-}}\left(\frac{1}{2h^-} + R\right)^3 RC_2.$$

Remark. We have required that (3.22) hold for an infinite time interval in order to simplify the exposition though it would be sufficient to require it for a time interval of approximate length $(2\mu/\sqrt{2\varepsilon})R$ which is that necessary to pass through the R-ball.

4. Lemmas

First of all we give an explicit expression for the perturbed vector field $\tilde{\mathfrak{X}}$ on the manifold of osculating elements. We shall drop all tildes in this section.

Lemma 3 (*The equations in the osculating elements*). *The differential equations in the reduced coordinates*

$$\frac{dQ}{dT}=V,\quad \frac{dV}{dT}=-\frac{Q}{|Q|^3}+\Phi(Q,t,\mu),\quad \frac{dt}{dT}=\mu \tag{4.1}$$

transform into

$$\left.\begin{aligned}\frac{dh}{d\tau}&=-\frac{\operatorname{sh}\mathscr{E}}{2h}\langle\Phi,\omega\rangle+\frac{\operatorname{ch}\mathscr{E}}{\sqrt{2h}}\langle\Phi,[c,\omega]\rangle\\ \frac{dc}{d\tau}&=\frac{(e\operatorname{ch}\mathscr{E}-1)}{(2h)^{\frac{5}{2}}}(e-\operatorname{ch}\mathscr{E})[\omega,\Phi]+\frac{(e\operatorname{ch}\mathscr{E}-1)}{(2h)^2}\operatorname{sh}\mathscr{E}[[c,\omega],\Phi]\end{aligned}\right\} \tag{4.2}$$

$$\left.\begin{aligned}\frac{d\omega}{d\tau}&=-\frac{(e\operatorname{ch}\mathscr{E}-1)}{(2h)^{\frac{3}{2}}}\frac{\langle\Phi,\omega\rangle}{e}[c,\omega]\\ &\quad+\frac{\operatorname{sh}\mathscr{E}}{2he}\left\{-\frac{1}{\sqrt{2h}}\operatorname{sh}\mathscr{E}\langle\omega,\Phi\rangle+\operatorname{ch}\mathscr{E}\langle[c,\omega],\Phi\rangle\right\}[c,\omega]\\ &\quad-\frac{(e\operatorname{ch}\mathscr{E}-1)}{(2h)^2}\operatorname{sh}\mathscr{E}(\Phi-\langle\Phi,\omega\rangle\omega)\\ \frac{d\mathscr{E}}{d\tau}&=1+\frac{\langle\Phi,\omega\rangle}{(2h)^2e}[(e^2-1)\operatorname{ch}^2\mathscr{E}-2(e\operatorname{ch}\mathscr{E}-1)]\\ &\quad+\frac{\langle\Phi,[c,\omega]\rangle}{(2h)^{\frac{3}{2}}e}\operatorname{sh}\mathscr{E}\operatorname{ch}\mathscr{E}\\ \frac{dt}{d\tau}&=\mu\frac{e\operatorname{ch}\mathscr{E}-1}{(2h)^{\frac{3}{2}}}.\end{aligned}\right\} \tag{4.3}$$

When passing to the $(\xi,\mathscr{E},t)$ coordinates, the pair of equations (4.2) are replaced by the equivalent ones:

$$\begin{aligned}\frac{d\xi_1}{d\tau}&=\frac{d\log h}{d\tau}=\frac{1}{h}\frac{dh}{d\tau},\\ \frac{d\xi_2}{d\tau}&=\frac{d\sqrt{2h}c}{d\tau}=\frac{1}{\sqrt{2h}}\frac{dh}{d\tau}c+\sqrt{2h}\frac{dc}{d\tau}.\end{aligned} \tag{4.4}$$

Remark. In the following we shall also need the equation

$$\frac{1}{e}\frac{de}{d\tau}=-\frac{(e^2-1)}{(2h)^2e}\operatorname{ch}\mathscr{E}\operatorname{sh}\mathscr{E}\langle\omega,\Phi\rangle-\frac{(\operatorname{ch}^2\mathscr{E}-2e\operatorname{ch}\mathscr{E}+1)}{(2h)^{\frac{3}{2}}e}\langle[c,\omega],\Phi\rangle. \tag{4.5}$$

This will be obtained in the proof of the lemma:

Proof. We differentiate formulae (3.10) which describe h, c, f as functions of the reduced coordinates with respect to the reduced time T. In compliance with equation (4.1) we have

$$\frac{dh}{dT}=\langle V,\Phi\rangle,\quad \frac{dc}{dT}=[Q,\Phi],\quad \frac{df}{dT}=[\Phi,c]+[V,[Q,\Phi]].$$

Using (3.13) we obtain h and c

$$\begin{aligned}\frac{dh}{dT}&=\langle V,\Phi\rangle=-\frac{1}{\sqrt{2h}}\frac{\operatorname{sh}\mathscr{E}}{|Q|}\langle\Phi,\omega\rangle+\frac{\operatorname{ch}\mathscr{E}}{|Q|}\langle\Phi,[c,\omega]\rangle\\ \frac{dc}{dT}&=[Q,\Phi]=\frac{1}{2h}(e-\operatorname{ch}\mathscr{E})[\omega,\Phi]+\frac{1}{\sqrt{2h}}\operatorname{sh}\mathscr{E}[[c,\omega],\Phi].\end{aligned}\tag{4.6}$$

Let us consider the equation for ω. In order to obtain $d\omega/dT$ we have to compute $\frac{1}{|f|}\frac{df}{dT}$ and eliminate its ω-component. We carry out the calculations by recalling that $|f|=e$ and that $c=[\omega,[c,\omega]]$ (because of $c\perp\omega$ and $|\omega|=1$). We have

$$\frac{1}{|f|}\frac{df}{dT}=\frac{1}{|f|}(\omega\langle\Phi,[c,\omega]\rangle-[c,\omega]\langle\Phi,\omega\rangle+Q\langle V,\Phi\rangle-\Phi\langle V,Q\rangle).\tag{4.7}$$

Then (3.13) implies that

$$\langle V,Q\rangle=\frac{e\operatorname{sh}\mathscr{E}}{\sqrt{2h}}.\tag{4.8}$$

Eliminating the ω-component in (4.7) and using (3.13) again we have

$$\begin{aligned}\frac{d\omega}{dT}=&-\frac{[c,\omega]}{e}\langle\Phi,\omega\rangle-\frac{\operatorname{sh}\mathscr{E}}{\sqrt{2h}}(\Phi-\langle\Phi,\omega\rangle\omega)\\ &+\frac{1}{e}\left(-\frac{1}{\sqrt{2h}}\frac{\operatorname{sh}\mathscr{E}}{|Q|}\langle\omega,\Phi\rangle+\frac{\operatorname{ch}\mathscr{E}}{|Q|}\langle[c,\omega],\Phi\rangle\right)\frac{1}{\sqrt{2h}}\operatorname{sh}\mathscr{E}[c,\omega].\end{aligned}\tag{4.9}$$

Furthermore, the ω-component of $\frac{1}{|f|}\frac{df}{dT}$ is nothing but $\frac{1}{e}\frac{de}{dT}$. So from (4.7), using (3.13) and (4.8), we have

$$\begin{aligned}\frac{1}{e}\frac{de}{dT}=&\frac{1}{e}\langle[c,\omega],\Phi\rangle\\ &+\frac{1}{2h}\frac{(e-\operatorname{ch}\mathscr{E})}{e}\left\{-\frac{1}{\sqrt{2h}}\frac{\operatorname{sh}\mathscr{E}}{|Q|}\langle\omega,\Phi\rangle+\frac{\operatorname{ch}\mathscr{E}}{|Q|}\langle[c,\omega],\Phi\rangle\right\}-\langle\Phi,\omega\rangle\frac{e\operatorname{sh}\mathscr{E}}{e\sqrt{2h}}.\end{aligned}\tag{4.10}$$

Before deriving the equation for $\mathscr{E}$, it is convenient to pass to the time τ in the equations for h, c, ω. Multiplying the pair of equations (4.6), the equations (4.9) and (4.10) by $|Q|/\sqrt{2h}$ we obtain (4.2) and the first of the equations (4.3) from the lemma and, after collecting the similar terms, the equation (4.5).

We differentiate the equality

$$\operatorname{sh}\mathscr{E}=-\sqrt{2h}|Q|\langle V,\omega\rangle$$

defining $\mathscr{E}$ (cf. (3.11)) with respect to T:

$$\begin{aligned}\operatorname{ch}\mathscr{E}\frac{d\mathscr{E}}{dT}=&-\frac{1}{\sqrt{2h}}\frac{dh}{dT}|Q|\langle V,\omega\rangle-\sqrt{2h}\left\langle V,\frac{Q}{|Q|}\right\rangle\langle V,\omega\rangle\\ &-\sqrt{2h}|Q|\left\langle-\frac{Q}{|Q|^3}+\Phi,\omega\right\rangle-\sqrt{2h}|Q|\left\langle V,\frac{d\omega}{dT}\right\rangle.\end{aligned}$$

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Rearranging the terms and passing to the time τ, we find

$$\operatorname{ch}\mathscr{E}\frac{d\mathscr{E}}{dT}=\left\langle\frac{Q}{|Q|},\omega\right\rangle-\langle V,\omega\rangle\langle V,Q\rangle-\frac{1}{\sqrt{2h}}|Q|\langle V,\omega\rangle\frac{dh}{d\tau}$$
$$-|Q|^2\langle\Phi,\omega\rangle-\sqrt{2h}|Q|\left\langle V,\frac{d\omega}{d\tau}\right\rangle. \tag{4.11}$$

Here the sum of the first two terms is equal, of course, to ch $\mathscr{E}$. The third, fourth and fifth terms are equal respectively to:

$$-\frac{1}{\sqrt{2h}}|Q|\langle V,\omega\rangle\frac{dh}{d\tau}=\frac{1-\operatorname{ch}^2\mathscr{E}}{(2h)^2}\langle\Phi,\omega\rangle+\frac{\operatorname{ch}\mathscr{E}\operatorname{sh}\mathscr{E}}{(2h)^{\frac{3}{2}}}\langle\Phi,[c,\omega]\rangle,$$
$$-|Q|^2\langle\Phi,\omega\rangle=-\frac{e^2\operatorname{ch}^2\mathscr{E}-2e\operatorname{ch}\mathscr{E}+1}{(2h)^2}\langle\Phi,\omega\rangle,$$
$$-\sqrt{2h}|Q|\left\langle V,\frac{d\omega}{d\tau}\right\rangle=-\sqrt{2h}\operatorname{ch}\mathscr{E}\left\{-\frac{|Q|}{\sqrt{2h}}\frac{\langle\Phi,\omega\rangle}{e}c^2+\frac{c^2}{2he}\operatorname{sh}\mathscr{E}\left(\frac{1}{\sqrt{2h}}\operatorname{sh}\mathscr{E}\langle\omega,\Phi\rangle\right.\right.$$
$$\left.\left.+\operatorname{ch}\mathscr{E}\langle[c,\omega],\Phi\rangle\right)-\frac{|Q|}{2h}\operatorname{sh}\mathscr{E}\langle\Phi,[c,\omega]\rangle\right\}.$$

Summing up these expressions, cancelling ch $\mathscr{E}$, and collecting similar terms we obtain the second equation of (4.3); the third is obvious. This completes the proof of the lemma. In the proof we showed that, in particular, the vector field $\tilde{\mathfrak{X}}$ defined by (4.4), (4.3) is regular on the whole manifold $\mathfrak{M}$.

Now our next goal is to estimate the right-hand sides of the equations (4.2)–(4.5) in terms of the energy h and of the modulus of position vector $|Q|$. From (3.14) and the definition of ω, we obtain estimates for the elementary expressions entering into the equations under discussion:

$$\begin{array}{ll} e\operatorname{ch}\mathscr{E}=1+2h|Q| & e\le 1+2h|Q| \\ |e\operatorname{sh}\mathscr{E}|\le 1+2h|Q| & \dfrac{1}{e}\le 1 \\ \operatorname{ch}\mathscr{E}\le 1+2h|Q| & \left|\dfrac{\sqrt{2h}c}{e}\right|=\dfrac{\sqrt{2h}c}{\sqrt{1+2hc^2}}\le 1 \\ |\operatorname{sh}\mathscr{E}|\le 1+2h|Q| & |\omega|=1. \end{array} \tag{4.12}$$

LEMMA 4. (*The estimates for the right-hand side of equations in the osculating elements.*) *The following estimates are valid for the right-hand sides of the equations*

(*a*) $$\left|\frac{dh}{d\tau}\right|\le 2\frac{(1+2h|Q|)}{2h}|\Phi|$$

(*b*) $$\left|\frac{dc}{d\tau}\right|\le 2\frac{(1+2h|Q|)^2}{(2h)^{\frac{5}{2}}}|\Phi|$$

(*c*) $$\left|\frac{d\omega}{d\tau}\right|\le 3\frac{(1+2h|Q|)^2}{(2h)^2}|\Phi|$$

$$(d) \qquad \left|\frac{d\mathscr{E}}{d\tau}-1\right| \le 2\,\frac{(1+2h|Q|)^2}{(2h)^2}|\Phi| \tag{4.13}$$

$$(e) \qquad \left|\frac{dt}{d\tau}\right| \le \mu\,\frac{(1+2h|Q|)}{(2h)^{\frac{3}{2}}}$$

$$(f) \qquad \left|\frac{d\xi_1}{d\tau}\right| \le 4\,\frac{1+2h|Q|)}{(2h)^2}|\Phi|$$

$$(g) \qquad \left|\frac{d\xi_2}{d\tau}\right| \le 4\,\frac{(1+2h|Q|)^2}{(2h)^2}|\Phi|$$

of the entire system

$$(h) \qquad |\tilde{\mathfrak{X}}-\mathfrak{X}| < 6.71\,\frac{(1+2h|Q|)^2}{(2h)^2}|\Phi|$$

as well as for $\hat{\tilde{\mathfrak{X}}}-\hat{\mathfrak{X}} = \mathrm{pr}_{(\xi,t)}(\tilde{\mathfrak{X}}-\mathfrak{X})$

$$(i) \qquad |\hat{\tilde{\mathfrak{X}}}-\hat{\mathfrak{X}}| \le 6.41\,\frac{(1+2h|Q|)^2}{(2h)^2}|\Phi|$$

and

$$(j) \qquad \frac{1}{e}\frac{de}{d\tau} \le 2\,\frac{(1+2h|Q|)^2}{(2h)^2}|\Phi|.$$

Proof. The estimates (a)–(c), (e) are more or less obvious consequences of equations (4.2), (4.3), the formulae (3.13), (3.14) and the estimates (4.12). When obtaining (d), one should take into account the inequality

$$|(e^2-1)\,\mathrm{ch}^2\,\mathscr{E} - 2(e\,\mathrm{ch}\,\mathscr{E}-1)| \le e^2\,\mathrm{ch}^2\,\mathscr{E}$$

and when obtaining (j) the inequality

$$|\mathrm{ch}^2\,\mathscr{E} - 2e\,\mathrm{ch}\,\mathscr{E}+1| < e^2\,\mathrm{ch}^2\,\mathscr{E}.$$

Also, (h), (i) are consequences of (c), (d), (f), (g).

