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## ON THE EXCURSIONS OF REFLECTED LOCAL-TIME PROCESSES AND STOCHASTIC FLUID QUEUES

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# ON THE EXCURSIONS OF REFLECTED LOCAL-TIME PROCESSES AND STOCHASTIC FLUID QUEUES

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## Abstract

In this paper we extend our previous work. We consider the local-time process  $L$  of a strong Markov process  $X$ , add negative drift to  $L$ , and reflect it à la Skorokhod to obtain a process  $Q$ . The reflection of  $X$ , together with  $Q$ , is, in some sense, a macroscopic model for a service system with two priorities. We derive an expression for the joint law of the duration of an excursion, the maximum value of the process on it, and the time between successive excursions. We work with a properly constructed stationary version of the process. Examples are also given in the paper.

*Keywords:* Lévy process; local time; Skorokhod reflection; stationary process

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## 1. Introduction

In this paper we consider a model for which special cases have been studied in [8], [9], and [11], consisting of a priority queueing system where the high priority class is a stochastic process denoted by  $X$  and the low priority class is a process denoted by  $Q$ . Our interest is in studying  $Q$ . It turns out that the problem can be expressed in general terms via an underlying strong Markov process  $X$  and its local time  $L$  at 0, a process which is considered as an input to  $Q$ . For further motivation to the physical problem, we refer the reader to [8], [9], and [11]. In what follows, we express the problem in mathematical terms.

Consider a stationary strong Markov process  $X = (X_t, t \in \mathbb{R})$ , defined on some filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t, t \in \mathbb{R}))$ , with values in  $\mathbb{R}_+$  almost surely (a.s.) càdlàg paths, and adapted to  $(\mathcal{F}_t)$ . In this paper, the local time  $L$  of the process  $X$  at  $x = 0$  is considered as an  $(\mathcal{F}_t)$ -adapted stationary random measure that regenerates jointly with  $X$  at every (stopping) time that  $X$  hits 0. More precisely:

- (A1)  $L$  assigns a nonnegative random variable  $L(B, \omega)$  to each  $B \in \mathcal{B}(\mathbb{R})$  such that  $L(\cdot, \omega)$  is a Radon measure for each  $\omega \in \Omega$ ,
- (A2) for any a.s. finite  $(\mathcal{F}_t)$ -stopping time  $T$  at which  $X_T = 0$ , the process  $((X_{T+t}, L(T, T+t)), t \geq 0)$  is independent of  $\mathcal{F}_T$ .

We take the broader perspective with regard to the process  $L$  and we allow for the case that it is a local time of an irregular point (in which case  $L$  has discontinuous paths) as well as the case that 0 is a sticky point (in which case  $L$  is absolutely continuous with respect to the Lebesgue measure with density  $c \mathbf{1}(X_t = 0)$  for some  $c > 0$ ). We refer the reader to [1, Chapter IV] (in

particular Corollary 6), [3, Section V.3], and [10, Chapter 6] for further discussion. For each  $s \in \mathbb{R}$ , define the inverse local-time process with respect to  $t$  by

$$L_{s;u}^{-1} := \inf\{t > 0: L[s, s + t] > u\}, \quad u \geq 0. \tag{1}$$

What is important is that, owing to this definition, the inverse of the cumulative local time is a Lévy process in the following sense.

**Lemma 1.** *If  $L$  is continuous then, for every a.s. finite  $(\mathcal{F}_t)$ -stopping time  $T$  such that  $X_T = 0$ , the process  $(L_{T;u}^{-1}, u \geq 0)$  is a subordinator with  $L_{T;0}^{-1} = 0$ .*

If  $L$  is not continuous, that is, if 0 is an irregular point for  $X$ , then Lemma 1 is taken as an additional requirement to the definition of  $L$ . This is easily arranged by choosing  $L$  to be a modification of the counting process on  $\mathcal{Z}$ , the discrete set of times that  $X$  visits 0, so that the inverse is a subordinator. To do this, we assign, to each element of  $\mathcal{Z}$ , an independent and identically distributed (i.i.d.) unit-mean exponentially distributed weight. Then let the local time on an interval  $I$  to be the sum of all the weights of the points of  $\mathcal{Z}$  in  $I$ .

We summarise this as an assumption, in addition to (A1)–(A2) above.

(A3) If  $L$  is discontinuous then we require that, for every a.s. finite  $(\mathcal{F}_t)$ -stopping time  $T$  such that  $X_T = 0$ , the process  $(L_{T;u}^{-1}, u \geq 0)$  is a subordinator.

We will also need the following assumption.

(A4) The stationary random measure  $L$  has finite rate not exceeding 1, i.e.

$$E L(0, t) = \mu t,$$

where  $0 < \mu < 1$ .

Then, as in [8], [9], [11], and [13], we define a stationary process  $Q = (Q_t, t \in \mathbb{R})$  by

$$Q_t = \sup_{-\infty < s \leq t} \{L(s, t] - (t - s)\}, \quad t \in \mathbb{R}. \tag{2}$$

Furthermore,  $Q$  is ergodic (its invariant  $\sigma$ -field is trivial). Note that  $Q$  also satisfies, pathwise,

$$\begin{aligned} Q_t &= Q_s + L(s, t] - (t - s) - \inf_{s \leq r \leq t} (Q_s + L(s, r] - (r - s)) \wedge 0 \\ &= \sup_{s \leq r \leq t} (L(r, t] - (t - r)) \vee (Q_s + L(s, t] - (t - s)) \end{aligned} \tag{3}$$

for all  $-\infty < s < t < \infty$ . It is worth recalling [8] that if we consider (3) as a fixed point equation for  $Q$  then the process defined by (2) is the unique stationary and ergodic solution of (3). A typical sample path of  $Q$  is depicted in Figure 2 in Section 2. It consists of isolated excursions away from 0 (also called ‘busy periods’), followed by intervals of time at which  $Q$  stays at 0 (called ‘idle periods’). In this respect, the process  $Q$  is thought of as the workload in a stochastic fluid queue. Amongst other things in [8], [9], and [13], expressions are derived for the marginal distribution of  $Q$  and the Laplace transform of the duration of typical idle and busy periods.

In this paper we shall derive an expression for the joint law of three random variables: the duration of a busy period, the duration of an idle period, and the maximum of  $Q$  over a busy period. The result is formulated as Theorem 1 in Section 3: (16) therein is new and extends

some of the results of [8]. The result is expressed in terms of the process  $\Lambda$ , which is in turn a function of the underlying Markov process  $X$ . Its construction and properties are given in Section 2. The approach in this paper is new (compared to [8] and [9]). In Section 4 we use Theorem 1 for finding marginal distributions. Subsequently, in Section 5, we prove Theorem 2 on the joint law of endpoints of an idle period. The formula given in Theorem 2 appeared in our previous paper [8], but the proof presented here is new. We then prove Theorem 3 on the joint law of endpoints of a busy period, together with the maximum of  $Q$  over this period. This is a new result which is also proved by means of applying Theorem 1 together with Palm calculus.

It is assumed, throughout, that  $(\Omega, \mathcal{F}, P)$  is endowed with a  $P$ -preserving measurable flow  $\theta_t: \Omega \rightarrow \Omega, t \in \mathbb{R}$ , with a measurable inverse  $\theta_t^{-1} = \theta_{-t}$ . In other words,  $P(\theta_t A) = P(A)$  for all  $A \in \mathcal{F}$  and  $t \in \mathbb{R}$ . All stationary random processes and measures can be constructed on  $\Omega$  in such a way that the flow commutes with the natural shift, e.g.  $Q_t(\theta_s \circ \omega) = Q_{t+s}(\omega)$ , and  $L(B, \theta_s \omega) = L(B + s, \omega)$  for all  $s, t \in \mathbb{R}$ , Borel sets  $B \subset \mathbb{R}$ , and  $\omega \in \Omega$ . The flow will be used explicitly in Section 5 to obtain distributions conditional on observing a positive (or a zero) value of  $Q_0$ .

### 2. A closer look at the reflected process

Consider now any a.s. finite  $(\mathcal{F}_t)$ -stopping time  $T$ , such that  $X_T = 0$ . Then  $(L_{T;t}^{-1}, t \geq 0)$  is a subordinator starting from 0 (owing to Lemma 1 or assumption (A3)) with a law that does not depend on  $T$ . It turns out that the process of interest is  $\Lambda_T = \{\Lambda_{T,t} : t \geq 0\}$ , where

$$\Lambda_{T;t} = t - L_{T;t}^{-1}, \quad t \geq 0. \tag{4}$$

Note that, irrespective of  $T$ , the process  $\Lambda_T$  obeys the law of the same bounded variation spectrally negative Lévy process which is issued from the origin at time 0. By (A4),  $L$  has rate  $\mu < 1$ ; hence,  $E \Lambda_{T;1} = 1 - 1/\mu < 0$ . Since  $L_{T;t}^{-1}$  is a subordinator, it has a well-defined, possibly nonzero, drift. If this drift is larger than or equal to unity then  $-\Lambda_T$  is a subordinator and, as it will turn out, this is a trivial case.

We therefore assume in the sequel that the drift of  $L_T^{-1}$  is less than unity or, equivalently, that

(A5) the drift  $\delta_\Lambda$  of the process  $\Lambda$  defined by (4) is strictly positive.

Under this assumption, the point 0 is irregular for  $(-\infty, 0)$  for  $\Lambda_T$  (this follows as a standard result for bounded variation spectrally negative Lévy processes—see [1, Chapter VII]).

In addition, under (A5), it is clear that the time taken for  $\Lambda_T$  to first enter  $(-\infty, 0)$  is a.s. strictly positive. It will be shown below (Lemma 3) that this implies that the excursions of the process  $Q$ , i.e. the busy periods, have strictly positive length with probability 1. It can be intuitively seen, via a geometric argument involving the reflection of the space–time path of  $\Lambda_T$  about the diagonal (see Figure 1) that the time taken for  $\Lambda_T$  to first enter  $(-\infty, 0)$  is a.s. equal to the length of the excursion of  $Q$  started at time  $T$ .

In this light, note also that  $\Lambda_T$  cannot creep downwards because it is spectrally negative with paths of bounded variation (cf. [1, Chapter VII]). Hence, the overshoot at first passage of  $\Lambda_T$  into  $(-\infty, 0)$  is a.s. strictly positive. It turns out (Lemma 1) that this overshoot agrees with the idle period following the aforementioned excursion of  $Q$ .

The above analysis implies that, on finite intervals of time,  $Q$  has finitely many excursions (busy periods) separated by positive-length idle periods. Denote by

$$\dots < g(-1) < g(0) < g(1) < g(2) < \dots$$

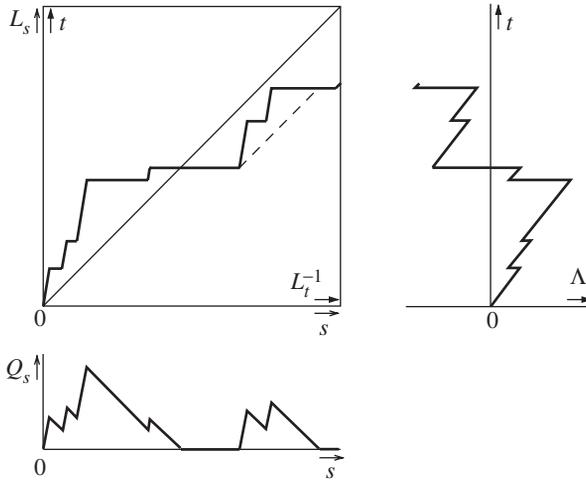


FIGURE 1: The construction of the process  $(\Lambda_{T;t}, t \geq 0)$  and related processes, assuming that  $T = 0$ . Note that  $\Lambda$  may have countably many jumps on finite intervals.

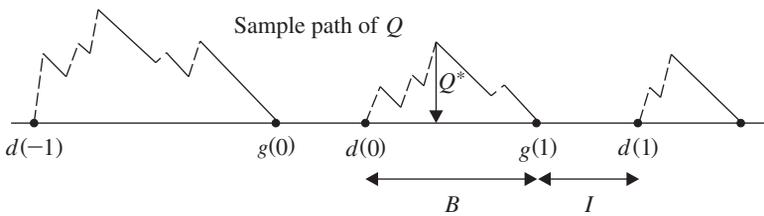


FIGURE 2: The definitions of  $g(n)$  and  $d(n)$ . By convention, the origin of time is between  $g(0)$  and  $g(1)$ , under the original measure  $P$ . Under  $P_d$ , the origin of time is at  $d(0)$ . Under  $P_g$ , the origin of time is at  $g(0)$ . The random variable  $Q^*$  is the maximum deviation from 0 of  $Q$  within the typical busy period.

the beginnings of the idle periods and by

$$\dots < d(-1) < d(0) < d(1) < d(2) < \dots$$

their ends; see Figure 2. We choose the indexing so that  $g(0) \leq 0 < g(1)$ . Let  $N_g$  and  $N_d$  respectively be the point processes with points  $\{g(n) : n \in \mathbb{Z}\}$  and  $\{d(n) : n \in \mathbb{Z}\}$ . As  $Q$  is a stationary process,  $N_g$  and  $N_d$  are jointly stationary with finite, nonzero intensity [8] denoted by  $\lambda$  (an expression for which is given by (24) and is derived in Section 4.3, below). Corresponding to point processes  $N_g$  and  $N_d$  we have the Palm probabilities  $P_g$  and  $P_d$ , respectively. Let us consider  $Q$  under the measure  $P_d$ . Then  $P_d(d(0) = 0) = 1$ , i.e. the origin of time is placed at the beginning of a busy period. By the strong Markov property, the ‘cycles’

$$C_n := \{Q_t : d(n) \leq t < d(n + 1)\}, \quad n \in \mathbb{Z},$$

are i.i.d. under the measure  $P_d$ . In particular, the pairs of random variables

$$(g(n + 1) - d(n), d(n + 1) - g(n + 1)), \quad n \in \mathbb{Z},$$

are i.i.d. under  $P_d$ . Consider the triple

$$(B, I, Q^*) := \left( g(1) - d(0), d(1) - g(1), \sup_{d(0) < t < g(1)} Q_t \right), \tag{5}$$

which is a function of cycle  $C_0$ . We are primarily interested in the  $P_d$ -law of  $(B, I, Q^*)$ . Since, under  $P_d$ , the origin of time is placed at  $d(0)$ , we interpret  $B, I$ , and  $Q^*$  as the typical busy period, the typical idle period, and the maximum value of  $Q$  over a typical busy period, respectively.

The next lemma is proved in [8].

**Lemma 2.** *Let  $D = \inf\{t > 0: X_t = 0\}$  and  $d = \inf\{t > 0: Q_t > 0\}$ . Then  $d = D$  a.s. on  $\{Q_0 = 0\}$ .*

We now obtain an alternative expression for  $B = g(1) - d(0)$  and  $I = d(1) - g(1)$  in terms of the inverse local time.

**Lemma 3.** *We have*

$$B = g(1) - d(0) = \inf\{u > 0: L_{d(0);u}^{-1} > u\}, \tag{6}$$

$$B + I = d(1) - d(0) = L_{d(0);g(1)-d(0)}^{-1}. \tag{7}$$

*Proof.* Since  $d(0)$  is the end of an idle period (and the beginning of a busy period), we have  $Q_{d(0)-} = 0$ . Then, using (3), we obtain

$$Q_t = L[d(0), t] - (t - d(0)), \quad d(0) \leq t < g(1),$$

which gives

$$B = g(1) - d(0) = \inf\{t > 0: L[d(0), d(0) + t] = t\}.$$

Consider now  $L_{d(0);u}^{-1}$ , defined by (1). By Lemma 2,  $d(0)$  is a point of increase of the function  $t \mapsto L[d(0), d(0) + t]$ . Hence,  $g(1) > d(0)$ . Also, when  $L[d(0), d(0) + t] - t$  decreases, it does so continuously. Therefore,

$$B = \inf\{t > 0: L[d(0), d(0) + t] \leq t\}.$$

Note also that, for all  $t, x > 0$ ,

$$L[d(0), d(0) + t] \leq x \iff t \leq L_{d(0);x+\varepsilon}^{-1} \quad \text{for all } \varepsilon > 0.$$

It follows that

$$B = \inf\{t > 0: L_{d(0);t}^{-1} > t\},$$

by the right continuity of  $t \mapsto L_{d(0);t}^{-1}$ . To prove the expression for  $B + I$ , note that, as a measure,  $L$  is not supported in the interval  $[g(1), d(1))$  because, by definition,  $Q$  is 0 for all  $t$  in this interval. This completes the proof.