Analogous estimates for the equations of variation which correspond to the system (4.4), (4.3) are far more cumbersome. To avoid a tedious calculation we make one observation.

The system (4.4), (4.3) may be written in the form

$$\frac{d\xi}{d\tau} = T_{(0,0,0,1)} + A(\xi)\Phi(Q(\xi), t, \mu)$$
$$\frac{dt}{d\tau} = \mu B(\xi) \tag{4.14}$$

where the operator-valued function $A(\xi)$ and the R-valued function $B(\xi)$ may be represented as polynomials of several elementary expressions.

Definition. We shall call the function

$$p(\xi) = P_\sigma^N\left(\mathrm{ch}\,\mathscr{E}, \mathrm{sh}\,\mathscr{E}, e\,\mathrm{ch}\,\mathscr{E}, e\,\mathrm{sh}\,\mathscr{E}, e, \frac{1}{e}, \frac{\sqrt{2hc}}{e}, \omega, \frac{1}{\sqrt{2h}}\right)$$

a pseudopolynomial of the class $\mathcal{P}_\sigma^N$ if, in the first place, P_σ^N is a polynomial, i.e.

$$P_\sigma^N(\alpha_1, \dots, \alpha_5, \beta_6, \beta_7, \beta_8, \gamma_9) = \sum_\nu L_\nu(\underbrace{\alpha_1, \dots, \alpha_1}_{\nu_1}, \dots, \underbrace{\gamma_9, \dots, \gamma_9}_{\nu_9})$$

where

$$\alpha_1, \dots, \alpha_5, \beta_6, \gamma_9 \in \mathbb{R};\ \beta_4, \beta_8 \in \mathbb{R}^3,$$

and where each L_ν is a polylinear function of its $|\nu| = \nu_1 + \cdots + \nu_9$ arguments. Moreover, in the second place, the degrees ν satisfy the conditions

(i) $$\nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5 \le N;$$

(ii) $$\tfrac{1}{2}\nu_9 = \sigma.$$

(The representation $p(\xi)$ in the form of a pseudopolynomial is not unique, of course.)

As we shall see, the functions $A(\xi)$, $B(\xi)$ in (4.14) are pseudopolynomials of the classes $\mathcal{P}_2^2$, $\mathcal{P}_{\frac{3}{2}}^1$ respectively. The following lemma allows us to estimate the pseudopolynomials and their derivatives in a simple way.

LEMMA 5. (*The estimates of pseudopolynomials and their derivatives.*) *Let* $p(\xi)$ *be a pseudopolynomial of class* $\mathcal{P}_\sigma^N$. *Then for all* $\xi \in \{(\xi_1, \dots, \xi_4) = (\log h, \sqrt{2hc}, \omega, \mathscr{E}) \mid h > 0, c \in \mathbb{R}^3, \omega \in \mathbb{R}^3, |\omega| = 1, \mathscr{E} \in \mathbb{R}\}$ *we have*

$$(a) \qquad |p(\xi)| \le \Big(\sum_\nu \|L_\nu\|\Big) \frac{(1+2h|Q|)^N}{(2h)^\sigma}$$

$$(b) \qquad \left|\frac{\partial p(\xi)}{\partial \xi_1}\right| \le \frac{(\sum_\nu \|L_\nu\| \nu_9)}{2} \frac{(1+2h|Q|)^N}{(2h)^\sigma}$$

$$(c) \qquad \left|\frac{\partial p}{\partial \xi_2}\right| \le \Big(\sum_\nu \|L_\nu\|(\nu_3+\nu_4+\nu_5+\nu_6+\nu_7)\Big) \frac{(1+2h|Q|)^N}{(2h)^\sigma} \qquad (4.15)$$

$$(d) \qquad \left|\frac{\partial p}{\partial \xi_3}\right| \le \Big(\sum_\nu \|L_\nu\| \nu_8\Big) \frac{(1+2h|Q|)^N}{(2h)^\sigma}$$

$$(e) \qquad \left|\frac{\partial p}{\partial \xi_4}\right| \le \Big(\sum_\nu \|L_\nu\|(\nu_1+\nu_2+\nu_3+\nu_4)\Big) \frac{(1+2h|Q|)^N}{(2h)^\sigma}.$$

Proof. We consider some pairs of inequalities ($j = 1, \dots, 4$)

$$\left.\begin{aligned} &\|\alpha_i\| \le (1+2h|Q|) \\ &\left\|\frac{\partial \alpha_i}{\partial \xi_j}\right\| \le \chi(i,j)(1+2h|Q|) \end{aligned}\right. \quad \text{where } i = 1, \dots, 5 \quad \chi(i,j) = \begin{cases} 0 \quad \text{for } \dfrac{\partial \alpha_i}{\partial \xi_j} \equiv 0 \\ 1 \text{ for } \dfrac{\partial \alpha_i}{\partial \xi_j} \not\equiv 0 \end{cases}$$

$$\left.\begin{aligned} &\|\beta_i\| \le 1 \\ &\left\|\frac{\partial \beta_i}{\partial \xi_j}\right\| \le \chi(i,j) \end{aligned}\right. \quad \text{where } i = 6, 7, 8 \quad \chi(i,j) = \begin{cases} 0 \text{ for } \dfrac{\partial \beta_i}{\partial \xi_j} \equiv 0 \\ 1 \text{ for } \dfrac{\partial \beta_i}{\partial \xi_j} \not\equiv 0 \end{cases}$$

$$\|\gamma_g\| = \frac{1}{\sqrt{2h}} \qquad \text{where} \qquad \chi(g,j) = \begin{cases} 0 \text{ for } \dfrac{\partial\gamma_g}{\partial\xi_j} \equiv 0 \\ 1 \text{ for } \dfrac{\partial\gamma_g}{\partial\xi_j} \not\equiv 0. \end{cases}$$

$$\left\|\frac{\partial\gamma_g}{\partial\xi_j}\right\| \le \frac{\chi(g,j)}{2}\frac{1}{\sqrt{2h}}$$

The first inequality in each pair is just (4.12). Each second inequality is a consequence of (4.12).

Explanation is perhaps needed for the estimation of $\partial\beta_7/\partial\xi_2$. The operator

$$\frac{1}{e}\left(\mathrm{id} - \frac{(\sqrt{2h}c)\otimes(\sqrt{2h}^T c)}{e^2}\right)$$

has one eigenvalue $1/e^3$ with an eigenvector which is co-linear to c and a double eigenvalue $1/e$ with eigenvectors which are orthogonal to c, so the norm of $\partial\beta_4/\partial\xi_2$ does not exceed 1.

Thus

$$\left|\frac{\partial}{\partial\xi_i} L_\nu(\underbrace{\alpha_1,\dots,\alpha_1}_{\nu_1},\dots,\underbrace{\beta_6,\dots,\beta_6}_{\nu_6},\dots,\underbrace{\gamma_9,\dots,\gamma_9}_{\nu_9})\right|$$

$$\le \|L_\nu\|\left\{\sum_{i=1}^{5}\nu_i\left\|\frac{\partial\alpha_i}{\partial\xi_j}\right\|\cdot\|\alpha_1\|^{\nu_1}\cdots\|\alpha_i\|^{\nu_i-1}\cdots\|\alpha_5\|^{\nu_5}\|\beta_6\|^{\nu_6}\cdots\|\beta_8\|^{\nu_8}\|\gamma_9\|^{\nu_9}\right.$$

$$+\sum_{i=6}^{8}\nu_i\left\|\frac{\partial\beta_i}{\partial\xi_j}\right\|\cdot\|\alpha_1\|^{\nu_1}\cdots\|\alpha_5\|^{\nu_5}\cdots\|\beta_i\|^{\nu_i-1}\cdots\|\gamma_9\|^{\nu_9}$$

$$\left.+\nu_9\left\|\frac{\partial\gamma_9}{\partial\xi_j}\right\|\cdot\|\alpha_1\|^{\nu_1}\cdots\|\alpha_5\|^{\nu_5}\|\beta_6\|^{\nu_6}\cdots\|\beta_8\|^{\nu_8}\|\gamma_9\|^{\nu_9-1}\right\}$$

$$\le \|L_\nu\|\left\{\frac{\nu_9}{2}\chi(g,j)+\sum_{i=1}^{8}\nu_i\chi(i,j)\right\}\frac{(1+2h|Q|)^{\nu_1+\cdots+\nu_5}}{(\sqrt{2h})^{\nu_9}}.$$

Substituting here the calculated values of χ, we obtain the estimates (4.15) (b)–(e). The estimate (a) is obvious.

Remark. When considering $p(\xi)$ as a function of h, c, ω, $\mathscr{E}$ we have

$$|p(\xi)| \le C\frac{(1+2h|Q|)^N}{(2h)^\sigma}, \quad \left|\frac{\partial p}{\partial h}\right| \le C_h\frac{(1+2h|Q|)^N}{(2h)^{\sigma+1}}, \quad \left|\frac{\partial p}{\partial c}\right| \le C_c\frac{(1+2h|Q|)^N}{(2h)^{\sigma-\frac{1}{2}}}.$$

We had passed to the normalized elements because of this and also because of our desire to have the estimates of the right-hand side of the equations in osculating elements (compare in (4.14) the estimates (a), (b) with (c), (d) and (f), (g) with (c), (d)).

LEMMA 6 (*The estimates for the right-hand side of equations of variation and others*). *The functions $A(\xi)$, $Q(\xi)$, $|Q(\xi)|$, $B(\xi)$ are the pseudopolynomials of the classes $\mathscr{P}_2^2$,*

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$\mathcal{P}^1_1$, $\mathcal{P}^1_1$, $\mathcal{P}^1_{\frac{3}{2}}$, *respectively and there hold the estimates*

$$\begin{aligned}
&(a) && \|A(\xi)\| \le 6.71\,\frac{(1+2h|Q|)^2}{(2h)^2} \\
&(b) && \left\|\frac{\partial A(\xi)}{\partial \xi}\right\| \le 28.11\,\frac{(1+2h|Q|)^2}{(2h)^2} \\
&(c) && \left\|\frac{\partial Q(\xi)}{\partial \xi}\right\| \le 4.25\,\frac{(1+2h|Q|)}{2h} \\
&(d) && \left\|\frac{\partial |Q|}{\partial \xi}\right\| \le 1.74\,\frac{(1+2h|Q|)}{2h} \qquad (4.16)\\
&(e) && \left\|\frac{\partial^2 |Q|}{\partial \xi\,\partial \xi}\right\| \le 3\,\frac{(1+2h|Q|)}{2h} \\
&(f) && \left\|\frac{\partial B(\xi)}{\partial \xi}\right\| \le 2.04\,\frac{(1+2h|Q|)}{(2h)^{\frac{3}{2}}} \\
&(g) && \left\|\frac{\partial^2 B(\xi)}{\partial \xi\,\partial \xi}\right\| \le 4.25\,\frac{(1+2h|Q|)}{(2h)^{\frac{3}{2}}}.
\end{aligned}$$

Proof. To verify that the above functions are of the indicated classes, one should rewrite the expressions (4.4), (4.3), (3.13), (3.14) in the form of polynomials in α, β, γ and count the corresponding degrees. We obtain (a) in a manner similar to $(4.13h)$. We estimate (b) via

$$\sqrt{\sum_{i,j=1}^{4}\left|\frac{\partial A_i}{\partial \xi_j}\right|^2}.$$

We estimate the derivatives of each component A_i, with respect to ξ_2, ξ_4 by means of lemma 5 and in turn with respect to ξ_1, ξ_3 by means of lemma 4 to achieve less generous estimates. We use the fact that the polynomials under consideration are homogeneous with respect to ξ_1, ξ_9 and that the estimates in lemma 4 have been obtained at the expense of an estimate of polynomial coefficients in ξ_1 and ξ_3 monomials.

We obtain the other estimates by direct computation and by use of (4.12) and the estimate

$$\left|\frac{1}{e}\left(\mathrm{id}-\frac{\xi_2 \otimes^T \xi_2}{e^2}\right)\right| \le \frac{1}{e}.$$

As we have seen, the estimates of the right-hand side for both the equation of elements and the equations of variation contain the expression $(1+2h|Q|)^2$ with $\alpha \ge 1$ only. So the following lemma plays a basic role in the sequel, being, in particular, the main tool of the so-called 'continuous induction' (see §§ 5, 6). The assumptions we formulate below, i.e. that the comet retains its hyperbolic character within an R-ball and others on the magnitude of osculating elements' variations during this motion, are the *a priori* assumptions.

LEMMA 7 (*The estimation of a typical integral*). *Let a solution of the system* (4.2), (4.3) *be defined on a closed interval* $[\tau^-, \tau^+]$ *and satisfy for some* ε_0, $\bar{h}$, R, κ *the* *following inequalities*

(*a*) $$\frac{d\mathscr{E}}{d\tau} \geq \frac{1}{1+\varepsilon_0}$$

(*b*) $$0 < h(\tau)$$

(*c*) $$h(\tau) \leq \bar{h}$$

(*d*) $$|Q(\tau)| \leq R$$

(*e*) $$\frac{e(\tau)}{e^-}, \frac{e(\tau)}{e^+} \leq \kappa.$$

Then for $\alpha \geq 1$ *we have*

$$\int_{\tau^-}^{\tau^+} (1+2h(\tau)|Q(\tau)|)^\alpha \, d\tau \leq 2\kappa(1+\varepsilon_0)(1+2\bar{h}R)^\alpha.$$

Proof. Changing the independent variable in the integral and applying (*a*), (*b*), we have

$$\int_{\tau^-}^{\tau^+} (1+2h|Q(\tau)|)^\alpha \, d\tau = \int_{\mathscr{E}^-}^{\mathscr{E}^+} (1+2h|Q(\tau)|)^\alpha \frac{d\tau}{d\mathscr{E}} \, d\mathscr{E}$$
$$\leq (1+\varepsilon_0) \int_{\mathscr{E}^-}^{\mathscr{E}^+} (1+2h|Q(\tau)|)^\alpha \, d\mathscr{E}.$$

Now, if we first decompose the integrand into the product $(1+2h|Q|)^{\alpha-1} \times (1+2h|Q|)$, and then majorize the first factor with regard to (*c*), and (*d*) by a constant and use the definition of $|Q|$, we have

$$(1+\varepsilon_0) \int_{\mathscr{E}^-}^{\mathscr{E}^+} (1+2h|Q(\tau)|)^\alpha \, d\mathscr{E} \leq (1+\varepsilon_0(1+2\bar{h}R)^{\alpha-1} \int_{\mathscr{E}^-}^{\mathscr{E}^+} (1+2h|Q(\tau)|) \, d\mathscr{E}$$
$$= (1+\varepsilon_0)(1+2\bar{h}R)^{\alpha-1} \int_{\mathscr{E}^-}^{\mathscr{E}^+} e(\tau) \operatorname{ch} \mathscr{E}(\tau) \, d\mathscr{E}.$$

Let us introduce the notation $\bar{e} = \max_{\tau^- \leq \tau \leq \tau^+} e(\tau)$:

We have

$$(1+\varepsilon_0)(1+2\bar{h}R)^{\alpha-1} \int_{\mathscr{E}^-}^{\mathscr{E}^+} e(\tau) \operatorname{ch} \mathscr{E}(\tau) \, d\mathscr{E} \leq (1+\varepsilon_0)(1+2\bar{h}R)^{\alpha-1} \bar{e} \int_{\mathscr{E}^-}^{\mathscr{E}^+} \operatorname{ch} \mathscr{E} \, d\mathscr{E}$$
$$= (1+\varepsilon_0)(1+2\bar{h}R)^{\alpha-1} \bar{e} (\operatorname{sh} \mathscr{E}^+ - \operatorname{sh} \mathscr{E}^-)$$
$$\leq (1+\varepsilon_0)(1+2\bar{h}R)^{\alpha-1} \bar{e} (\operatorname{ch} \mathscr{E}^+ + \operatorname{ch} \mathscr{E}^-).$$

The condition (*e*) implies $\bar{e} \leq \kappa e^-$, $\bar{e} \leq \kappa e^+$ and, therefore,

$$\bar{e}(\operatorname{ch} \mathscr{E}^+ + \operatorname{ch} \mathscr{E}^-) \leq \kappa(e^+ \operatorname{ch} \mathscr{E}^+ + e^- \operatorname{ch} \mathscr{E}^-).$$

We apply the definition of $|Q|$ and conditions (*c*), (*d*):

$$e^\pm \operatorname{ch} \mathscr{E}^\pm = 1 + 2h^\pm |Q^\pm| \leq 1 + 2\bar{h}R$$

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in order to finish the proof of the lemma:

$$\int_{\tau^-}^{\tau^+} (1+2h(\tau)|Q(\tau)|)^\alpha \, d\tau \le (1+\varepsilon_0)(1+2\bar{h}R)^{\alpha-1} 2\kappa(1+2\bar{h}R)$$
$$\le 2(1+\varepsilon_0)\kappa(1+2\bar{h}R)^\alpha.$$

5. *Beginning of the basic theorem's proof: The existence of Poincaré's mapping*

Let a point $p^- = (\xi_1^-, \xi_2^-, \omega^-, \mathscr{E}^-, t^-)$ belong to $\mathfrak{S}_R^-(\varepsilon)$. We fix the positive constants ε_0 and $\varepsilon_1, \varepsilon_2 < 1$ whose values we shall specify below and denote by

$$G = G_{p^-} = \{(\xi_1, \xi_2, \omega, \mathscr{E}, t) : |h - h^-| \le \varepsilon_1 h^-, |e - e^-| \le \varepsilon_2 e^-, |\mathbf{Q}| \le R\}$$

a tube, or rather a layer, in the manifold of osculating elements, depending on p^-. In compliance with (4.13(*d*)) and (3.7), if $p(\tau) = \{\xi_1(\tau), \xi_2(\tau), \omega(\tau), \mathscr{E}(\tau), t(\tau)\}$ belongs to this set, then we have the inequalities

$$0 < \varepsilon(1-\varepsilon_1) \le \underline{h} = h^-(1-\varepsilon_1) \le h(\tau) \le h^-(1+\varepsilon_1) = \bar{h} \tag{5.1}$$

$$\underline{e} = e^-(1-\varepsilon_2) \le e(\tau) \le e^-(1+\varepsilon_2) = \bar{e} \tag{5.2}$$

or

$$\frac{e(\tau_1)}{e(\tau_2)} \le \frac{e^-(1+\varepsilon_2)}{e^-(1-\varepsilon_2)} = \frac{1+\varepsilon_2}{1-\varepsilon_2} = \kappa \tag{5.3}$$

(for τ_1, τ_2 such that $p(\tau_1), p(\tau_2) \in G$), and

$$\left|\frac{d\mathscr{E}}{d\tau} - 1\right| \le \frac{2(1+2h|Q|)^2}{(2h)^2} |\Phi| \le 2\left(\frac{1+\varepsilon_1}{1-\varepsilon_1}\right)^2 \left(\frac{1}{2h^-} + R\right)^2 C_0 \mu^2 R.$$

Thus, if

$$2\left(\frac{1+\varepsilon_1}{1-\varepsilon_1}\right)^2 \left(\frac{1}{2h^-} + R\right)^2 C_0 \mu^2 R \le \frac{\varepsilon_0}{1+\varepsilon_0} \tag{5.4}$$

then

$$\frac{d\mathscr{E}}{d\tau} \ge 1 - \frac{\varepsilon_0}{1+\varepsilon_0} = \frac{1}{1+\varepsilon_0}. \tag{5.5}$$

Hence, while the trajectory stays in G, the eccentric anomaly increases monotonically.

Because of (4.8) and (3.18)

$$\frac{d|Q|}{d\tau} = \frac{|Q|}{\sqrt{2h}} \frac{d|Q|}{dT} = \frac{|Q|}{\sqrt{2h}} \left\langle \frac{Q}{|Q|}, V \right\rangle = \frac{\mathscr{E}}{2h}. \tag{5.6}$$

Therefore the sign of $d|Q|/d\tau$ coincides with the sign of $\mathscr{E}$.

According to the definition of $\mathfrak{S}_R^-(\varepsilon)$ one has at all points on this set

$$|Q| = R \quad \text{and} \quad \frac{d|Q|}{d\tau} = \frac{1}{\sqrt{2h}} \langle Q, V \rangle < 0.$$

Hence $\mathscr{E}^- < 0$ and, moreover, $|Q| < R$ for $\tau > \tau^-$, sufficiently close to τ^-. Since the first two inequalities which appear in the definition of G hold strictly, the point $p(\tau)$ belongs to G for τ which are greater than τ^- and sufficiently close to it. The

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right-hand side of the system (4.14) is bounded on G due to lemma 4 since h is separated from zero and so either the trajectory remains within int G for all $\tau > \tau^-$ or else $p(\tau) \in \partial G$ for some $\hat{\tau} > \tau^-$ and $p(\tau) \in \operatorname{int} G$ for $\tau \in (\tau^-, \hat{\tau})$.

We shall show that the first possibility does not occur. In fact, let us take $\mathscr{E}_1 > 0$ sufficiently large to satisfy the inequality

$$\operatorname{ch} \mathscr{E}_1 - 1 > 2h^-(1+\varepsilon_1)R.$$

Now, if $|h' - h^-| \leq \varepsilon_1 h^-$ and $|\mathscr{E}'| > \mathscr{E}_1$ then at the point $p' = (\xi_1', \xi_2', \omega', \mathscr{E}', t')$ we have the inequality

$$|Q| = \frac{e \operatorname{ch} \mathscr{E}' - 1}{2h'} \geq \frac{\operatorname{ch} \mathscr{E}_1 - 1}{2h^-(1+\varepsilon_1)} > R.$$

Thus $p' \notin G$ for arbitrary ξ_2', ω', t'. This implies that $|\mathscr{E}(\tau)| \leq \mathscr{E}_1$ in G. By virtue of (5.5) we may also conclude that the trajectory stays in G during the interval $\Delta\tau$ whose length does not exceed $2\mathscr{E}_1(1+\varepsilon_0)$.

Now we shall find conditions which guarantee that the trajectory meets the part of the boundary of G where $|Q(\hat{\tau})| = R$, so that the trajectory entering the R-ball will necessarily leave it. Thus we have to exclude two possibilities:

$$|h(\hat{\tau}) - h^-| = \varepsilon_1 h^- \quad \text{and} \quad |e(\hat{\tau}) = e^-| = \varepsilon_2 e^-$$

for which purpose lemma 7 will be of use.

The requirement that the trajectory segment $[\tau^-, \hat{\tau}]$ belongs to the region G, or, in other words, the requirement that $|Q| \leq R$, together with (5.1) and (5.3), shows that the above lemma applies and gives the following estimates for $|h(\hat{\tau}) - h^-|$ and $|e(\hat{\tau}) - e^-|$. Using (4.13a), the inequality (3.7) and lemma 7, we obtain

$$\begin{aligned}
\frac{1}{h^-}|h(\hat{\tau}) - h^-| &\leq \frac{1}{h^-}\int_{\tau^-}^{\hat{\tau}} \left|\frac{dh}{d\tau}\right| d\tau \leq \frac{1}{h^-}\int_{\tau^-}^{\hat{\tau}} \frac{2(1+2h(\tau)|Q|)}{2h(\tau)} |\Phi(Q, t, \mu)|\, d\tau \\
&\leq \frac{2}{h^-(2h)}\left(\int_{\tau^-}^{\hat{\tau}} (1+2h|Q|)\, d\tau\right) \max_{|Q| \leq R} |\Phi(Q, t, \mu)| \\
&\leq \frac{4}{(2h^-)^2(1-\varepsilon_1)} 2\kappa(1+\varepsilon_0)(1+2\bar{h}R)C_0\mu^2 R \\
&\leq 8\left(\frac{1+\varepsilon_2}{1-\varepsilon_2}\right)\left(\frac{1+\varepsilon_1}{1-\varepsilon_1}\right)(1+\varepsilon_0)\left(\frac{1}{2h^-}+R\right)^2 C_0\mu^2 R. \qquad (5.7)
\end{aligned}$$

Hence, if we choose ε_i to satisfy

$$8\,\frac{1+\varepsilon_2}{1-\varepsilon_2}\,\frac{1+\varepsilon_1}{1-\varepsilon_1}(1+\varepsilon_0)\left(\frac{1}{2h^-}+R\right)^2 C_0\mu^2 R < \varepsilon_1 \qquad (5.8)$$

then the inequality

$$|h(\hat{\tau}) - h^-| < \varepsilon_1 h^-$$

is valid.

According to the assumption (5.2) one has

$$e(\hat{\tau})/e^- \leq 1 + \varepsilon_2$$

and, therefore

$$\left|\frac{e(\hat\tau)}{e^-}-1\right|\le(1+\varepsilon_2)\left|\log\frac{e(\hat\tau)}{e^-}\right|\le(1+\varepsilon_1)\int_{\tau^-}^{\hat\tau}\frac{1}{e}\left|\frac{de}{d\tau}\right|d\tau.$$

As above, but using (4.13*j*) instead of (4.13*a*), one may obtain the following estimate

$$(1+\varepsilon_2)\int_{\tau^-}^{\hat\tau}\frac{1}{e}\left|\frac{de}{d\tau}\right|d\tau\le 4\frac{(1+\varepsilon_2)^2}{1-\varepsilon_2}\left(\frac{1+\varepsilon_1}{1-\varepsilon_1}\right)^2(1+\varepsilon_0)\left(\frac{1}{2h^-}+R\right)^2C_0\mu^2R \quad (5.9)$$

which implies that $|e(\hat\tau)-e^-|<\varepsilon_2 e^-$ as long as

$$4\frac{(1+\varepsilon_2)^2}{1-\varepsilon_2}\left(\frac{1+\varepsilon_1}{1-\varepsilon_1}\right)^2(1+\varepsilon_0)\left(\frac{1}{2h^-}+R\right)^2C_0\mu^2R<\varepsilon_2. \quad (5.10)$$

Comparing the inequalities (5.4), (5.8), (5.10), we find that

$$\left(\frac{1}{2\varepsilon}+R\right)^2C_0\mu^2R$$
$$<\min\left\{\frac{\varepsilon_0}{2(1+\varepsilon_0)}\left(\frac{1-\varepsilon_1}{1+\varepsilon_1}\right)^2,\frac{\varepsilon_1}{8(1+\varepsilon_0)}\frac{1-\varepsilon_1}{1+\varepsilon_1}\frac{1-\varepsilon_2}{1+\varepsilon_2},\frac{\varepsilon_2}{4(1+\varepsilon_0)}\left(\frac{1-\varepsilon_1}{1+\varepsilon_1}\right)^2\frac{1-\varepsilon_2}{(1+\varepsilon_2)^2}\right\}$$

which proves the existence of the Poincaré mapping $\mathfrak{S}_R^-(\varepsilon)\to\mathfrak{S}_R^+$. Specifying the values of ε_i one may make this condition more explicit. For example, if we let

$$\varepsilon_0=\tfrac{1}{17},\quad \varepsilon_1=\varepsilon_2=\tfrac{1}{4} \quad (5.11)$$

then we get the condition

$$\left(\frac{1}{2\varepsilon}+R\right)^2C_0\mu^2R<\tfrac{1}{100}. \quad (5.12)$$

6. *Continuation of the proof of the theorem: The estimates for the perturbation of the Poincaré mapping*

For the point (ξ^-,t^-) of entry into the R-ball, we will denote the parameters characterizing the point of exit from the R-ball for the *unperturbed* trajectory by ξ^+, Q^+, t^+, τ^+ and by $\tilde\xi^+, \tilde Q^+, \tilde t^+, \tilde\tau^+$ for the *perturbed* one. It is clear that

$$\xi_1^+=\xi_1^-,\quad \xi_2^+=\xi_1^-,\quad \omega^+=\omega^-,\quad \mathscr{E}^+=-\mathscr{E}^-,$$
$$h^+=h^-,\quad e^+=e^-,\quad \mathscr{E}^+-\mathscr{E}^-=\tau^+-\tau^-.$$