Henceforth it will be convenient to work with the process  $\Lambda = (\Lambda_t, t \geq 0)$ , where

$$\Lambda_t := t - L_{d(0);t}^{-1}, \quad t \geq 0.$$

Note also that  $d(0)$  is an  $(\mathcal{F}_t)$ -stopping time at which  $X$  takes the value 0 and, hence, in our earlier notation,  $\Lambda_t = \Lambda_{d(0);t}$ .

From (6), and as discussed at the beginning of Section 2, we see that  $B$  is simply the first time at which  $\Lambda$  enters  $(-\infty, 0)$ ,

$$B = \inf\{t > 0: \Lambda_t < 0\}, \tag{8}$$

which is necessarily strictly positive thanks to the irregularity of 0 for  $(-\infty, 0)$  for  $\Lambda$ . From (6) and (7), we see that

$$I = L_{d(0);g(1)}^{-1} - B = L_{d(0);B}^{-1} - B = -\Lambda_B, \tag{9}$$

i.e.  $I$  is, in absolute value, equal to the value of  $\Lambda$  at the first time it becomes negative. Again, we recall from the discussion at the beginning of Section 2 that  $\Lambda$  cannot creep downwards and, hence,  $I > 0$  a.s.

Consider now the random variable  $Q^* = \sup_{d(0) < t < g(1)} Q_t$ . If we define

$$\tau_x := \inf\{t > 0: \Lambda_t > x\} = \inf\{t > 0: \Lambda_t = x\}, \tag{10}$$

we immediately see that

$$\{Q^* < x\} = \{B < \tau_x\}. \tag{11}$$

### 3. The triple law

Recall that  $P_d$  is the Palm probability with respect to the point process  $\{d(n), n \in \mathbb{Z}\}$ . The function

$$H(\alpha, \beta, x) = E_d[e^{-\alpha B - \beta I} \mathbf{1}(Q^* \leq x)]$$

characterises the joint law of the triple  $(B, I, Q^*)$  under  $P_d$ . Since  $P_d(d(0) = 0) = 1$ , we have

$$\Lambda_t = t - L_{0;t}^{-1} \quad \text{with } \Lambda_0 = 0, P_d\text{-a.s.} \tag{12}$$

Recalling (8), (9), and (11) for  $B, I$ , and  $Q^*$ , respectively, we write

$$H(\alpha, \beta, x) = E_d[e^{-\alpha B + \beta \Lambda_B} \mathbf{1}(B \leq \tau_x)]. \tag{13}$$

Since our primary object is the process  $\Lambda$  defined in (12), and in view of (4) and (13), it makes sense to consider the process on its canonical probability space and denote its law by  $\hat{P}_0$ . Then

$$H(\alpha, \beta, x) = \hat{E}_0[e^{-\alpha B + \beta \Lambda_B} \mathbf{1}(B < \tau_x)]. \tag{14}$$

The latter function may now be expressed in terms of so-called scale functions for spectrally negative Lévy processes. To define the latter, let

$$\psi_\Lambda(\theta) := \log \hat{E}_0 e^{\theta \Lambda_1}, \quad \theta \geq 0,$$

be the Laplace exponent of  $\Lambda$  under  $\hat{P}_0$ . Then the, so-called,  $q$ -scale function for  $(\Lambda, \hat{P}_0)$ , denoted by  $W^{(q)}(x)$ , satisfies  $W^{(q)}(x) = 0$  for  $x < 0$  and on  $[0, \infty)$  it is the unique continuous (right continuous at the origin) monotone increasing function whose Laplace transform is given by

$$\int_0^\infty e^{-\theta x} W^{(q)}(x) dx = \frac{1}{\psi_\Lambda(\theta) - q} \quad \text{for } \beta > \Phi_\Lambda(q), \tag{15}$$

where

$$\Phi_\Lambda(q) = \sup\{\theta \geq 0: \psi_\Lambda(\theta) = q\}$$

is the right inverse of  $\psi_\Lambda$ . (See, for example, the discussion in Chapter 9 of [10].)

**Theorem 1.** Let  $\Lambda$  be the process defined by (12), let  $B$  be its first entry time to  $(-\infty, 0)$  as in (8), and let  $\tau_x$  be the first hitting time of  $\{x\}$  as in (10). For  $\alpha, \beta, x \geq 0$ , we have

$$\begin{aligned}
 H(\alpha, \beta, x) &= \hat{E}_0[e^{-\alpha B + \beta \Lambda_B} \mathbf{1}(B < \tau_x)] \\
 &= 1 - \frac{1}{\delta_\Lambda} \frac{1 + (\alpha - \psi_\Lambda(\beta)) \int_0^x e^{-\beta y} W^{(\alpha)}(y) dy}{e^{-\beta x} W^{(\alpha)}(x)}. \tag{16}
 \end{aligned}$$

*Proof.* Let  $\mathcal{G}_t := \sigma(\Lambda_s, s \leq t)$ , and define, for all  $\beta \geq 0$ , the exponential  $(\mathcal{G}_t)$ -martingale

$$M_t^\beta := e^{\beta \Lambda_t - \psi_\Lambda(\beta)t}, \quad t \geq 0.$$

On the canonical space of  $\Lambda$ , let  $\hat{P}_0^\beta$  be a probability measure, absolutely continuous with respect to  $\hat{P}_0$  on  $\mathcal{G}_t$  for each  $t$ , with Radon–Nikodým derivative

$$\left. \frac{d\hat{P}_0^\beta}{d\hat{P}_0} \right|_{\mathcal{G}_t} := M_t^\beta.$$

Note that  $\Lambda$  is still a Lévy process under  $\hat{P}_0^\beta$  with Laplace exponent

$$\psi_\Lambda^\beta(\theta) = \log \hat{E}_0^\beta e^{\theta \Lambda_1} = \psi_\Lambda(\beta + \theta) - \psi_\Lambda(\beta).$$

It is straightforward to check from the above formula that, under  $\hat{P}_0^\beta$ ,  $\Lambda$  is spectrally negative, with bounded variation paths and drift coefficient equal to  $\delta_\Lambda$ . Since on the stopped  $\sigma$ -field  $\mathcal{G}_B$  we have  $(d\hat{P}_0^\beta/d\hat{P}_0)|_{\mathcal{G}_B} = M_B^\beta$ , we may substitute

$$e^{\beta \Lambda_B} = M_B^\beta e^{\psi_\Lambda(\beta)B}$$

in (14) for  $H$  to obtain

$$H(\alpha, \beta, x) = \hat{E}_0[M_B^\beta e^{\psi_\Lambda(\beta)B} e^{-\alpha B} \mathbf{1}(B < \tau_x)] = \hat{E}_0^\beta[e^{-(\alpha - \psi_\Lambda(\beta))B} \mathbf{1}(B < \tau_x)].$$

Let

$$q := \alpha - \psi_\Lambda(\beta),$$

and assume that  $q \geq 0$ . It follows from [10, Theorem 8.1(iii)] that

$$H(\alpha, \beta, x) = \hat{E}_0^\beta[e^{-qB} \mathbf{1}(B < \tau_x)] = Z_\beta^{(q)}(0) - Z_\beta^{(q)}(x) \frac{W_\beta^{(q)}(0)}{W_\beta^{(q)}(x)}, \tag{17}$$

where  $W_\beta^{(q)}$  is the  $q$ -scale function for  $(\Lambda, \hat{P}_0^\beta)$  and  $Z_\beta^{(q)}$  is given by

$$Z_\beta^{(q)}(x) = 1 + q \int_0^x W_\beta^{(q)}(t) dt.$$

It is easy to see [10, Lemma 8.4] that the Laplace transform of  $W_\beta^{(q)}(\cdot)$  is the Laplace transform of  $W^{(q)}(\cdot)$  shifted by  $\beta$  and this ensures that

$$W_\beta^{(q)}(x) = e^{-\beta x} W^{(q)}(x).$$

Moreover, since  $\Lambda$  still has drift coefficient  $\delta_\Lambda$  under  $\hat{P}_0^\beta$ , [10, Lemma 8.6] tells us that, irrespective of the values of  $q$  and  $\beta$ ,  $W_\beta^{(q)}(0) = 1/\delta_\Lambda$ . Putting the pieces together, this gives us the desired expression for  $\alpha \geq \psi_\Lambda(\beta)$ . However [10, Lemma 8.3], since  $W^{(q)}(x)$  is analytic in  $q$ , the condition on  $\alpha$  can be relaxed to  $\alpha \geq 0$  by using a straightforward analytic extension argument. This completes the proof.