From now on we assume that the condition (5.12) is fulfilled and ε_i are fixed in accordance with (5.11). The following estimates may be obtained by the same reasoning as in (5.7), (5.9), i.e. by virtue of lemmas 4, 7:

$$|\tilde\xi_1^+-\xi_1^+|=|\tilde\xi_1^+-\xi_1^-|\le 8(1+\varepsilon_0)\frac{1+\varepsilon_1}{(1-\varepsilon_1)^2}\cdot\frac{(1+\varepsilon_2)}{(1-\varepsilon_2)}\left(\frac{1}{2h^-}+R\right)^2C_0\mu^2R$$
$$\le 31.38\left(\frac{1}{2h^-}+R\right)^2C_0\mu^2R; \quad (6.1)$$

$$\left|\frac{\tilde h^+}{h^+}-1\right|\le 23.53\left(\frac{1}{2h^-}+R\right)^2C_0\mu^2R; \quad (6.2)$$

$$\int_{\tau^-}^{\tilde\tau^+}\left|\frac{dh}{d\tau}\right|d\tau<11.77\left(\frac{1}{2h^-}+R\right)C_0\mu^2R; \quad (6.3)$$

$$|\tilde{\xi}_2^+ - \xi_2^+| \le 39.22\left(\frac{1}{2h^-} + R\right)^2 C_0\mu^2 R; \qquad (6.4)$$

$$|\tilde{\omega}^+ - \omega^+| \le 29.42\left(\frac{1}{2h^-} + R\right)^2 C_0\mu^2 R; \qquad (6.5)$$

$$\left|\frac{\tilde{e}^+}{e^+} - 1\right| = \left|\frac{\tilde{e}^+}{e^-} - 1\right| < 24.51\left(\frac{1}{2h^-} + R\right)^2 C_0\mu^2 R; \qquad (6.6)$$

$$\int_{\tau^-}^{\tilde{\tau}^+} \frac{1}{e}\left|\frac{de}{d\tau}\right| d\tau \le 19.61\left(\frac{1}{2h^-} + R\right)^2 C_0\mu^2 R. \qquad (6.7)$$

The quantities $\tilde{\mathscr{E}}^+ - \mathscr{E}^+$ and $\tilde{t}^+ - t^+$ should be estimated in a different way, since $\mathscr{E}$ and t are not fixed along the unperturbed trajectory. Although it does not appear in the theorem, the estimate for $\tilde{\mathscr{E}}^+ - \mathscr{E}^+$ will be derived, as we intend to exploit these results later.

We denote by $\mathscr{E}_R(\tau)$ the value of the eccentric anomaly when an orbit leaves the R-ball at time τ. It is obvious that $\mathscr{E}_R(\tau^-) = \mathscr{E}^+$, $\mathscr{E}_R(\tilde{\tau}^+) = \tilde{\mathscr{E}}^+$. The element $\mathscr{E}_R(\tau)$ is associated with the other ones by the relation

$$\operatorname{ch} \mathscr{E}_R(\tau) = \frac{1 + 2h(\tau)R}{e(\tau)}.$$

The difference $\operatorname{ch} \mathscr{E}_R(\tau) - \operatorname{ch} \mathscr{E}^+$ may be evaluated

$$\operatorname{ch} \mathscr{E}_R(\tau) - \operatorname{ch} \mathscr{E}^+ = \int_{\tau^-}^{\tau} \left[\frac{d}{d\sigma} \operatorname{ch} \mathscr{E}_R(\sigma)\right] d\sigma \quad \tau^- \le \tau \le \tilde{\tau}^+$$

$$= \int_{\tau^-}^{\tau} \left[2\frac{R}{e(\sigma)}\frac{dh}{d\sigma} - \frac{1 + 2h(\sigma)R}{e}\frac{1}{e}\frac{de}{d\sigma}\right] d\sigma \qquad (6.8)$$

and estimated by

$$|\operatorname{ch} \mathscr{E}_R(\tau) - \operatorname{ch} \mathscr{E}^+| \le \frac{2R}{\underline{e}} \int_{\tau^-}^{\tau} \left|\frac{dh}{d\sigma}\right| d\sigma + \frac{1 + 2\bar{h}R}{\underline{e}} \int_{\tau^-}^{\tau} \frac{1}{e}\left|\frac{de}{d\sigma}\right| d\sigma.$$

Using (6.3), (6.7) and $\underline{e} = e^-(1 - \varepsilon_2)$ (see (5.2)) we find

$$|\operatorname{ch} \mathscr{E}_R(\tau) - \operatorname{ch} \mathscr{E}^+| \le \frac{2R}{e^-(1 - \varepsilon_2)} 11.77\left(\frac{1}{2h^-} + R\right) C_0\mu^2 R$$

$$+ \frac{(1 + 2h^- R)}{e^-} \frac{1 + \varepsilon_1}{1 - \varepsilon_2} 19.61\left(\frac{1}{2h^-} + R\right)^2 C_0\mu^2 R.$$

Enlarging the factor $2R$ in the first term up to

$$(1 + 2h^- R)\left(\frac{1}{2h^-} + R\right) > 2R$$

and summing, we obtain

$$|\operatorname{ch} \mathscr{E}_R(\tau) - \operatorname{ch} \mathscr{E}^+| \le \frac{1 + 2h^- R}{e^-} 48.39\left(\frac{1}{2h^-} + R\right)^2 C_0\mu^2 R$$

$$= \operatorname{ch} \mathscr{E}^- 48.39\left(\frac{1}{2h^-} + R\right)^2 C_0\mu^2 R \qquad (6.9)$$

or

$$\left|\frac{\operatorname{ch}\mathscr{E}_R(\tau)}{\operatorname{ch}\mathscr{E}^+}-1\right|\le 48.39\left(\frac{1}{2h^-}+R\right)^2 C_0\mu^2 R. \tag{6.10}$$

Because $\operatorname{ch}\mathscr{E}_R(\tau)-\operatorname{ch}\mathscr{E}^+=\operatorname{sh}\mathscr{E}'(\mathscr{E}_R(\tau)-\mathscr{E}^+)$ holds for some $\mathscr{E}'$, which lies between $\mathscr{E}^+$ and $\mathscr{E}_R(\tau)$ and because $\mathscr{E}^+$ and $\mathscr{E}_R(\tau)$ have the same signs, (6.9) implies

$$|\mathscr{E}_R(t)-\mathscr{E}^+|\le\frac{\operatorname{ch}\mathscr{E}^+}{\min_{\mathscr{E}'=\mathscr{E}^+,\mathscr{E}_R(\tau)}|\operatorname{sh}\mathscr{E}'|}48.39\left(\frac{1}{2h^-}+R\right)^2 C_0\mu^2 R. \tag{6.11}$$

This estimate leaves much to be desired for small $\mathscr{E}$ when

$$\frac{d}{d\mathscr{E}}\operatorname{ch}\mathscr{E}\sim 0,$$

that is for trajectories which just barely meet the R-ball with small entrance/exit angles. However, this case is of little interest, so we exclude it from consideration by imposing the condition

$$|\cos(\widehat{V^-,Q^-})|\ge k \quad (0<k<1) \tag{6.12}$$

on the entrance angle. This restricts the domain of Poincaré mapping to $\mathfrak{S}_R^-(\varepsilon,k)$. We also make the *a priori* assumption

$$\left|\frac{\cos(\widehat{V_R(\tau),Q_R(\tau)})}{\cos(\widehat{V^+,Q^+})}-1\right|\le\tfrac{1}{2} \tag{6.13}$$

about the exit angles of all osculating orbits.

Let us try to formulate a more verifiable condition which provides the validity of (6.13). To achieve this, we give an estimate for

$$\frac{\cos(\widehat{V_R,Q_R})}{\cos(\widehat{V^-,Q^+})}-1.$$

The equations (3.14) and (4.8) and the well-known formula

$$V^2=2h\frac{e\operatorname{ch}\mathscr{E}+1}{e\operatorname{ch}\mathscr{E}-1}, \tag{6.14}$$

(which may be easily obtained from (3.13) and (3.14)) imply that

$$\cos(\widehat{V,Q})=\frac{\langle V,Q\rangle}{|V|\cdot|Q|}=\frac{e\operatorname{sh}\mathscr{E}}{\sqrt{e^2\operatorname{ch}^2\mathscr{E}-1}}. \tag{6.15}$$

It follows that

$$\frac{\cos(\widehat{V_R,Q_R})}{\cos(\widehat{V^+,Q^+})}=\frac{e}{e^+}\frac{\operatorname{sh}\mathscr{E}_R}{\operatorname{sh}\mathscr{E}^+}\sqrt{\frac{h^+}{h}}\sqrt{\frac{e^+\operatorname{ch}\mathscr{E}^++1}{e\operatorname{ch}\mathscr{E}_R+1}}. \tag{6.16}$$

(We have taken into account here that $\dfrac{e^+\operatorname{ch}\mathscr{E}^+-1}{2h^+}=|Q^+|=\dfrac{e\operatorname{ch}\mathscr{E}_R-1}{2h}=R.$)

The difference of the first factor and 1 is given by (6.6). To estimate the difference of the third one and 1, we apply

$$\left|1-\frac{h^+}{h}\right|<\frac{1}{1-\varepsilon_1}$$

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(see (5.1)) and (6.2):

$$\left|\sqrt{\frac{h^+}{h}}-1\right|<\frac{1}{2}\frac{1}{\sqrt{1-\varepsilon_1/(1-\varepsilon_1)}}\frac{1}{1-\varepsilon_1}\left|\frac{h}{h^+}-1\right|$$
$$\leq 19.22\left(\frac{1}{2h^-}+R\right)^2 C_0\mu^2 R. \tag{6.17}$$

We obtain the estimate for the fourth factor from

$$\left|\frac{e^+\operatorname{ch}\mathscr{E}^++1}{e\operatorname{ch}\mathscr{E}_R+1}-1\right|=\left|\frac{1+2h^+|Q^+|-1-2hR}{2+2hR}\right|\leq\frac{|2h^+-2h|R}{2hR}=\left|\frac{h^+}{h}-1\right|$$
$$\leq\frac{1}{1-\varepsilon_1}\left|\frac{h}{h^+}-1\right|$$

having in mind $|(h/h^+)-1|<\varepsilon$:

$$\left|\sqrt{\frac{e^+\operatorname{ch}\mathscr{E}^++1}{e\operatorname{ch}\mathscr{E}_R+1}}-1\right|\leq\frac{1}{2}\frac{1}{\sqrt{1-\varepsilon_1/(1-\varepsilon_1)}}\frac{1}{1-\varepsilon_1}\left|\frac{h}{h^+}-1\right|$$
$$\leq 19.22\left(\frac{1}{2h^-}+R\right)^2 C_0\mu^2 R. \tag{6.18}$$

It remains to estimate the difference between the factor $\operatorname{sh}\mathscr{E}_R/\operatorname{sh}\mathscr{E}^+$ in (6.16) and 1. Applying the Cauchy formula and inequality (6.9), we obtain

$$\left|\frac{\operatorname{sh}\mathscr{E}_R(\tau)}{\operatorname{sh}\mathscr{E}^+}-1\right|=\frac{1}{|\operatorname{sh}\mathscr{E}^+|}\left|\frac{\operatorname{sh}\mathscr{E}_R-\operatorname{sh}\mathscr{E}^+}{\operatorname{ch}\mathscr{E}_R-\operatorname{ch}\mathscr{E}^+}\right||\operatorname{ch}\mathscr{E}_R-\operatorname{ch}\mathscr{E}^+|$$
$$\leq\frac{\operatorname{ch}\mathscr{E}^+}{\operatorname{sh}\mathscr{E}^+}\left|\frac{\operatorname{ch}\mathscr{E}'}{\operatorname{sh}\mathscr{E}'}\right|\cdot 48.39\left(\frac{1}{2h^-}+R\right)^2 C_0\mu^2 R \tag{6.19}$$

with $\mathscr{E}'\in[\mathscr{E}^+,\mathscr{E}_R(\tau)]$. We rewrite the inequality (6.12), squaring both its parts and taking into account (6.15),

$$\frac{(e^-)^2\operatorname{sh}^2\mathscr{E}^-}{(e^-)^2\operatorname{ch}^2\mathscr{E}^--1}\geq k^2.$$

This implies

$$\frac{(e^-)^2\operatorname{sh}^2\mathscr{E}^-}{(e^-)^2\operatorname{ch}^2\mathscr{E}^-}\geq k^2\left(\frac{e^-\operatorname{ch}\mathscr{E}^--1}{e^-\operatorname{ch}\mathscr{E}^-}\right)\left(\frac{e^-\operatorname{ch}\mathscr{E}^-+1}{e^-\operatorname{ch}\mathscr{E}^-}\right)>k^2\left(\frac{e^-\operatorname{ch}\mathscr{E}^--1}{e^-\operatorname{ch}\mathscr{E}^-}\right)$$

and

$$\frac{\operatorname{ch}^2\mathscr{E}^+}{\operatorname{sh}^2\mathscr{E}^+}=\frac{\operatorname{ch}^2\mathscr{E}^-}{\operatorname{sh}^2\mathscr{E}^-}\leq\frac{1}{k^2}\frac{1+2h^-|Q|}{2h^-|Q|}=\frac{1}{k^2}\frac{((1/2h^-)+R)}{R}. \tag{6.20}$$

In a similar way, via assumption (6.13), we obtain

$$\frac{\operatorname{ch}^2\mathscr{E}'}{\operatorname{sh}^2\mathscr{E}'}=\frac{\operatorname{ch}^2\mathscr{E}_R(\tau')}{\operatorname{sh}^2\mathscr{E}_R(\tau)}\leq\left(\frac{2}{k}\right)^2\frac{((1/2h(\tau))+R)}{R}$$
$$\leq\frac{4}{1-\varepsilon_1}\frac{1}{k^2}\frac{((1/2h^-)+R)}{R} \tag{6.21}$$

where τ' is chosen to satisfy $\mathscr{E}'=\mathscr{E}_R(\tau')$. Taking into account (6.20) and (6.21), we

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obtain from (6.19)

$$\left|\frac{\operatorname{sh}\mathscr{E}_R(\tau)}{\operatorname{sh}\mathscr{E}^+}-1\right|\leq\frac{2\cdot 48.39}{\sqrt{1-\varepsilon_1}}\,\frac{1}{k^2}\left(\frac{1}{2h^-}+R\right)^3C_0\mu^2$$
$$\leq\frac{111.76}{k^2}\left(\frac{1}{2h^-}+R\right)^3C_0\mu^2. \tag{6.22}$$

Now, having obtained the estimates of the differences of every factor in (6.16) and 1, we may estimate the difference for the whole product. To this end we use the elementary inequality

$$\left|\prod_{i=1}^{n}(1+\beta_i)-1\right|<\max\left\{\exp\left(\sum_{i=1}^{n}\alpha_i\right)\sum_{i=1}^{n}\alpha_i,\ \sum_{i=1}^{n}\frac{\alpha_i}{1-\alpha_i}\right\} \tag{6.23}$$

which is valid when $|\beta_i|\leq\alpha_i<1$. We assume

$$\beta_i=\frac{\operatorname{sh}\mathscr{E}_R}{\operatorname{sh}\mathscr{E}^+}-1,\quad \beta_2=\frac{e}{e^+}-1,$$
$$\beta_3=\sqrt{\frac{h^+}{h}}-1,\quad \beta_4=\sqrt{\frac{e^+\operatorname{ch}\mathscr{E}^++1}{e\operatorname{ch}\mathscr{E}_R+1}} \tag{6.24}$$

and

$$\alpha_1=111.76\,\frac{1}{k^2}\left(\frac{1}{2h^-}+R\right)^3C_0\mu^2,\quad \alpha_2=24.51\,\frac{1}{k^2}\left(\frac{1}{2h^-}+R\right)^3C_0\mu^2,$$
$$\alpha_3=\alpha_4=19.22\,\frac{1}{k^2}\left(\frac{1}{2h^-}+R\right)^3C_0\mu^2.$$

The inequalities $|\beta_i|\leq\alpha_i$ are valid as long as (6.13) holds, as one may see from (6.22), (6.16), (6.17), and (6.18). That $\alpha_i<1$ may be regarded as an independent condition. If

$$\left(\frac{1}{2\varepsilon}+R\right)^3C_0\mu^2\leq\frac{k^2}{500} \tag{6.25}$$

this condition is certainly met. Thus the *a posteriori* estimate of the ratio of sines of exit angles (see (6.16), (6.23), (6.24))

$$\left|\frac{\cos(\widehat{V_R,Q_R})}{\cos(\widehat{V^+,Q^+})}-1\right|=\left|\prod_{i=1}^{4}(1+\beta_i)-1\right|$$
$$\leq 247.95\,\frac{1}{k^2}\left(\frac{1}{2h^-}+R\right)^3C_0\mu^2\leq\frac{247.95}{500} \tag{6.26}$$

is stronger than the *a priori* estimate (6.13). As the 'continuous induction principle' states (see, e.g. [**1**]), if for a continuous function (

$$\frac{\cos(\widehat{V_R(\tau),Q_R(\tau)})}{\cos(\widehat{V^+,Q^+})}$$

here) of a scalar argument (τ here), the *a posteriori* estimate is stronger than the *a priori* one, then both estimates are valid. Therefore, if (6.25) is fulfilled, then both (6.13) and its consequences (6.21), (6.22), (6.26) are valid.

To finish the estimation of perturbation of $\mathscr{E}^+$ started by (6.11) we must use the inequality

$$\frac{\operatorname{sh}\mathscr{E}^+}{\min\limits_{\mathscr{E}'=\tilde{\mathscr{E}}^+,\,\mathscr{E}^+}\operatorname{sh}\mathscr{E}'}<1.29,$$

which is a consequence of (6.22), (6.25). Moreover, (6.20) gives

$$\begin{aligned}|\tilde{\mathscr{E}}^+-\mathscr{E}^+|&\le\frac{\operatorname{ch}\mathscr{E}^+}{\operatorname{sh}\mathscr{E}^+}\frac{\mathscr{E}}{\min\limits_{\mathscr{E}'=\tilde{\mathscr{E}}^+,\,\mathscr{E}^+}\operatorname{sh}\mathscr{E}'}\cdot 48.39\Big(\frac{1}{2h^-}+R\Big)^2C_0\mu^2R\\&\le\frac{1}{k}\frac{\sqrt{(1/2h^-)+R}}{\sqrt{R}}62.43\Big(\frac{1}{2h^-}+R\Big)^2C_0\mu^2R\\&\le 62.43\frac{1}{k}\Big(\frac{1}{2h^-}+R\Big)^3C_0\mu^2.\end{aligned}$$

It remains to estimate the perturbations of the exit time from the R-ball. Let us consider once more the Kepler orbit osculating about the perturbed one at the instant τ, which corresponds to the instant $t(\tau)$ of the physical (unreduced) time. By the standard formula for Kepler motion (it can be easily obtained also by integrating (4.2), (4.3) with $\Phi=0$) the physical elapsed time between the instants of osculation and the exit from the R-ball if the comet moves along the Kepler osculating orbit, is described by the relation

$$t_R(\tau)-t(\tau)=\mu\frac{e(\tau)\operatorname{sh}\mathscr{E}_R-\mathscr{E}_R(\tau)}{(2h(\tau))^{3/2}}-\mu\frac{e(\tau)\operatorname{sh}\mathscr{E}(\tau)-\mathscr{E}(\tau)}{(2h(\tau))^{3/2}}. \tag{6.27}$$

Naturally, $t_R(\tau^-)=t^+$, $t_R(\tilde{\tau}^+)=\tilde{t}^+$, which implies

$$\tilde{t}^+-t^+=\int_{\tau^-}^{\tau^+}\frac{d}{d\tau}t_R(\tau)\,d\tau. \tag{6.28}$$

The derivative, entering here as integrand, is evaluated using (6.27), taking into account the fact that

$$\frac{d}{dt}t(\tau)=\frac{\mu e(\tau)\operatorname{ch}\mathscr{E}(\tau)-1}{(2h(\tau))^{3/2}}:$$

$$\begin{aligned}\frac{d}{d\tau}t_R&=\mu\frac{e\operatorname{ch}\mathscr{E}-1}{(2h)^{3/2}}-\frac{d}{d\tau}\Big[\mu\frac{e\operatorname{sh}\mathscr{E}-\mathscr{E}}{(2h)^{3/2}}\Big]+\frac{d}{d\tau}\Big[\frac{\operatorname{sh}\mathscr{E}_R-\mathscr{E}_R}{(2h)^{3/2}}\Big]\\&=\mu\frac{e\operatorname{ch}\mathscr{E}-1}{(2h)^{3/2}}\Big(1-\frac{d\mathscr{E}}{d\tau}\Big)-\mu\frac{e\operatorname{sh}\mathscr{E}}{(2h)^{3/2}}\frac{1}{e}\frac{de}{d\tau}\\&\quad+3\mu\frac{e\operatorname{sh}\mathscr{E}-\mathscr{E}}{(2h)^{3/2}}\frac{1}{2h}\frac{dh}{d\tau}+\mu\frac{e\operatorname{ch}\mathscr{E}_R-1}{(2h)^{3/2}}\frac{d\mathscr{E}_R}{d\tau}\\&\quad+\mu\frac{e\operatorname{sh}\mathscr{E}_R}{(2h)^{3/2}}\frac{1}{e}\frac{de}{d\tau}-3\mu\frac{e\operatorname{sh}\mathscr{E}_R-\mathscr{E}_R}{(2h)^{3/2}}\frac{1}{2k}\frac{dh}{d\tau}.\end{aligned} \tag{6.29}$$

The fourth term contains the quantity $d\mathscr{E}_R/d\tau$, which has not figured in our estimates yet. From the equality of integrands in (6.8) we note that

$$\operatorname{sh}\mathscr{E}_R\frac{d\mathscr{E}_R}{d\tau}=2\frac{R}{e}\frac{dh}{d\tau}-\frac{1+2hR}{e}\frac{1}{e}\frac{de}{d\tau}.$$

This, together with (6.21), (4.13(*a*), (*j*)) yields

$$\begin{aligned}\left|\frac{d\mathscr{E}_R}{d\tau}\right|&<\frac{(1+2hR)(1/2h+R)}{e\operatorname{sh}\mathscr{E}_R}\cdot 2\frac{(1+2h|Q|)}{2h}|\Phi|\\&\quad+\frac{1+2hR}{e\operatorname{sh}\mathscr{E}_R}\frac{2(1+2h|Q|)^2}{(2h)^2}|\Phi|\\&\le\frac{2}{\sqrt{1-\varepsilon_1}}\frac{1}{k}\sqrt{\frac{(1/2h^-)+R}{R}}\cdot 4\frac{(1+2hR)(1+2h|Q|)}{(2h)^2}C_0\mu^2R\\&\le 9.24\frac{1}{k}\left(\frac{1}{2h^-}+R\right)\frac{(1+2hR)(1+2h|Q|)}{(2h)^2}C_0\mu^2.\end{aligned}\tag{6.30}$$

Finally, by virtue of lemma 4, we may estimate the sum of all terms in (6.29) except the fourth one by

$$\begin{aligned}&\frac{18\mu}{\sqrt{2h^-}}\frac{(1+2hR)(1+2h|Q|)^2}{(2h)^3}C_0\mu^2R|\tilde{t}^+-t^+|\\&\le\int_{\tau^-}^{\tau^+}\left|\frac{d}{dt}t_R(\tau)\right|d\tau\\&\le\left(18\frac{1+\varepsilon_1}{1-\varepsilon_1}+9.24\right)\left(\frac{1+\varepsilon_1}{1-\varepsilon_1}\right)^2\cdot 2\frac{1+\varepsilon_2}{1-\varepsilon_2}\frac{(1+\varepsilon_R)}{\sqrt{1-\varepsilon_1}}\frac{\mu}{\sqrt{2h^-}}\frac{1}{k}\left(\frac{1}{2h^-}+R\right)^3C_0\mu^2R\\&\le 444.28\frac{\mu}{\sqrt{2h^-}}\frac{1}{k}\left(\frac{1}{2h^-}+R\right)^3C_0\mu^2R.\end{aligned}\tag{6.31}$$

7. *The end of the proof: estimates of perturbation of derivative of the Poincaré mapping*

Let us evaluate the derivative of the Poincaré mapping for the perturbed and unperturbed flows in the coordinates (ξ, t)† at an arbitrary point

$$(\xi_0^-, \mathscr{E}_R^-(\xi_0^-), t_0^-)\in\mathfrak{S}_R^-.$$

This mapping $(\xi^-, t^-)\to(\tilde{\xi}^+, \tilde{t}^+)$ can be represented as a composition of four mappings:

(1) The inclusion

$$\iota:\mathbb{R}\times T\mathbb{S}^2\times\mathbb{R}\supset U\to\mathfrak{S}_R^-(\varepsilon,k),\quad \iota:(\xi^-,t^-)\mapsto(\xi^-,\mathscr{E}_R^-(\xi^-),t^-).$$

(2) The shift $\tilde{\theta}$ (θ resp.) along the integral curves of $\tilde{\mathfrak{X}}$ ($\mathfrak{X}$ resp.) for the fixed 'time' $\tilde{\tau}_0^+-\tau_0^-$ ($\tau_0^+-\tau_0^-$ resp.) where

$$\xi(\tau_0^-)=\xi_0^-,\quad t(\tau_0^-)=t_0^-$$

and

$$(\tilde{\xi}(\tilde{\tau}_0^+),\tilde{\mathscr{E}}(\tilde{\tau}_0^+),\tilde{t}(\tilde{\tau}_0^+))\in\mathfrak{S}_R^+\quad((\xi(\tau_0^+),\mathscr{E}(\tau_0^+),t(\tau_0^+))\in\mathfrak{S}_R^+\text{ resp.});$$

† The term 'coordinates' is a rather relative one: ξ runs over a region in $\mathbb{R}\times T\mathbb{S}^2$.

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(3) The projection along the integral curves of $\tilde{\mathfrak{X}}$ ($\mathfrak{X}$ resp.) onto the hypersurface $\mathfrak{S}_R^+$;

(4) The projection $\mathrm{pr}_{(\xi,t)}$ of $\mathfrak{M}$ onto (ξ, t)-space $\mathbb{R}\times T\mathbb{S}^2\times\mathbb{R}$.

The composition of the two latter mappings we denote by $\tilde{\pi}$ (resp. π). We shall write the derivatives in matrix form, using the fact that $\mathfrak{M}$ is embedded in the ambient Euclidean space.