In view of (8), (9), (11), and (13), we obtain the following corollary.

**Corollary 1.** (The joint law of typical  $B$ ,  $I$ , and  $Q^*$ .) *Assume that (A1)–(A5) hold. Then the joint law of the length  $B$  of a typical busy period, the length  $I$  of a typical idle period, and the maximum  $Q^*$  of  $Q$  over the typical busy period are expressed by the formula*

$$E_d[e^{-\alpha B - \beta I} \mathbf{1}(Q^* \leq x)] = 1 - \frac{1}{\delta_\Lambda} \frac{1 + (\alpha - \psi_\Lambda(\beta)) \int_0^x e^{-\beta y} W^{(\alpha)}(y) dy}{e^{-\beta x} W^{(\alpha)}(x)}, \tag{18}$$

where  $\alpha, \beta, x \geq 0$ .

#### 4. Marginal distributions

Clearly, (18) can be used to extract more detailed information about the typical behaviour of  $Q$ . Let us first derive the distribution (Laplace transform) of the pair  $(B, I)$  under the measure  $P_d$ . We have

$$E_d[e^{-\alpha B} e^{-\beta I}] = \hat{E}_0[e^{-\alpha B} e^{\beta \Lambda_B}] = \lim_{x \rightarrow \infty} H(\alpha, \beta, x).$$

To derive the limit, let us temporarily assume that  $q = \alpha - \psi_\Lambda(\beta) > 0$  and  $\beta \geq 0$ . Consider (16) in the form (17), and use the limiting result

$$\lim_{x \rightarrow \infty} \frac{Z_\beta^{(q)}(x)}{W_\beta^{(q)}(x)} = \frac{q}{\Phi_\Lambda^\beta(q)}$$

from [10, Exercise 8.5], where the function  $\Phi_\Lambda^\beta$  is the right inverse of  $\psi_\Lambda^\beta$ . That is,

$$\begin{aligned} \Phi_\Lambda^\beta(q) &= \sup\{\theta \geq 0: \psi_\Lambda^\beta(\theta) = q\} \\ &= \sup\{\theta \geq 0: \psi_\Lambda^\beta(\theta) = \alpha - \psi_\Lambda(\beta)\} \\ &= \sup\{\theta \geq 0: \psi_\Lambda(\theta + \beta) = \alpha\} \\ &= \Phi_\Lambda(\alpha) - \beta. \end{aligned}$$

This gives

$$E_d[e^{-\alpha B} e^{-\beta I}] = 1 - \frac{1}{\delta_\Lambda} \frac{\alpha - \psi_\Lambda(\beta)}{\Phi_\Lambda(\alpha) - \beta}. \tag{19}$$

To remove the restriction that  $\alpha > \psi_\Lambda(\beta)$  in (19) and replace it instead by just  $\alpha \geq 0$ , we may again proceed with an argument involving analytical extension, taking care to note that, for the case where  $\alpha = \psi_\Lambda(\beta)$ ,

$$\lim_{|\alpha - \psi_\Lambda(\beta)| \rightarrow 0} \frac{\alpha - \psi_\Lambda(\beta)}{\Phi_\Lambda(\alpha) - \beta} = \lim_{|\alpha - \psi_\Lambda(\beta)| \rightarrow 0} \frac{\psi_\Lambda^\beta(\Phi_\Lambda(\alpha) - \beta)}{\Phi_\Lambda(\alpha) - \beta} = \psi_\Lambda^{\beta'}(0+) = \psi_\Lambda'(\beta).$$

##### 4.1. Busy period

Letting  $\beta = 0$  in (19), we find the  $P_d$ -law of  $B$ . That is,

$$E_d[e^{-\alpha B}] = 1 - \frac{1}{\delta_\Lambda} \frac{\alpha}{\Phi_\Lambda(\alpha)}.$$

This formula is consistent with the result of [8, Proposition 8] and, moreover, we see that the mean duration of the busy period is given by

$$E_d[B] = \frac{1}{\delta_\Lambda \Phi_\Lambda(0)}. \tag{20}$$

**4.2. Idle period**

To find the  $P_d$ -law of  $I$ , we need to set  $\alpha = 0$ . Recall, however, from the beginning of Section 2 that  $E_d(\Lambda_1) < 0$ . This implies that  $\Phi_\Lambda(0) > 0$  and, hence, we have

$$E_d[e^{-\beta I}] = 1 - \frac{1}{\delta_\Lambda} \frac{\psi_\Lambda(\beta)}{\beta - \Phi_\Lambda(0)}. \tag{21}$$

It follows that the mean idle period is thus equal to

$$E_d[I] = \frac{-\psi'_\Lambda(0+)}{\delta_\Lambda \Phi_\Lambda(0)}, \tag{22}$$

where  $\psi'_\Lambda(0+) = E_d(\Lambda_1) < 0$ .

**4.3. Rates**

A cycle of the process  $Q$  is defined as the interval from the beginning of a busy period until the beginning of the next busy period. We therefore have

$$\text{mean cycle length} = E_d[B + I] = \frac{1 - \psi'_\Lambda(0+)}{\delta_\Lambda \Phi_\Lambda(0)}. \tag{23}$$

We can express the common rate,  $\lambda$ , of  $N_g$  and  $N_d$  as the inverse of the mean cycle length:

$$\lambda := E N_d(0, 1) = E N_g(0, 1) = \frac{1}{E_d[B + I]} = \frac{\delta_\Lambda \Phi_\Lambda(0)}{1 - \psi'_\Lambda(0+)}. \tag{24}$$

**4.4. The maximum over a busy period**

We now derive the  $P_d$ -distribution of  $Q^*$ . Letting  $\alpha = \beta = 0$  in (16), we obtain

$$P_d(Q^* \leq x) = 1 - \frac{1}{\delta_\Lambda W(x)},$$

where  $W(x) \equiv W^{(0)}(x)$  is defined through its Laplace transform

$$\int_0^\infty e^{-\theta x} W(x) dx = \frac{1}{\psi_\Lambda(\theta)} \quad \text{for } \theta > \Phi_\Lambda(0). \tag{25}$$

An immediate observation is that  $\lim_{x \rightarrow 0} P_d(Q^* \leq x) = 0$ , since  $W(0) = \lim_{\theta \rightarrow 0} \theta / \psi_\Lambda(\theta) = 1 / \delta_\Lambda$ . So, under  $P_d$ , the random variable  $Q^*$  has no atom at 0—which is, of course, expected.

We now show that  $Q^*$  has exponential tail under  $P_d$  and derive the precise asymptotics. To do this, let

$$\beta^* := \Phi_\Lambda(0).$$

Then it follows from (25) that the Laplace transform of  $x \mapsto e^{-\beta^* x} W(x)$  is  $\theta \mapsto 1 / \psi_\Lambda(\beta^* + \theta)$ . From the final value theorem for Laplace transforms,

$$\lim_{x \rightarrow \infty} e^{-\beta^* x} W(x) = \lim_{\theta \rightarrow 0} \frac{\theta}{\psi_\Lambda(\beta^* + \theta)} = \frac{1}{\psi'_\Lambda(\beta^*)},$$

where we have used the fact that  $\psi_\Lambda(\beta^*) = 0$ . It follows that

$$P_d(Q^* > x) \sim \frac{\psi'_\Lambda(\Phi_\Lambda(0))}{\delta_\Lambda} e^{-\Phi_\Lambda(0)x}$$

as  $x \rightarrow \infty$ .

### 5. Cycle formulae

We now show how the use of cycle formulae of Palm calculus enable us to find (see Theorem 2 below) the joint law of the endpoints of an idle period conditional on the event that the idle period contains the origin of time. Also, we characterise (see Theorem 3 below) the joint law of the endpoints of a busy period, together with the maximum of  $Q$  over this busy period, conditional on the event that the busy period contains the origin of time.