One may easily evaluate the derivative $\mathscr{I}$ of ι

$$\mathscr{I} = {}^T\begin{pmatrix} \mathrm{id}_\xi & \dfrac{\partial \mathscr{E}_R^-}{\partial \xi} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

by finding $\partial\mathscr{E}_R^-/\partial\xi$ as the derivative of an implicit function from the relation $\Psi(\xi, \mathscr{E}_R^-) = |Q(\xi, \mathscr{E}_R^-)| - R = 0$:

$$\frac{\partial \mathscr{E}_R^-}{\partial \xi} = -\left(\frac{\partial\Psi}{\partial\mathscr{E}}\right)^{-1}\frac{\partial\Psi}{\partial\xi} = -\frac{2h}{e\,\mathrm{sh}\,\mathscr{E}}\left(-\frac{e\,\mathrm{ch}\,\mathscr{E}-1}{2h}, \frac{{}^T\xi_2\,\mathrm{ch}\,\mathscr{E}}{2he}, 0\right).$$

The first component of the latter can be estimated in $\mathfrak{S}_R^-(\varepsilon, k)$ by means of (4.8)

$$\left|\frac{e\,\mathrm{ch}\,\mathscr{E}-1}{e\,\mathrm{sh}\,\mathscr{E}}\right| = \frac{\sqrt{2h}|Q|}{|\langle V, Q\rangle|} - \frac{|V|\cdot|Q|}{|\langle V, Q\rangle|}\frac{\sqrt{2h}}{|V|} \le \frac{1}{k}$$

and the second by means of (4.8), (6.14) and (3.14)

$$\left|\frac{{}^T\xi_2\,\mathrm{ch}\,\mathscr{E}}{e^2\,\mathrm{sh}\,\mathscr{E}}\right| = \frac{|V|\cdot|Q|}{|\langle V, Q\rangle|}\frac{|\xi_2|\,\mathrm{ch}\,\mathscr{E}}{e\sqrt{2h}|V||Q|}$$
$$\le \frac{1}{k}\frac{|\xi_2|}{e\sqrt{i+\xi_2^2-(1/\mathrm{ch}^2\,\mathscr{E})}} \le \frac{1}{k}\frac{|\xi_2|}{e\sqrt{1+\xi_2^2-1}} \le \frac{1}{k}.$$

We can write, therefore, the majorant for $\mathscr{I}$ in (ξ, ξ)-, (ξ, t)-, (t, ξ)-, and (t, t)-components

$$\mathscr{I} \prec \begin{pmatrix} \sqrt{1+2/k^2} & 0 \\ 0 & 1 \end{pmatrix} \prec \begin{pmatrix} 1.74/k & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.1}$$

The derivative $\tilde{\Theta}$ of $\tilde{\theta}$ is given by the solution of the equations of variation for (4.14):

$$\begin{aligned} \frac{d}{dt}(\delta\tilde{\xi}) &= \frac{\partial A(\tilde{\xi})}{\partial \xi}(\delta\tilde{\xi})\Phi(Q(\tilde{\xi}), \tilde{t}, \mu) + A(\tilde{\xi})\frac{\partial\Phi}{\partial Q}\frac{\partial Q}{\partial\xi}(\delta\tilde{\xi}) + A(\tilde{\xi})\frac{\partial\Phi}{\partial t}(\delta\tilde{t}) \\ \frac{d}{dt}(\delta\tilde{t}) &= \mu\frac{\partial B}{\partial\xi}(\delta\tilde{\xi}) \end{aligned} \tag{7.2}$$

along the trajectory $(\tilde{\xi}(\tau), \tilde{t}(\tau))$ in the interval $(\tau^-, \tilde{\tau}^+)$. The derivative Θ of θ is the solution of the same system with $\Phi = 0$ along the trajectory $(\xi(\tau), t(\tau))$ in the interval (τ^-, τ^+). Since

$$\delta\xi(\tau^+) = \delta\xi(\tau^-),$$

it is sufficient to estimate

$$\delta\tilde{\xi}(\tilde{\tau}^+) - \delta\tilde{\xi}(\tau^-) = \delta\tilde{\xi}(\tilde{\tau}^+) - \delta\xi(\tau^+)$$

in order to estimate the perturbation of the derivative of the coordinates ξ. As for the estimate of the variation of t, we shall not need it.

Let us rewrite the system (7.2) for the sake of brevity in the form

$$\begin{aligned} \frac{d}{d\tau}(\delta\xi) &= \alpha(\tau)\delta\xi + \beta(\tau)\delta t \\ \frac{d}{d\tau}(\delta t) &= \gamma(\tau)\delta\xi \end{aligned} \tag{7.3}$$

and integrate it

$$\begin{aligned} \delta\tilde{\xi}(\tau) &= \delta\tilde{\xi}(\tau^-) + \int_{\tau^-}^{\tau} \alpha(\sigma)\delta\tilde{\xi}(\sigma)\,d\sigma + \int_{\tau-}^{\tau} \beta(\sigma)\left\{\delta\tilde{t}(\tau^-) + \int_{\tau^-}^{\sigma} \gamma(s)\delta\tilde{\xi}(s)\,ds\right\} d\sigma \\ &= \delta\tilde{\xi}(\tau^-) + \left[\int_{\tau^-}^{\tau} \beta(\sigma)\,d\sigma\right]\delta\tilde{t}(\tau^-) + \int_{\tau^-}^{\tau}\left[\alpha(s) + \left[\int_s^{\tau}\beta(\sigma)\,d\sigma\right]\gamma(s)\right]\delta\tilde{\xi}(s)\,ds. \end{aligned}$$

Introducing the notations

$$\begin{aligned} &\mathscr{X}(\tau) = \|\delta\tilde{\xi}(\tau) - \delta\tilde{\xi}(\tau^-)\|, \\ &\mathscr{A} = \int_{\tau^-}^{\tau^+} \|\alpha(s)\|\,ds, \quad \mathscr{B} = \int_{\tau^-}^{\tau^+} \|\beta(s)\|\,ds, \quad \Gamma = \int_{\tau^-}^{\tau^+} \|\gamma(s)\|\,ds \end{aligned}$$

we write down the integral inequality for $\mathscr{X}(\tau)$:

$$\begin{aligned} \mathscr{X}(\tau) \le &\left[\int_{\tau^-}^{\tau}\left\{\|\alpha(s) + \left[\int_s^{\tau}\|\beta(\sigma)\|d\sigma\right]\|\gamma(s)\|\right\} ds\right]\|\delta\tilde{\xi}(\tau^-)\| \\ &+ \left[\int_{\tau^-}^{\tau}\|\beta(\sigma)\|\,d\sigma\right]\|\delta\tilde{t}(\tau^-)\| \\ &+ \int_{\tau^-}^{\tau}\left\{\|\alpha(s)\| + \left[\int_s^{\tau}\|\beta(\sigma)\|\,d\sigma\right]\|\gamma(s)\|\right\}\mathscr{X}(s)\,ds \\ \le\, &(\mathscr{A} + \mathscr{B}\Gamma)\|\delta\tilde{\xi}(\tau^-)\| + \mathscr{B}\|\delta t(\tau^-)\| + \int_{\tau^-}^{\tau}\{\|\alpha(s)\| + \mathscr{B}\|\gamma(s)\|\}\mathscr{X}(s)\,ds. \end{aligned}$$

Furthɜr, applying Gronwall's lemma, we obtain

$$\begin{aligned} \|\delta\tilde{\xi}(\tau) - \delta\tilde{\xi}(\tau^-)\| &= \mathscr{X}(\tau) \\ &\le \{(\mathscr{A} + \mathscr{B}\Gamma)\|\delta\tilde{\xi}(\tau^-)\| + \mathscr{B}\|\delta\tilde{t}(\tau^-)\|\}\exp(\mathscr{A} + \mathscr{B}\Gamma). \end{aligned} \tag{7.4}$$

To estimate $\mathscr{A}$, $\mathscr{B}$, Γ at $\tau^- \le \tau \le \tilde{\tau}^+$ we must use lemmas 6 and 7

$$\begin{aligned} \mathscr{A} &= \int_{\tau^-}^{\tilde{\tau}^+}\|\alpha(\tau)\|\,d\tau \le \int_{\tau^-}^{\tilde{\tau}^+}\left[\left\|\frac{\partial\mathscr{A}(\xi)}{\partial\xi}\right\|\cdot\|\Phi\| + \|\mathscr{A}(\xi)\|\cdot\left\|\frac{\partial\Phi}{\partial Q}\right\|\cdot\left\|\frac{\partial Q}{\partial\xi}\right\|\right] d\tau \\ &\le \left\{2\kappa(1+\varepsilon_0)\left[\left(\frac{1+\varepsilon_1}{1-\varepsilon_1}\right)^2 28.11 + \left(\frac{1+\varepsilon_1}{1-\varepsilon_1}\right)^3 6.71\cdot 4.25\right]\right\}\left(\frac{1}{2h^-} + R\right)^3 C_0\mu^2 \\ &\le 741.57\left(\frac{1}{2h^-} + R\right)^3 C_0\mu^2; \end{aligned}$$

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$$\mathcal{B}=\int_{\tau^-}^{\tilde{\tau}^+}\|\beta(\tau)\|\,d\tau\le\int_{\tau^-}^{\tilde{\tau}^+}\|\mathcal{A}(\xi)\|\cdot\left\|\frac{\partial\Phi}{\partial t}\right\|d\tau$$

$$\le\left[2\kappa(1+\varepsilon_0)\left(\frac{1+\varepsilon_1}{1-\varepsilon_1}\right)^2 6.71\right]\left(\frac{1}{2h^-}+R\right)^2 C_2\mu^2R$$

$$\le 65.79\left(\frac{1}{2h^-}+R\right)^2 C_2\mu^2R;$$

$$\Gamma=\int_{\tau^-}^{\tilde{\tau}^+}\|\gamma(\tau)\|\,d\tau\le\mu\int_{\tau^-}^{\tilde{\tau}^+}\left\|\frac{\partial\mathcal{B}}{\partial\xi}\right\|d\tau$$

$$\le\left[2\kappa(1+\varepsilon_0)\frac{1+\varepsilon_1}{(1-\varepsilon_1)^{\frac{3}{2}}}\cdot 2.07\right]\left(\frac{1}{2h^-}+R\right)\frac{\mu}{\sqrt{2h^-}}$$

$$\le 14.06\left(\frac{1}{2h^-}+R\right)\frac{\mu}{\sqrt{2h^-}}.$$

Imposing the condition

$$\mathcal{A}+\mathcal{B}\Gamma\le 741.57\left(\frac{1}{2h^-}+R\right)^3 C_0\mu^2+925.01\left(\frac{1}{2h^-}+R\right)^3 C_2\frac{\mu^3R}{\sqrt{2h^-}}$$

$$<\frac{925.01}{500} \tag{7.5}$$

or, the stronger one,

$$\mu^2\left(\frac{1}{2\varepsilon}+R\right)^3\left(\frac{C_0}{k^2}+\frac{\mu R}{\sqrt{2\varepsilon}}C_2\right)<\frac{1}{500} \tag{7.6}$$

we obtain from (7.4) the estimates of the variation ($\exp(A+B\Gamma)\le 6.36$):

$$\left\|\frac{\partial\tilde{\xi}(\tilde{\tau}^+)}{\partial\tilde{\xi}(\tau^-)}-\frac{\partial\xi(\tau^+)}{\partial\xi(\tau^-)}\right\|=\left\|\frac{\partial\tilde{\xi}(\tilde{\tau}^+)}{\partial\tilde{\xi}(\tau^-)}-\mathrm{id}\right\|\le 5884\mu^2\left(\frac{1}{2h^-}+R\right)^3\left(C_0+\frac{\mu R}{\sqrt{2h^-}}C_2\right). \tag{7.7}$$

$$\left\|\frac{\partial\tilde{\xi}(\tilde{\tau}^+)}{\partial\tilde{t}(\tau^-)}-\frac{\partial\xi(\tau^+)}{\partial t(\tau^-)}\right\|=\left\|\frac{\partial\tilde{\xi}(\tilde{\tau}^+)}{\partial\tilde{t}(\tau^-)}\right\|\le B\exp(A+B\Gamma)\le 419\left(\frac{1}{2h^-}+R\right)^2 C_2\mu^2R. \tag{7.8}$$

It is easy to see that the derivatives Π and $\tilde{\Pi}$ of the mappings π, $\tilde{\pi}$ are the composition of projection pr onto (ξ, t)-space and respectively of the projections onto the hyperplanes tangent to $\Psi=0$ along the vectors $\mathfrak{X}(\xi^+, t^+)$, $\tilde{\mathfrak{X}}(\tilde{\xi}^+, \tilde{t}^+)$:

$$\Pi=\mathrm{pr}-\frac{\hat{\mathfrak{X}}\otimes\dfrac{\partial\Psi}{\partial(\xi,t)}}{\left\langle\mathfrak{X},\dfrac{\partial\Psi}{\partial(\xi,t)}\right\rangle},\qquad\tilde{\Pi}=\mathrm{pr}-\frac{\hat{\tilde{\mathfrak{X}}}\otimes\dfrac{\partial\tilde{\Psi}}{\partial(\xi,t)}}{\left\langle\tilde{\mathfrak{X}},\dfrac{\partial\Psi}{\partial(\xi,t)}\right\rangle}. \tag{7.9}$$

We have used here the simpler notation

$$\frac{\partial\tilde{\Psi}}{\partial(\xi,t)}=\frac{\partial\Psi(\tilde{\xi}^+,\tilde{t}^+)}{\partial(\xi,t)}.$$

All objects appearing in the first (resp., second) formula are taken at the point (ξ^+, t^+) (resp., $(\tilde{\xi}^+, \tilde{t}^+)$). The cap $\hat{}$, as in lemma 4 denotes the projection into (ξ, t)-space.

Let us transform $\tilde{\Pi}-\Pi$ to the form convenient for estimation

$$\tilde{\Pi}-\Pi=\frac{\hat{\tilde{\mathfrak{X}}}\otimes\dfrac{\partial\tilde{\Psi}}{\partial(\xi,t)}}{\left\langle\tilde{\mathfrak{X}},\dfrac{\partial\tilde{\Psi}}{\partial(\xi,t)}\right\rangle}\left(\frac{\left\langle\tilde{\mathfrak{X}},\dfrac{\partial\tilde{\Psi}}{\partial(\xi,t)}\right\rangle}{\left\langle\tilde{\mathfrak{X}},\dfrac{\partial\Psi}{\partial(\xi,t)}\right\rangle}-1\right)$$

$$-\frac{1}{\left\langle\mathfrak{X},\dfrac{\partial\Psi}{\partial(\xi,t)}\right\rangle}(\hat{\tilde{\mathfrak{X}}}-\hat{\mathfrak{X}})\otimes\frac{\partial\tilde{\Psi}}{\partial(\xi,t)}-\frac{\hat{\mathfrak{X}}}{\left\langle\mathfrak{X},\dfrac{\partial\Psi}{\partial(\xi,t)}\right\rangle}\otimes\left(\frac{\partial\tilde{\Psi}}{\partial(\xi,t)}-\frac{\partial\Psi}{\partial(\xi,t)}\right). \tag{7.10}$$

By virtue of (5.6), (4.8) and (6.12) we obtain

$$\left|\left\langle\mathfrak{X},\frac{\partial\Psi}{\partial(\xi,t)}\right\rangle\right|^{-1}=\left|\frac{2h^{+}}{e^{+}\operatorname{sh}\mathscr{E}^{+}}\right|\le\frac{1}{|Q^{+}|\cos(\widehat{V^{+},Q^{+}})}<\frac{1}{kR}. \tag{7.11a}$$

Similarly, (see (6.13)), we have

$$\left|\left\langle\tilde{\mathfrak{X}},\frac{\partial\tilde{\Psi}}{\partial(\xi,t)}\right\rangle\right|^{-1}<\frac{2}{kR}. \tag{7.11b}$$

Below we shall use the notation

$$a=\left(\frac{1}{2h^{-}}+R\right)^{3}C_0\mu^2,\quad b=\left(\frac{1}{2h^{-}}+R\right)^{2}C_2\mu^2R,\quad g=\left(\frac{1}{2h^{-}}+R\right)\frac{\mu}{\sqrt{2h^{-}}}.$$