Let  $(\Omega, \mathcal{F}, P)$  be endowed with a P-preserving flow  $(\theta_t, t \geq 0)$  (see the end of Section 1). Consider a random measure  $M$  with finite intensity  $\lambda_M$ , and a point process  $N$  with finite intensity  $\lambda_N$  such that  $M(B, \theta_t\omega) = M(B + t, \omega)$  and  $N(B, \theta_t\omega) = N(B + t, \omega)$  for  $t \in \mathbb{R}$ , a Borel subset  $B$  of  $\mathbb{R}$ , and  $\omega \in \Omega$ . (In other words,  $M$  and  $N$  are jointly stationary.) Then, for any nonnegative measurable  $Z: \Omega \rightarrow \mathbb{R}$ , we have

$$\lambda_M E_M[Z] = \lambda_N E_N \int_{T_k}^{T_{k+1}} Z \circ \theta_t M(dt), \tag{26}$$

where  $P_M, E_M$  and  $P_N, E_N$  respectively denote the Palm probability and expectation with respect to  $M$  and  $N$ ,  $T_0$  is the first atom of  $N$  which is less than or equal to 0, and  $T_k$  and  $T_{k+1}$  are any two successive atoms of  $N$ . (See [12] for a special case.)

The next result can be found for some special cases in [8] (Lévy processes) and [9] (diffusions), and the general expression is derived in [13]. Here we offer a new proof in the general case based on (26).

**Theorem 2.** (The joint law of the endpoints of the idle period.) *Assume that (A1)–(A5) hold. Then, conditional on  $Q_0 = 0$ , the left endpoint,  $g(0)$ , and the right endpoint,  $d(0)$ , of the idle period containing  $t = 0$  have joint Laplace transform given by*

$$E[e^{-\alpha d(0) + \beta g(0)} \mid Q_0 = 0] = \frac{\Phi_\Lambda(0)}{-\psi'_\Lambda(0+)} \frac{1}{\alpha - \beta} g \left( \frac{\psi_\Lambda(\alpha)}{\alpha - \Phi_\Lambda(0)} - \frac{\psi_\Lambda(\beta)}{\beta - \Phi_\Lambda(0)} g \right)$$

for nonnegative  $\alpha$  and  $\beta$  ( $\alpha \neq \beta$ ).

*Proof.* Let  $M_I$  be the restriction of the Lebesgue measure on the idle periods:

$$M_I(A) = \int_A \mathbf{1}(Q_t = 0) dt, \quad A \in \mathcal{B}(\mathbb{R}).$$

Then  $E_{M_I}[Z] = E[Z \mid Q_0 = 0]$  for all nonnegative random variables  $Z$ . Apply (26) with  $M = M_I, N = N_d$ , and  $Z = e^{-\alpha d(0) + \beta d(0)}$ :

$$\lambda_{M_I} E_{M_I}[e^{-\alpha d(0) + \beta g(0)}] = \lambda E_d \int_{d(-)}^{d(0)} e^{-\alpha d(0) \circ \theta_t + \beta g(0) \circ \theta_t} M_I(dt).$$

Here  $\lambda$  is the rate of  $N_d$  and is given by (24). The rate  $\lambda_{M_I}$  is given by

$$\lambda_{M_I} = \frac{E_d[I]}{E_d[B + I]}.$$

Hence,

$$\frac{\lambda}{\lambda_{M_I}} = \frac{1}{E_d[I]} = \frac{\delta_\Lambda \Phi_\Lambda(0)}{-\psi'_\Lambda(0+)},$$

where we have used (22) and (23). Now,  $P_d(d(0) = 0) = 1$ . To compute the integral above, note that  $M_I$  is 0 on the interval  $(d(-1), g(0))$  and that,  $P_d$ -a.s., for  $g(0) \leq t \leq d(0) = 0$ , we have  $d(0) \circ \theta_t = -t$  and  $g(0) \circ \theta_t = g(0) - t$ . So,  $P_d$ -a.s.,

$$\int_{d(-1)}^{d(0)} e^{-\alpha d(0) \circ \theta_t + \beta g(0) \circ \theta_t} M_I(dt) = \int_{g(0)}^0 e^{(\alpha-\beta)t - \alpha g(1)} dt = \frac{e^{\beta g(0)} - e^{\alpha g(0)}}{\alpha - \beta}.$$

Combining the above we obtain

$$E[e^{-\alpha d(0) + \beta g(0)} \mid Q_0 = 0] = \frac{\Phi_\Lambda(0)}{-\psi'_\Lambda(0+)} \frac{E_d[e^{\beta g(0)}] - E_d[e^{\alpha g(0)}]}{\alpha - \beta}.$$

Since  $E_d[e^{\beta g(0)}] = E_d[e^{-\beta I}]$ , the result is obtained by using (21).

**Theorem 3.** (The joint law of the endpoints of the busy period and the maximum over it.) *Assume that (A1)–(A5) hold. Then, conditional on  $Q_0 > 0$ , the left endpoint,  $d(0)$ , and the right endpoint,  $g(1)$ , of the busy period containing  $t = 0$ , together with the maximum of  $Q_s$  for  $s$  ranging over this busy period, have a joint law which is characterised by*

$$\begin{aligned} & E[e^{-\alpha g(1) + \beta d(0)} \mathbf{1}(Q^* \leq x) \mid Q_0 > 0] \\ &= \frac{\Phi_\Lambda(0)}{\alpha - \beta} g \left( \frac{1 + \alpha \int_0^x W^{(\alpha)}(y) dy}{W^{(\alpha)}(x)} - \frac{1 + \beta \int_0^x W^{(\beta)}(y) dy}{W^{(\beta)}(x)} g \right) \end{aligned} \tag{27}$$

for nonnegative  $\alpha$  and  $\beta$  ( $\alpha \neq \beta$ ).

*Proof.* Let  $M_B$  be the restriction of the Lebesgue measure on the busy periods:

$$M_B(A) = \int_A \mathbf{1}(Q_t > 0) dt, \quad A \in \mathcal{B}(\mathbb{R}).$$

Then  $E_{M_B}[Z] = E[Z \mid Q_0 > 0]$  for all random variables  $Z \geq 0$ . Apply (26) to obtain

$$\begin{aligned} & \lambda_{M_B} E_{M_B}[e^{-\alpha g(1) + \beta d(0)} \mathbf{1}(Q^* \leq x)] \\ &= \lambda E_d \int_{d(0)}^{d(1)} e^{-\alpha g(1) \circ \theta_t + \beta d(0) \circ \theta_t} \mathbf{1}(Q^* \circ \theta_t \leq x) M_B(dt) \\ &= \lambda E_d \int_0^{g(1)} e^{-\alpha(g(1)-t) - \beta t} \mathbf{1}(Q^* \leq x) dt \\ &= \lambda E_d g \left[ \mathbf{1}(Q^* \leq x) e^{-\alpha g(1)} \frac{e^{(\alpha-\beta)g(1)} - 1}{\alpha - \beta} g \right] \\ &= \frac{\lambda}{\alpha - \beta} g (E_d[e^{-\beta g(1)} \mathbf{1}(Q^* \leq x)] - E_d[e^{-\alpha g(1)} \mathbf{1}(Q^* \leq x)] g) \\ &= \frac{\lambda}{\alpha - \beta} (H(\beta, 0, x) - H(\alpha, 0, x)), \end{aligned} \tag{28}$$

where  $H(\alpha, \beta, x)$  is the right-hand side of (18). Using (20), (23), and (24), we have

$$\frac{\lambda}{\lambda_{M_B}} = \frac{1}{E_d[B]} = \delta_\Lambda \Phi_\Lambda(0).$$

Combining the above we obtain the announced formula.

Theorem 3 yields Corollary 2, below, which recovers a result obtained in [13] using different methods (for special cases, see [8] and [9]). Clearly, Corollary 2 could also be proved analogously to Theorem 2.

**Corollary 2.** *Assume that (A1)–(A5) hold. Then, conditional on  $Q_0 > 0$ , the left endpoint,  $d(0)$ , and the right endpoint,  $g(1)$ , of the busy period containing  $t = 0$  have joint Laplace transform given by*

$$E[e^{-\alpha g(1) + \beta d(0)} \mid Q_0 > 0] = \frac{\Phi_\Lambda(0)}{\alpha - \beta} g \left( \frac{\alpha}{\Phi_\Lambda(\alpha)} - \frac{\beta}{\Phi_\Lambda(\beta)} g \right)$$

for nonnegative  $\alpha$  and  $\beta$  ( $\alpha \neq \beta$ ).