The matrices (7.9) will be considered as consisting of (ξ,ξ)-, (t,ξ)-, (ξ,t)-, (t,t)-components. It follows from (7.11), (4.16*d*) and (4.13*i*) that

$$\frac{\hat{\tilde{\mathfrak{X}}}\otimes\dfrac{\partial\Psi}{\partial(\xi,t)}}{\left\langle\tilde{\mathfrak{X}},\dfrac{\partial\tilde{\Psi}}{\partial(\xi,t)}\right\rangle}\prec\frac{1}{k}\begin{pmatrix}52.91a & 0\\ 5.36g & 0\end{pmatrix}. \tag{7.12}$$

Hence, by means of (7.6), we have

$$\tilde{\Pi}\prec\begin{pmatrix}1.11 & 0\\ 5.36\,g/k & 0\end{pmatrix}. \tag{7.13}$$

Since

$$\left|\frac{\left\langle\tilde{\mathfrak{X}},\dfrac{\partial\tilde{\Psi}}{\partial(\xi,t)}\right\rangle}{\left\langle\mathfrak{X},\dfrac{\partial\Psi}{\partial(\xi,t)}\right\rangle}-1\right|=\left|\frac{\tilde{e}^{+}}{e^{+}}\cdot\frac{h^{+}}{\tilde{h}^{+}}\cdot\frac{\operatorname{sh}\tilde{\mathscr{E}}^{+}}{\operatorname{sh}\mathscr{E}^{+}}-1\right|,$$

we obtain by virtue of (6.23), using (6.6), (6.2), (6.22) and (7.6):

$$\left|\frac{\left\langle\mathfrak{X},\dfrac{\partial\tilde{\Psi}}{\partial(\xi,t)}\right\rangle}{\left\langle\mathfrak{X},\dfrac{\partial\Psi}{\partial(\xi,t)}\right\rangle}-1\right|\le 235.55\,\frac{a}{k^2}.$$

Multiplying (7.12) by this estimate we obtain a majorant for the first term in (7.9)

$$\frac{\left\langle \mathring{\tilde{\mathfrak{X}}} \otimes \frac{\partial \tilde{\Psi}}{\partial(\xi, t)} \right\rangle}{\left\langle \tilde{\mathfrak{X}}, \frac{\partial \tilde{\Psi}}{\partial(\xi, t)} \right\rangle} \left(\frac{\left\langle \tilde{\mathfrak{X}}, \frac{\partial \tilde{\Psi}}{\partial(\xi, t)} \right\rangle}{\left\langle \mathfrak{X}, \frac{\partial \Psi}{\partial(\xi, t)} \right\rangle} - 1 \right) \prec \frac{a}{k^3} \begin{pmatrix} 12463a & 0 \\ 1263g & 0 \end{pmatrix}. \tag{7.14}$$

The t-component of

$$\mathring{\tilde{\mathfrak{X}}}(\tilde{\xi}^+, t^+) - \hat{\mathfrak{X}}(\xi^+, t^+)$$

is equal to

$$\frac{\mu R}{\sqrt{2\tilde{h}^+}} - \frac{\mu R}{\sqrt{2h^+}} = \frac{\mu R}{\sqrt{2h^-}} \left(\frac{h^+}{\sqrt{\tilde{h}^+}} - 1 \right)$$

and, in fact, has already been estimated (6.17); the estimate of ξ-component follows from (4.13i); the estimate of $\frac{\partial \Psi}{\partial(\xi, t)}$ from (4.16d). Thus, we have a majorant of the second term in (7.9):

$$\frac{1}{\left\langle \mathfrak{X}, \frac{\partial \Psi}{\partial(\xi, t)} \right\rangle} (\mathring{\tilde{\mathfrak{X}}} - \hat{\mathfrak{X}}) \otimes \frac{\partial \tilde{\Psi}}{\partial(\xi, t)} \prec \frac{1}{k} \begin{pmatrix} 26.44a & 0 \\ 44.60ag & 0 \end{pmatrix}. \tag{7.15}$$

To estimate

$$\left\| \frac{\partial \tilde{\Psi}}{\partial(\xi, t)} - \frac{\partial \Psi}{\partial(\xi, t)} \right\| = \left\| \frac{\partial |Q(\tilde{\xi}^+)|}{\partial \xi} - \frac{\partial |Q(\xi^+)|}{\partial \xi} \right\|$$

we shall proceed as we did several times before.

$$\frac{\partial |Q(\tilde{\xi}^+, \tilde{\mathscr{E}}^+)|}{\partial \xi} - \frac{\partial |Q(\xi^+, \mathscr{E}^+)|}{\partial \xi} = \frac{\partial |Q(\xi(\tau), \mathscr{E}_R(\tau))|}{\partial \xi} \Bigg|_{\tau=\tau^-}^{\tau=\tilde{\tau}^+}$$

$$= \int_{\tau^-}^{\tau^+} \left(\frac{\partial^2 |Q(\xi, \mathscr{E}_R)|}{\partial \xi \, \partial \xi} \frac{d\xi}{d\tau} + \frac{\partial^2 |Q(\xi, \mathscr{E}_R)|}{\partial \xi \, \partial \mathscr{E}} \frac{d\mathscr{E}_R}{d\tau} \right) d\tau.$$

Because of $\partial |Q| / \partial \omega = 0$ the integrand may be estimated by

$$\left\| \frac{\partial^2 |Q|}{\partial \xi \, \partial \xi} \right\| \sqrt{\left(\frac{d\xi_1}{d\tau} \right)^2 + \left(\frac{d\xi_2}{d\tau} \right)^2 + \left(\frac{d\mathscr{E}_R}{d\tau} \right)^2},$$

which, in turn, may be estimated by virtue of (4.16e), (4.13f, g) and (6.30). Finally applying lemma 7, we obtain

$$\left| \frac{\partial |Q(\hat{\xi}^+)|}{\partial \xi} - \frac{\partial |Q(\xi^+)|}{\partial \xi} \right| \le 341 \left(\frac{1}{2h^-} + R \right) \frac{a}{k}.$$

This gives the majorant of the last term in (7.9):

$$\frac{\mathring{\tilde{\mathfrak{X}}}}{\left\langle \mathfrak{X}, \frac{\partial \Psi}{\partial(\xi, t)} \right\rangle} \otimes \left(\frac{\partial \tilde{\Psi}}{\partial(\xi, t)} - \frac{\partial \Psi}{\partial(\xi, t)} \right) \prec \begin{pmatrix} 0 & 0 \\ 341\, ag/k^2 & 0 \end{pmatrix}. \tag{7.16}$$

Summing up (7.14)–(7.16) and taking into account (7.6) we obtain in accordance with (7.10)

$$\tilde{\Pi}-\Pi \prec \begin{pmatrix} 52a/k & 0 \\ 1649ag/k^3 & 0 \end{pmatrix}. \tag{7.17}$$

All that now remains is to estimate the perturbation of the derivatives of the Poincaré mapping. As we have seen, the difference between the derivatives at the point (ξ_0^-, t_0^-) may be represented in the form

$$\tilde{S}_* - S_* = \tilde{\Pi}\tilde{\Theta}\mathcal{I} - \Pi\Theta\mathcal{I} = (\tilde{\Pi}(\tilde{\Theta}-\Theta) + (\tilde{\Pi}-\Pi)\Theta)\mathcal{I}. \tag{7.18}$$

Majorants for Θ and $\tilde{\Theta}-\Theta$ have the form

$$\Theta \prec \begin{pmatrix} 1 & 0 \\ \left\|\frac{\partial t^+}{\partial \xi^-}\right\| & 1 \end{pmatrix}, \quad \tilde{\Theta}-\Theta \prec \begin{pmatrix} 4717a+5884bg & 419b \\ \left\|\frac{\partial \tilde{t}^+}{\partial \xi^-}-\frac{\partial t^+}{\partial \xi^-}\right\| & \left\|\frac{\partial \tilde{t}^+}{\partial t^-}-\frac{\partial t^+}{\partial t^-}\right\| \end{pmatrix}. \tag{7.19}$$

(The latter represents, as a matter of fact, the pair of inequalities (7.7) and (7.8).) From (7.18), we obtain the following majorant

$$\tilde{S}_* - S_* \prec \begin{pmatrix} 9202\frac{a}{k^2}+11366\frac{bg}{k} & 466b \\ 46864\frac{ag}{k^4}+54878\frac{bg^2}{k^2} & 2246\frac{bg}{k} \end{pmatrix} \tag{7.20}$$

which does not depend on $(\xi_0^-, t_0^-) \in \mathfrak{S}_R^-(\varepsilon, k)$.

We resume our arguments. At each stage of selecting the small parameter (see (5.12), (6.25), (7.6)), we required the fulfilment of a condition more restrictive than the preceding one. So, if (7.6) is fulfilled, then the Poincaré mapping exists and we have the estimates (6.1), (6.4), (6.5), (6.31), (7.20) which is the assertion of the theorem.

8. *The first terms of asymptotic expansion*

There often arises in practice the need to have a more precise approximation to the real orbit than Kepler's approximation. Below we construct the asymptotic approximation to this orbit, restricting ourselves by the first non-trivial terms of an expansion with respect to the integral powers of μ. We estimate further the degree of the approximation obtained in the $O(\mu^{\frac{1}{3}})$-neighbourhood of Jupiter.

We rewrite our basic system of equations (4.14) in a slightly modified form

$$\begin{cases} \dfrac{d\xi}{d\tau} = T_{(0,0,0,1)} + \mu A(\xi)\varphi(\mu Q(\xi), t^- + \mu T, \mu) \\ \dfrac{dT}{d\tau} = B(\xi). \end{cases} \tag{8.1}$$

Let

$$\begin{aligned} \xi(\tau) &= \xi_{(0)}(\tau) + \mu\xi_{(1)}(\tau) + \mu^2\xi_{(2)}(\tau) + \cdots \\ T(\tau) &= T_{(0)}(\tau) + \mu T_{(1)}(\tau) + \mu^2 T_{(2)}(\tau) + \cdots. \end{aligned} \tag{8.2}$$

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We substitute these expressions into (8.1) emphasizing the terms up to μ^2

$$\frac{d}{d\tau}(\xi_{(0)}(\tau)+\mu\xi_{(1)}(\tau)+\mu^2\xi_{(2)}(\tau)+0(\mu^3))$$

$$=T_{(0,0,0,1)}+\mu[A(\xi_{(0)})+0(\mu)]\cdot\left[\frac{\partial\varphi(0,t^-,0)}{\partial q}\mu Q(\xi_{(0)})\right.$$

$$\left.+\frac{\partial\varphi(0,t^-,0)}{\partial t}\mu T_{(0)}+\frac{\partial\varphi(0,t^-,0)}{\partial\mu}\mu+0(\mu^2)\right];$$

$$\frac{d}{d\tau}(T_{(0)}(\tau)+\mu T_{(1)}(\tau)+\mu^2 T_{(2)}(\tau)+0(\mu^3))$$

$$=B(\xi_{(0)})+\frac{\partial B(\xi_{(0)})}{\partial\xi}(\mu\xi_{(1)}+\mu^2\xi_{(2)}+0(\mu^3))$$

$$+\frac{\partial^2 B(\xi_{(0)})}{\partial\xi\,\partial\xi}(\mu\xi_{(1)}+0(\mu^2),\mu\xi_{(1)}+0(\mu^2)).$$

One has here

$$\frac{\partial\varphi(0,t^-,0)}{\partial t}=\frac{\partial\varphi(0,t^-,0)}{\partial\mu}=0$$

because $\varphi(0,t,\mu)\equiv 0$. When comparing the terms of the same order we obtain the equations

$$(1)\quad\begin{cases}\dfrac{d\xi_{(0)}}{d\tau}=T_{(0,0,0,1)},\\[2mm] \dfrac{dT_{(0)}}{d\tau}=B(\xi_{(0)});\end{cases}$$

$$(2)\quad\begin{cases}\dfrac{d\xi_{(1)}}{d\tau}=0,\\[2mm] \dfrac{dT_{(1)}}{d\tau}=\dfrac{\partial B(\xi_{(0)})}{\partial\xi}\xi_{(1)};\end{cases}\tag{8.3}$$

$$(3)\quad\begin{cases}\dfrac{d\xi_{(2)}}{d\tau}=A(\xi_{(0)})\dfrac{\partial\varphi(0,t^-,0)}{\partial q}Q(\xi_{(0)}),\\[2mm] \dfrac{dT_{(2)}}{d\tau}=\dfrac{\partial B(\xi_{(0)})}{\partial\xi}\xi_{(2)}+\dfrac{\partial^2 B(\xi_{(0)})}{\partial\xi\,\partial\xi}(\xi_1,\xi_1).\end{cases}$$

Taking $\xi(\tau^-)=\xi^-$, $T(\tau^-)=0$ as initial conditions for the system (8.1), we obtain from (8.2) the following initial conditions for the systems (8.3):

$$\xi_{(0)}(\tau^0)=\xi^-;\quad \xi_{(1)}(\tau^-)=0;\quad \xi_{(2)}(\tau^-)=0;$$
$$T_{(0)}(\tau^-)=0;\quad T_{(1)}(\tau^-)=0;\quad T_{(2)}(\tau^-)=0.$$

This implies that $\xi_{(1)}(\tau)\equiv 0$ and $T_{(1)}(\tau)\equiv 0$ and therefore simplifies the equations (8.3)

$$\begin{aligned}
&(1)\quad \begin{cases} \dfrac{d\xi_{(0)}}{d\tau}=T_{(0,0,0,1)} \\ \dfrac{dT_{(0)}}{d\tau}=B(\xi_{(0)}); \end{cases} \\
&(2)\quad \begin{cases} \dfrac{d\xi_{(2)}}{d\tau}=A(\xi_{(0)})\dfrac{\partial\varphi(0,t^{-},0)}{\partial q}Q(\xi_{(0)}) \\ \dfrac{dT_{(2)}}{d\tau}=\dfrac{\partial B(\xi_{(0)})}{\partial\xi}\xi_{(2)}. \end{cases}
\end{aligned} \tag{8.4}$$

Now we begin the estimation of the deviation of the approximate solution

$$\begin{aligned}\hat{\xi}(\tau)&=\xi_{(0)}(\tau)+\mu^2\xi_{(2)}(\tau),\\ \hat{T}(\tau)&=T_{(0)}(\tau)+\mu^2T_{(2)}(\tau)\end{aligned} \tag{8.5}$$

from the true one. We transform the differences $\xi(\tau)-\hat{\xi}(\tau)$, $T(\tau)-\hat{T}(\tau)$ bearing in mind that all functions defined on $\mathfrak{M}$ can be extended into $\mathbb{R}^9$ (and, in particular, onto the convex hull of $\mathfrak{M}$):

$$\begin{aligned}
&\xi(\tau)-\hat{\xi}(\tau)\\
&\quad=\int_{\tau^-}^{\tau}\left[\int_0^1\frac{\partial A(\xi_{(0)}+s(\xi-\xi_{(0)}))}{\partial\xi}(\xi-\xi_{(0)})\,ds\right]\mu\varphi(\mu Q,t^-+\mu T,\mu)\,d\sigma\\
&\qquad+\int_{\tau^-}^{\tau}A(\xi_{(0)})\left[\mu\varphi(\mu Q,t^-+\mu T,\mu)-\mu\frac{\partial\varphi(0,t^-,0)}{\partial q}\mu Q\right]d\sigma\\
&\qquad+\int_{\tau^-}^{\tau}A(\xi_{(0)})\mu\frac{\partial\varphi(0,t^-,0)}{\partial q}\mu\left[\int_0^1\frac{\partial Q(\xi_{(0)}+s(\xi-\xi_{(0)}))}{\partial\xi}(\xi-\xi_{(0)})\,ds\right]d\sigma;\\
&T(\tau)-\hat{T}(\tau)\\
&\quad=\int_{\tau^-}^{\tau}\left\{\int_0^1 s\left[\int_0^1\frac{\partial^2B(\xi_{(0)}+l_s(\xi-\xi_{(0)}))}{\partial\xi\,\partial\xi}(\xi-\xi_{(0)},\xi-\xi_{(0)})\,dl\right]ds\right\}d\sigma\\
&\qquad+\int_{\tau^-}^{\tau}\frac{\partial B(\xi_{(0)})}{\partial\xi}(\xi-\xi_{(0)}-\mu^2\xi_{(2)})\,d\sigma.
\end{aligned}$$

Estimates of these expressions are based on the following facts.