*Proof.* The argument proceeds as in the proof of Theorem 3 by omitting the factor  $\mathbf{1}(Q^* \leq x)$ , i.e. by formally replacing  $x$  with  $+\infty$ . The last line of (28) will give  $(\lambda/(\alpha - \beta))(H(\beta, 0, \infty) - H(\alpha, 0, \infty))$ , where  $H(\alpha, \beta, \infty)$  is given by the right-hand side of (19).

**Corollary 3.** *Assume that (A1)–(A5) hold. Then, conditional on  $Q_0 > 0$ , the maximum of  $Q$  over the busy period containing  $t = 0$  has distribution*

$$P(Q^* \leq x \mid Q_0 > 0) = \Phi_\Lambda(0) \frac{W(x) \int_0^x W(y) dy - \int_0^x W(x - y)W(y) dy}{W(x)^2} \tag{29}$$

for  $x \geq 0$ .

*Proof.* Letting  $\alpha, \beta \rightarrow 0$  in (27) yields

$$P(Q^* \leq x \mid Q_0 > 0) = \Phi_\Lambda(0) \lim_{\alpha \rightarrow 0} \frac{\partial \hat{H}}{\partial \alpha}(\alpha, 0, x),$$

where

$$\hat{H}(\alpha, 0, x) = 1 - \frac{1 + \alpha \int_0^x W^{(\alpha)}(y) dy}{W^{(\alpha)}(x)}.$$

Next, recall that, for each  $x > 0$ ,  $W^{(\alpha)}(x)$  is an entire function in the variable  $\alpha$  and, in particular,

$$W^{(\alpha)}(x) = \sum_{k \geq 0} \alpha^k W^{*(k+1)}(x),$$

where  $W^{*(k+1)}(x)$  is the  $(k + 1)$ th convolution of  $W$  (cf. [2]). From this we easily deduce that

$$\left. \frac{\partial}{\partial \alpha} W^{(\alpha)}(x) \right|_{\alpha=0} = \int_0^x W(y)W(x - y) dy.$$

The result now follows from straightforward differentiation.

**6. Example: local-time storage from reflected Brownian motion with negative drift**

Let  $X = \{X_t, t \in \mathbb{R}\}$  be a reflected Brownian motion with drift  $-c < 0$  in stationary state living on  $I = [0, \infty)$ , and let  $P_0$  denote the probability measure associated with  $X$  when initiated from 0 at time 0. Its local time (at 0) for  $s < t$  is given by

$$L(s, t] := \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_s^t \mathbf{1}_{[0, \varepsilon)}(X_u) du. \tag{30}$$

It is a standard result that the law of  $(L(0, t], t \geq 0)$  coincides with the law of  $(-\inf_{0 \leq s \leq t} X_s, t \geq 0)$ . (This is the reason we have the factor  $\frac{1}{2}$  in front of the integral in (30).) Let  $Q$  be the stationary process defined as in (2):

$$Q_t := \sup_{s \leq t} \{L(s, t] - (t - s)\}.$$

This particular example of fluid queues was introduced and analysed in [11] and further studied in [8] and [9].

Recall that  $E_0 L(0, 1] = c$ , and, hence,  $Q$  is well defined if and only if  $0 < c < 1$ . Here we make this example more complete by finding the  $\alpha$ -scale function associated with the process  $\Lambda_t := t - L_t^{-1}, t \geq 0$ , where

$$L_t^{-1} := \inf\{s : L(0, s] > t\}, \quad t \geq 0,$$

is the inverse local-time process. As seen from (16) and (29), the  $\alpha$ -scale function is the key ingredient needed for computing the distribution of the maximum of  $Q$  over a busy period and related random variables.

To begin with, we recall some basic formulae. When normalising as in (30) (see [4, p. 22] and [7, p. 214]), it holds that

$$\begin{aligned} E_0(\exp\{-\theta L_t^{-1}\}) &= \exp\left\{-t \int_0^\infty (1 - e^{-\theta u}) \frac{1}{\sqrt{2\pi u^3}} e^{-c^2 u/2} du\right\} \\ &= \exp\left\{-\frac{t}{G_\theta(0, 0)}\right\}, \end{aligned}$$

where

$$G_\theta(0, 0) := \frac{1}{\sqrt{2\theta + c^2 - c}}$$

is the resolvent kernel (Green kernel) of  $X$  at  $(0, 0)$ ; see [4, p. 129]. Consequently, we have

$$E_0(\exp\{\theta \Lambda_t\}) = \exp\left\{t \left(\theta - \frac{1}{G_\theta(0, 0)}\right)\right\} = \exp\{t \psi_\Lambda(\theta)\},$$

where

$$\psi_\Lambda(\theta) := \theta - \sqrt{2\theta + c^2} + c, \quad \theta \geq 0.$$

Recall (cf. (15)) that the  $\alpha$ -scale function ( $\alpha \geq 0$ ) associated with  $\Lambda$  is defined for  $x \geq 0$  via

$$\int_0^\infty e^{-\theta x} W^{(\alpha)}(x) dx = \frac{1}{\psi_\Lambda(\theta) - \alpha}; \tag{31}$$

for  $x < 0$ , we set  $W^{(\alpha)}(x) = 0$ . The 0-scale function is simply called the scale function and denoted by  $W$ . For the next proposition, introduce

$$\text{Erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt,$$

and note that  $\text{Erfc}(0) = 1, \text{Erfc}(+\infty) = 0$ , and  $\text{Erfc}(-\infty) = 2$ .

**Proposition 1.** The  $\alpha$ -scale function  $W^{(\alpha)}$  of  $\Lambda$  is, for  $x \geq 0$ , given by

$$W^{(\alpha)}(x) = \frac{e^{-c^2x/2}}{\lambda_1 - \lambda_2} \left( \lambda_1 e^{\lambda_1^2 x/2} \operatorname{Erfc} \left( -\lambda_1 \sqrt{\frac{x}{2}} \right) - \lambda_2 e^{\lambda_2^2 x/2} \operatorname{Erfc} \left( -\lambda_2 \sqrt{\frac{x}{2}} \right) \right), \tag{32}$$

where

$$\lambda_1 := 1 + \sqrt{(1 - c)^2 + 2\alpha}, \quad \lambda_2 := 1 - \sqrt{(1 - c)^2 + 2\alpha}. \tag{33}$$

In particular,

$$W(x) = \frac{e^{-c^2x/2}}{2(1 - c)} \left( (2 - c)e^{(2-c)^2x/2} \operatorname{Erfc} \left( -(2 - c)\sqrt{\frac{x}{2}} \right) - ce^{c^2x/2} \operatorname{Erfc} \left( -c\sqrt{\frac{x}{2}} \right) \right), \tag{34}$$

and  $W(0) = 1$ .

*Proof.* From (31) we have

$$\int_0^\infty e^{-\theta x} W^{(\alpha)}(x) dx = \frac{1}{\theta - \sqrt{2\theta + c^2} + c - \alpha}. \tag{35}$$

To invert this Laplace transform, introduce  $\lambda := 2\theta + c^2$ . With this notation,

$$\begin{aligned} \frac{1}{\theta - \sqrt{2\theta + c^2} + c - \alpha} &= \frac{2}{\lambda - 2\sqrt{\lambda} + 2(c - \alpha) - c^2} \\ &= \frac{2}{(\sqrt{\lambda} - \lambda_1)(\sqrt{\lambda} - \lambda_2)} \\ &= \frac{2}{\lambda_1 - \lambda_2} \left( \frac{1}{\sqrt{\lambda} - \lambda_1} - \frac{1}{\sqrt{\lambda} - \lambda_2} \right), \end{aligned} \tag{36}$$

where  $\lambda_{1,2}$  are the roots of the equation  $z^2 - 2z + 2(c - \alpha) - c^2 = 0$ , i.e. as in (33). Next, recall the following Laplace inversion formula (cf. [5, p. 233]) valid for  $\lambda - \beta^2 > 0$ :

$$\mathcal{L}^{-1} \left( \frac{1}{\sqrt{\lambda} + \beta} \right) = \frac{1}{\sqrt{\pi x}} - \beta e^{\beta^2 x} \operatorname{Erfc}(\beta\sqrt{x}). \tag{37}$$

Since

$$\int_0^\infty e^{-\theta x} W^{(\alpha)}(x) dx = \int_0^\infty e^{-\lambda y} e^{c^2 y} W^{(\alpha)}(2y) 2 dy,$$

we obtain, using (37),

$$2e^{c^2 y} W^{(\alpha)}(2y) = \frac{2}{\lambda_1 - \lambda_2} (\lambda_1 e^{\lambda_1^2 y} \operatorname{Erfc}(-\lambda_1 \sqrt{y}) - \lambda_2 e^{\lambda_2^2 y} \operatorname{Erfc}(-\lambda_2 \sqrt{y})),$$

which is (32). In particular, when  $\alpha = 0$ , it holds that  $\lambda_1 = 2 - c$  and  $\lambda_2 = c$ , yielding (34).