(a) It is easy to see that the estimate

$$\left|\varphi(q,t,\mu)-\frac{\partial\varphi(0,t^-,0)}{\partial q}q\right|\le C_{12}|q|^2\quad(C_{12}=\text{const.})$$

is valid in the ball $|q|\le\mu R$ when t belongs to a finite interval and μ is sufficiently small.

(b) The estimates of lemma 5 for pseudopolynomials obtained under assumption $(\xi,t)\in\mathfrak{M}$ also hold in the convex hull of $\mathfrak{M}$, provided one replaces $1+2h|Q|$ by

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e ch $\mathscr{E}$ (which is not the same because

$$|[\xi_2', \omega']\leq|\xi_2'|\cdot|\omega'|, \quad |\omega'|\leq 1$$

and

$$1+2h'|Q'|\leq e' \text{ ch } \mathscr{E}'$$

is true in the convex hull).

(c) If the conditions of the basic theorem are fulfilled, then for a certain Δ the inequality

$$|\xi(\tau)-\xi_{(0)}(\tau)|\leq\Delta$$

holds. We assert that if ξ' lies in the segment whose end-points are $\xi(\tau)$ and $\xi_{(0)}(\tau)$, then the quantities

$$f_1(\xi')=e' \text{ ch } \mathscr{E}' \quad \text{and} \quad f_2(\xi')=\frac{1}{2h^-}$$

can be estimated by means of their values at $\xi_{(0)}(\tau)$. Namely,

$$e' \text{ ch } \mathscr{E}'<\frac{\exp(\Delta)}{1-\Delta}(1+2h_{(0)}|Q_{(0)}|)$$

$$\frac{1}{2h'}=\frac{\exp(-\xi_1')}{2}\leq\frac{\exp(\Delta)}{2h_{(0)}}.$$

(The first inequality is a consequence of

$$(e')^2\leq\frac{1+(|\xi_{2(0)}|+\Delta)^2}{1+(\xi_{2(0)})^2}e_{(0)}^2\leq\frac{1}{(1-\Delta)^2}e_{(0)}^2$$

and

$$\text{ch } \mathscr{E}'\leq\exp(\Delta) \text{ ch } \mathscr{E}_{(0)}).$$

For any τ for which

$$|Q(\xi_{(0)}(\tau))|\leq R \quad \text{and} \quad |Q(\xi(\tau))|<R$$

hold simultaneously, the estimates

$$|\xi(\tau)-\hat{\xi}(\tau)|\leq C\mu^2\left(\frac{1}{2h^-}+R\right)^2 R\left(\mu^2\left(\frac{1}{2h^-}+R\right)^3+\mu R\right)$$

$$|T(\tau)-\hat{T}(\tau)|\leq C\frac{1}{\sqrt{2h^-}}\left(\frac{1}{2h^-}+R\right)\left\{\left[\mu^2\left(\frac{1}{2h^-}+R\right)^3\right]^2+\mu^2\left(\frac{1}{2h^-}+R\right)^3\mu R\right\}$$

also hold.

Thus, if h is bounded away from zero while μ and r tend to zero, then the error of the approximate solution (8.5) is of order λ^2 for the elements ξ and $\mu^{\frac{1}{3}}\lambda^{2+\frac{1}{3}}$ in physical time. This should be compared with λ for ξ and $\mu^{\frac{1}{3}}\lambda^{1+\frac{1}{3}}$ for t in case of the zero order (i.e. Kepler's) approximation. The formulae describing $\xi_{(2)}$, $T_{(2)}$ can be easily obtained from the system (8.4(2)) by successive integration. The integrals which occur may be evaluated in explicit form because the integrands are polynomials of degree three and four in ch $\mathscr{E}_{(0)}=$ ch τ, sh $\mathscr{E}_{(0)}=$ sh τ with constant coefficients. This means, in particular, that the approximate solution (8.5) is parametrized by the eccentric anomaly of the unperturbed orbit, since

$$\xi_{(0)}(\tau) = (\xi_{(0)}(\tau), \mathscr{E}_{(0)}(\tau)) \equiv (\text{const. } \tau)$$

is no more than the Kepler orbit osculating at the instant τ^- (or, if one prefers, at the instant t^-).

These formulae are rather bulky and so we do not include them. They may be useful, however, in applied problems because of the speed of computation in comparison with numerical integration and because sufficient accuracy is guaranteed. As an example, the problem of the approximate determination of the true pericentre and time of passage through it is within the power of a microcalculator. It suffices to determine the root of equation

$$\mathscr{E}_{(0)}(\tau) + \mu^2 \mathscr{E}_{(2)}(\tau) = \tau + \mu^2 \mathscr{E}_{(2)}(\tau) = 0$$

by 1–2 iterations taking $\tau = 0$ as the initial approximation and to substitute the found value of τ into the formulae for the time and for the rest of the elements.

9. *Discussion of the result. Other possible applications of the estimation method*

In the present paper, we considered perturbations caused by the external gravitational forces. The peculiarity of perturbations of this kind is that they vanish at the origin of the coordinate system centred at Jupiter. One may expect that the perturbation of the elements is small when the energy is bounded away from zero and the magnitude of the perturbing force (of order r in the case under consideration) is small with respect to Jupiter's gravity (of order μ/r^2), i.e. when

$$\frac{r}{\mu/r^2} \ll 1$$

or in the $\lambda\mu^{\frac{1}{3}}$ vicinity of Jupiter with λ equal to a constant much less than one. The theorem proved in this paper confirms these expectations. From the purely technical viewpoint the requirement for r^3/μ to be small appears when estimating the right-hand side of the system (4.14):

$$A(\xi)\mu\varphi((q, t, \mu) \prec C\left(\frac{1}{2h^-} + \frac{r}{\mu}\right)^2 \mu r \sim C\frac{r^3}{\mu}.$$

Another important class is formed by perturbations which do not vanish at the point coinciding with Jupiter. These perturbations have a non-gravitational nature (they may be caused, for example, by light pressure). One may regard their absolute magnitude to be of the order 1, while the relative magnitude is of order $1/(\mu/r^2)$. The latter is small within $\lambda\mu^{\frac{1}{2}}$ vicinity of Jupiter when λ is a constant much less than one. The estimate of the right-hand side of (4.14) has the form

$$A(\xi)\mu\varphi(q, t, \mu) \prec C\left(\frac{1}{2h^-} + \frac{r}{\mu}\right)^2 \mu \sim C\frac{r^2}{\mu}$$

and the arguments leading to the proof of the theorem may be repeated *mutatis mutandis* in this case also. The same is true with respect to the asymptotic formulae of the preceding section.

One more important case where our method works is the non-local study of the orbit perturbation, i.e. not only during passage in Jupiter's vicinity, but also on the whole infinite time interval. It turns out to be possible to estimate the variation of the elements when the perturbing force decreases at infinity a little more rapidly than the Newtonian one, or, more exactly, when the perturbation admits the estimate $\varepsilon/(1+|r|)^\beta$ with $\beta>2$, $\varepsilon \ll 1$. One may consider, as an example, the perturbation caused by the non-sphericity of Jupiter. The value of μ does not play any role, so one may assume $\mu=1$. In these assumptions the estimate of the right-hand side of (4.14) is

$$A(\xi)\varphi(q,t) \prec \left(\frac{1}{2h}+|r|\right)^2 \frac{\varepsilon}{(1+|r|)^\beta} < \frac{\max(1,2h)}{2h} \frac{\varepsilon}{(1+2hr)^{\beta-2}}.$$

Lemma 7 must be reformulated.

LEMMA 7′. *Let a solution of* (4.14) *exist for* $-\infty<\tau<+\infty$ *and satisfy the conditions*:

$$(a)\ \frac{d\mathscr{E}}{d\tau}>\frac{1}{1+\varepsilon_0};\quad (b)\ h(\tau)>0;$$

$$(c)\ e=\frac{1}{\kappa}e^- \le e(\tau) \le \kappa e^-$$

for some $\varepsilon_0>0$, $\kappa>1$. *Then the integral*

$$\int_{-\infty}^{+\infty} \frac{d\tau}{(1+2h|Q|)^{\beta-2}}$$

with $\beta>2$ *converges and admits the estimate*

$$\int_{-\infty}^{\infty} \frac{d\tau}{(1+2h|Q|)^{\beta-2}} < \frac{1}{e^{\beta-2}}(1+\varepsilon_0)\frac{2^{\beta-1}}{\beta-2}.$$

Proof.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{d\tau}{(1+2h|Q(\tau)|)^{\beta-2}} &\le \int_{-\infty}^{\infty} \frac{1}{(1+2h|Q|)^{\beta-2}}\frac{d\tau}{d\mathscr{E}}\,d\mathscr{E}\\ &\le (1+\varepsilon_0)\int_{-\infty}^{\infty}\frac{d\mathscr{E}}{(e\ \mathrm{ch}\ \mathscr{E})^{\beta-2}}\\ &\le \frac{1+\varepsilon_0}{\exp(\beta-2)}\int_{-\infty}^{\infty}\frac{2^{\beta-2}}{\exp(\beta-2)|\mathscr{E}|}\,d\mathscr{E} = \frac{1+\varepsilon_0}{e^{\beta-2}}\frac{2^{\beta-1}}{\beta-2}.\end{aligned}$$

Now one may easily estimate the perturbation of the elements, following if not the letter, then the spirit of the arguments given in §§ 5–7; the perturbation φ for simplicity may be regarded as an autonomous one.

Afterword

The goal of this paper is to present the first part of the joint research of the late Professor V. M. Alexeyev and myself which was undertaken on V. M. Alexeyev's initiative. The task we set for ourselves was to confirm the hypothesis, going back to Laplace, of the possibility of a capture of a comet by the Sun–Jupiter system due to the passage of the comet close to Jupiter. More precisely, we intended to prove, for the case of the restricted three-body problem as a model, the possibility of a capture in the strict mathematical sense, i.e. for a semi-infinite time interval. This could not be expected to be a simple problem, if only because, as it is known, the set of the initial conditions leading to capture is of measure zero. Nevertheless, the experience gained by Professor V. M. Alexeyev in the study of trajectories with close passage as well as in the study of quasi-random motion in the Kolmogorov–Sitnikov example made him believe that this could indeed be realized. Quite soon we found that it was possible to prove the existence of capture in another well-known dynamical system which could be regarded as the next stage of idealization; this is a combination of two two-body problems where the perturbations of the motion caused by the Sun are eliminated in a neighbourhood of Jupiter and, *vice versa*, the perturbations caused by Jupiter are eliminated outside that neighbourhood. As should be expected, the essence of the capture phenomenon proves to be connected with the presence of the so-called hyperbolic (Perron) subset in the phase space. In this situation methods of symbolic dynamics and, in particular, Professor V. M. Alexeyev's method of itinerary schemes, sometimes called the method of vertical and horizontal strips, turned out to be applicable. The existence of a hyperbolic set is a stable phenomenon (it should be emphasized that, on the contrary, the trajectories constituting this set are highly unstable). So it is natural to study the restricted problem by approximating it with the above-mentioned idealized problem. The main difficulty we encountered was the choice of the size of the neighbourhood of Jupiter. For a long time it seemed to us that there existed a 'blank space', namely, a ring $\varepsilon\mu^{\frac{1}{2}} \le r \le {}^{1}\mu^{\frac{1}{2}}$ around Jupiter within which one cannot restrict oneself to a single body, either to the Sun or to Jupiter, when considering the variational equations. The idea of the regularization 'in terms of first integrals' presented in this paper made it possible to clarify this delusion by extending the sphere of Kepler's asymptotics from $\mu^{\frac{1}{2}}$ to $\mu^{\frac{1}{3}}$. This enabled us to achieve our aim. The proof of the existence of a capture in the restricted three-body problem based on the consideration of orbits with close passage is being prepared for publication. An example of such a capture is given by an orbit which in the past is hyperbolic and in the future is asymptotically approaching a periodic orbit of the second species in Poincaré's classification.

The content of this paper is the result of joint work done by Professor V. M. Alexeyev and myself. Professor Alexeyev had intended to take upon himself the entire task of presenting the results, however he only had time to write the introductory part. The rest of the presentation is mine. A number of imperfections in the presentation of the material have been corrected by A. I. Grünthal who, as well as myself, has studied under Professor V. M. Alexeyev. YU. S. OSIPOV

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