Using the scale function  $W$  and the fact that

$$\Phi_\Lambda(0) = \sup\{\theta > 0: \psi_\Lambda(\theta) = 0\} = 2(1 - c),$$

(29) yields the distribution of the maximum  $Q^*$  over an observed busy period (i.e. over a busy period containing the origin of time).

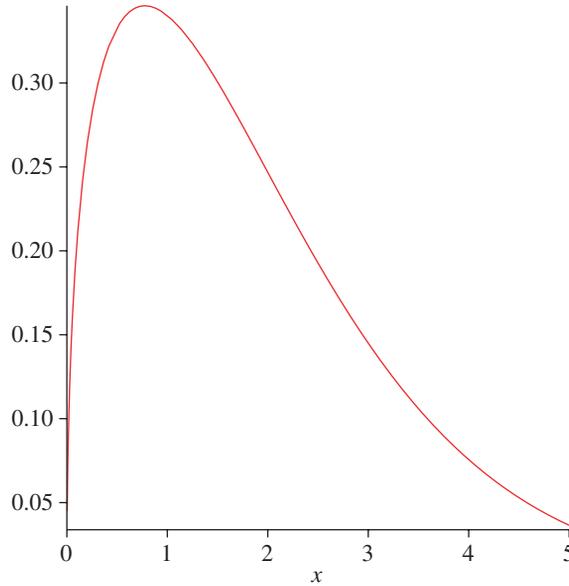


FIGURE 3: The density of  $Q^*$  conditional on  $\{Q_0 > 0\}$  for the example corresponding to Brownian motion with drift  $-c = -\frac{1}{2}$ .

**Proposition 2.** Let  $0 < c < 1$ . The distribution of the maximum  $Q^*$  over an observed busy period of a local-time storage associated with a reflected Brownian motion with drift  $-c$  is given by

$$P(Q^* \leq x \mid Q_0 > 0) = 2(1 - c) \frac{\int_0^x W(y)(W(x) - W(x - y)) \, dy}{W^2(x)}, \tag{38}$$

where the scale function  $W$  is given by (34).

We plot the derivative of (38) for  $c = \frac{1}{2}$  in Figure 3.

We recall some formulae from [9]. First

$$\begin{aligned} E[e^{\theta d(0) - \beta g(1)} \mid Q_0 > 0] &= \frac{8(1 - c)}{\sqrt{2\theta + (1 - c)^2} + \sqrt{2\beta + (1 - c)^2}} \\ &\quad \times \frac{1}{(\sqrt{2\theta + (1 - c)^2} + 1 + c)(\sqrt{2\beta + (1 - c)^2} + 1 + c)} \\ &=: F(\theta, \beta; 1 - c) \end{aligned} \tag{39}$$

and

$$\hat{E}[e^{\theta g(0) - \beta d(0)} \mid Q_0 = 0] = F(\theta, \beta; c). \tag{40}$$

Setting  $\beta = \theta$  in the right-hand side of (39) and (40), respectively, we obtain

$$E[e^{-\theta(g(1) - d(0))} \mid Q_0 > 0] = \frac{4(1 - c)}{\sqrt{2\theta + (1 - c)^2}(\sqrt{2\theta + (1 - c)^2} + 1 + c)^2} \tag{41}$$

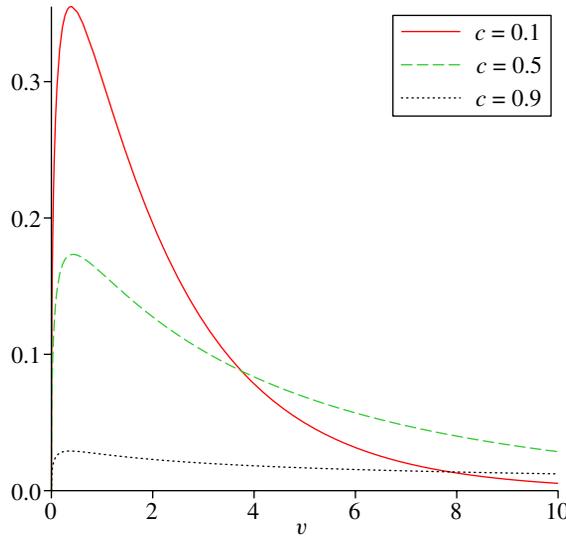


FIGURE 4: The density of the length of the busy period, given that  $Q_0 > 0$ , for three different values of  $c$ .

and

$$E[e^{-\theta(d(0)-g(0))} \mid Q_0 = 0] = \frac{4c}{\sqrt{2\theta + c^2}(\sqrt{2\theta + c^2} + 2 - c)^2}.$$

Taking the inverse Laplace transform of (41) (cf. [5, p. 234]), we obtain the density of the length of the busy period  $g(1) - d(0)$ , given  $Q_0 > 0$ , as

$$f_{g-b}(v) = 2(1 - c)e^{-(1-c)^2v/2} \left( \sqrt{\frac{2v}{\pi}} - (1 + c)ve^{(1+c)^2v/2} \operatorname{Erfc} \left( (1 + c)\sqrt{\frac{v}{2}} \right) \right).$$

Note that the density of the length of the idle period  $d(0) - g(0)$ , given that  $Q_0 = 0$ , is obtained from  $f_{g-b}(v)$  by substituting  $c$  for  $1 - c$ . In Figure 4 we plot  $f_{g-b}(v)$  for three different values of  $c$ . We note also that the mean busy period length has a simple expression:

$$E[g(1) - d(0) \mid Q_0 > 0] = \frac{2 - c}{(1 - c)^2}.$$

The joint density of  $d(0)$  and  $g(1)$  is given by

$$\begin{aligned} & \hat{P}[-d(0) \in dx, g(1) \in dy \mid Q_0 > 0] \\ &= 2(1 - c)e^{-(1-c)^2(x+y)/2} \\ & \times \left( \sqrt{\frac{2}{\pi(x+y)}} - (1 + c)e^{(1+c)^2(x+y)/2} \operatorname{Erfc} \left( (1 + c)\sqrt{\frac{x+y}{2}} \right) \right) \end{aligned}$$

and, again, the density for  $(g(0), d(0))$  is obtained by substituting  $c$  for  $1 - c$ .

Next we find the density of  $g(1)$  (recall that  $-d(0)$  is identical in law to  $g(1)$ ) by inverting the Laplace transform (obtained from (39) by choosing  $\theta = 0$ ):

$$E[e^{-\beta g(1)} \mid Q_0 > 0] = \frac{4(1 - c)}{(\sqrt{2\beta + (1 - c)^2} + 1 - c)(\sqrt{2\beta + (1 - c)^2} + 1 + c)}. \tag{42}$$

Letting  $\lambda := 2\beta + (1 - c)^2$  we rewrite (42) as

$$E[e^{-\beta g(1)} \mid Q_0 > 0] = \frac{2(1 - c)}{c} \left( \frac{1}{\sqrt{\lambda} + 1 - c} - \frac{1}{\sqrt{\lambda} + 1 + c} \right). \tag{43}$$

From (37),

$$\begin{aligned} &\mathcal{L}^{-1} \left( \frac{1}{\sqrt{\lambda} + 1 - c} - \frac{1}{\sqrt{\lambda} + 1 + c} \right) \\ &= (1 + c)e^{(1+c)^2x} \operatorname{Erfc}((1 + c)\sqrt{x}) - (1 - c)e^{(1-c)^2x} \operatorname{Erfc}((1 - c)\sqrt{x}). \end{aligned}$$

Consequently,

$$\begin{aligned} 2e^{-(1-c)^2x} f_{g(1)}(2x) &= \frac{2(1 - c)}{c} ((1 + c)e^{(1+c)^2x} \operatorname{Erfc}((1 + c)\sqrt{x}) \\ &\quad - (1 - c)e^{(1-c)^2x} \operatorname{Erfc}((1 - c)\sqrt{x})), \end{aligned} \tag{44}$$

where  $f_{g(1)}$  denotes the density of  $g(1)$  conditioned on  $\{Q_0 > 0\}$ . From (44) we obtain

$$\begin{aligned} f_{g(1)}(x) &= \frac{(1 - c)e^{-(1-c)^2x/2}}{c} \left( (1 + c)e^{(1+c)^2x/2} \operatorname{Erfc} \left( (1 + c)\sqrt{\frac{x}{2}} \right) \right. \\ &\quad \left. - (1 - c)e^{(1-c)^2x/2} \operatorname{Erfc} \left( (1 - c)\sqrt{\frac{x}{2}} \right) \right). \end{aligned} \tag{45}$$

Moreover, the density  $f_{d(0)}$  of  $d(0)$  conditional on  $\{Q_0 = 0\}$  is obtained from (45) by substituting  $c$  with  $1 - c$ :

$$f_{d(0)}(x) = \frac{ce^{-c^2x/2}}{1 - c} \left( (2 - c)e^{(2-c)^2x/2} \operatorname{Erfc} \left( (2 - c)\sqrt{\frac{x}{2}} \right) - ce^{c^2x/2} \operatorname{Erfc} \left( c\sqrt{\frac{x}{2}} \right) \right). \tag{46}$$

It is striking how similar formulae (34) and (46) are.

**Remark 1.** The scale function formulae (32) and (34) are clearly valid for all  $c \geq 0$ . In the  $c = 0$  case the process  $\{L(0, t]; t \geq 0\}$  is a version of the Brownian local time, and the  $\alpha$ -scale function  $W_0^{(\alpha)}$  of the corresponding process  $\Lambda$  is given by

$$\begin{aligned} W_0^{(\alpha)}(x) &= \frac{e^{(1+\alpha)x}}{2\sqrt{1+2\alpha}} \left( (1 + \sqrt{1+2\alpha})e^{x\sqrt{1+2\alpha}} \operatorname{Erfc} \left( -(1 + \sqrt{1+2\alpha})\sqrt{\frac{x}{2}} \right) \right. \\ &\quad \left. - (1 - \sqrt{1+2\alpha})e^{-x\sqrt{1+2\alpha}} \operatorname{Erfc} \left( -(1 - \sqrt{1+2\alpha})\sqrt{\frac{x}{2}} \right) \right). \end{aligned}$$

In particular,

$$W_0(x) = e^{2x} \operatorname{Erfc}(-\sqrt{2x}). \tag{47}$$

In the  $c = 1$  case it holds that  $\lambda_{1,2} = 1 \pm \sqrt{2\alpha}$  and, for  $\alpha \neq 0$ , (34) can be used directly. For the 0-scale function, we need to take the limit as  $c \rightarrow 1$  in (34):

$$W_1(x) = (1 + x) \operatorname{Erfc} \left( -\sqrt{\frac{x}{2}} \right) + \sqrt{\frac{2x}{\pi}} e^{-x/2}. \tag{48}$$

**Remark 2.** Here we display some formulae for Laplace transforms apparent from above and point out a misprint in [5].

First, from (35), (36), and (48), we have the following Laplace inversion formula valid for  $\lambda > 1$ :

$$\mathcal{L}^{-1}\left(\frac{1}{(\sqrt{\lambda} - 1)^2}\right) = (1 + 2x)e^x \operatorname{Erfc}(-\sqrt{x}) + \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

This can be ‘extended’ (for  $a > 0$ ) to

$$\mathcal{L}^{-1}\left(\frac{1}{(\sqrt{\lambda} - a)^2}\right) = (1 + 2a^2x)e^{a^2x} \operatorname{Erfc}(-a\sqrt{x}) + \frac{2a\sqrt{x}}{\sqrt{\pi}}. \tag{49}$$

Furthermore, it can be checked that (49) is valid for all  $a < 0$  by evaluating the Laplace transform of the right-hand side. This can be done term by term by using (see, e.g. [5, pp. 137, 177]) the well-known formulae:

$$\mathcal{L}(\sqrt{x}) = \frac{\sqrt{\pi}}{2}\lambda^{-3/2}, \quad \mathcal{L}(e^{a^2x} \operatorname{Erfc}(a\sqrt{x})) = \lambda^{-1/2}(\lambda^{1/2} + a)^{-1},$$

and

$$\begin{aligned} \mathcal{L}(xe^{a^2x} \operatorname{Erfc}(a\sqrt{x})) &= -\frac{\partial}{\partial \lambda} \mathcal{L}(e^{a^2x} \operatorname{Erfc}(a\sqrt{x})) \\ &= -\frac{\partial}{\partial \lambda} \lambda^{-1/2}(\lambda^{1/2} + a)^{-1} \\ &= \frac{1}{2a}(\lambda^{-3/2} - \lambda^{-1/2}(\lambda^{1/2} + a)^{-2}). \end{aligned}$$

We remark that Formula (10) of [5, p. 234],

$$\mathcal{L}^{-1}\left(\frac{1}{(\sqrt{\lambda} + \sqrt{b})^2}\right) = 1 - 2\sqrt{\frac{bx}{\pi}} + (1 - 2bx)e^{bx}(\operatorname{Erf}(\sqrt{bx}) - 1), \tag{50}$$

is *not* correct since it does not coincide with (49) (for  $a < 0$ ). Indeed, because

$$\operatorname{Erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

the right-hand side of (50) is 0 at 0, but the right-hand side of (49) is 1 at 0.

### 7. Further examples

In the previous example we derived a local-time process from a given Markov process. However, it is also possible to consider examples where just the local-time process  $L$ , or, equivalently, the subordinator  $L^{-1}$ , is specified. Indeed, the subordinator that will play the role of  $L^{-1}$  in this example has no drift and has Lévy measure given by

$$\Pi(x, \infty) = \frac{\gamma^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\gamma x} + \varphi \frac{\gamma^\nu}{\Gamma(\nu)} \int_x^\infty y^{\nu-1} e^{-\gamma y} dy,$$

where the constants  $\varphi, \gamma > 0$  and  $\nu \in (0, 1)$ . Note, in particular, that  $L^{-1}$  is the sum of two independent subordinators, one of which is a compound Poisson process with gamma

distributed jumps, and the other has infinite activity and is of the so-called tempered-stable type. Clearly,  $\Pi$  also describes the Lévy measure of  $-\Lambda$ .

According to [6], the process  $\Lambda$  belongs to the Gaussian tempered-stable convolution class and, moreover,

$$\psi_\Lambda(\theta) = (\theta - \varphi) \left( 1 - \left( \frac{\gamma}{\gamma + \theta} \right)^\nu \right) \quad \text{for } \theta \geq 0.$$

In particular,  $\delta_\Lambda = 1$  and  $\Phi_\Lambda(0) = \varphi$ . It is a straightforward exercise to show that

$$\hat{E}(\Lambda_1) = \psi'_\Lambda(0+) = -\varphi \frac{\nu}{\gamma},$$

and this implies that

$$\mu = \frac{1}{1 + \varphi\nu/\gamma} < 1,$$

as required.

From [6] we also know that

$$W(x) = e^{\varphi x} + \gamma^\nu e^{\varphi x} \int^x e^{-(\gamma+\varphi)y} y^{\nu-1} E_{\nu,\nu}(\gamma^\nu y^\nu) dy,$$

where

$$E_{\alpha,\beta}(x) := \sum_{n \geq 0} \frac{z^n}{\Gamma(\alpha n + \beta)}$$

is the two-parameter Mittag-Leffler function.

We may now deduce from the theory presented earlier that, for example,

$$P_d(Q^* \leq x) = \frac{1 - e^{-\varphi x} + \gamma^\nu \int^x e^{-(\gamma+\varphi)y} y^{\nu-1} E_{\nu,\nu}(\gamma^\nu y^\nu) dy}{1 + \gamma^\nu \int^x e^{-(\gamma+\varphi)y} y^{\nu-1} E_{\nu,\nu}(\gamma^\nu y^\nu) dy}$$

and

$$P_d(Q^* > x) \sim \left( 1 - \left( \frac{\gamma}{\gamma + \varphi} \right)^\nu \right) e^{-\varphi x}.$$

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