

METRICAL PROPERTIES OF BEST APPROXIMANTS

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Abstract

A rational number is called a best approximant of the irrational number ξ if it lies closer to ξ than all rational numbers with a smaller denominator. Metrical properties of these best approximants are studied. The main tool is the two-dimensional ergodic system, underlying the continued fraction expansion.

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1. Introduction

Let x denote a real irrational number and let

$$[B_0; B_1, B_2, \dots]$$

be its regular continued fraction expansion. Denote the corresponding sequence of convergents by

$$\left\{ \frac{P_n}{Q_n} \right\}_{n=0}^{\infty}.$$

These convergents are defined by

$$(1.1) \quad \frac{P_n}{Q_n} := [B_0; B_1, B_2, \dots, B_n], \quad n \geq 0.$$

The *approximation constant* $\Theta(p/q, x)$ of a fraction p/q with respect to x

is defined by the relation

$$(1.2) \quad \left| x - \frac{p}{q} \right| = \frac{\Theta\left(\frac{p}{q}, x\right)}{q^2}.$$

We shall write for abbreviation

$$(1.3) \quad \Theta_n = \Theta_n(x) = \Theta\left(\frac{P_n}{Q_n}, x\right), \quad n \geq 0.$$

The convergents provide very good rational approximations to x . This is expressed more precisely by the fact that the inequality $\Theta_n < 1$ always holds. One also has the following theorem which was conjectured by H. W. Lenstra and proved in [3]; see also [5] and [6].

(1.4) **THEOREM.** *For almost all x the limiting distribution of the sequence $\{\Theta_n\}_{n=0}^\infty$ exists and is equal to $(1/\log 2)f$ where*

$$f(z) = \begin{cases} 0 & \text{if } z \leq 0, \\ z & \text{if } 0 \leq z \leq \frac{1}{2}, \\ 1 - z + \log(2z) & \text{if } \frac{1}{2} \leq z \leq 1, \\ \log 2 & \text{if } 1 \leq z. \end{cases}$$

Further, P. Lévy has shown a result about the denominators of the convergents, which says that for almost all x

$$(1.5) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log Q_n = \frac{\pi^2}{12 \log 2};$$

see [1].

(1.6) **DEFINITION.** Let D/E be a fraction in its lowest terms with positive denominator. We call D/E a *best approximant* of x if

$$\left| \frac{p}{q} - x \right| \leq \left| \frac{D}{E} - x \right| \Rightarrow q > E$$

holds for every rational number $p/q \neq D/E$.

Since \mathbb{Q} is dense in \mathbb{R} there are infinitely many best approximants of x . Ordering and enumerating them by increasing denominator we obtain the sequence of best approximants belonging to x . We denote this sequence by

$$\left\{ \frac{D_k}{E_k} \right\}_{k=1}^\infty.$$

The question arises whether analogues of Theorem (1.4) and of Lévy’s result exist for the best approximants. As we will see later, the answer is yes: for almost all x and for all z

$$\frac{1}{k} \# \left\{ j \leq k : \Theta \left(\frac{D_j}{E_j}, x \right) \leq z \right\} \rightarrow 0 \quad \text{if } k \rightarrow \infty$$

and

$$\frac{1}{k} \log E_k \rightarrow 0 \quad \text{if } k \rightarrow \infty;$$

see Corollary (2.17) and Corollary (2.19). These results, however, give too little information about the distribution of the approximation constants of the best approximants respectively the growth of the denominators E_k . In order to obtain more refined results, the set of best approximants is replaced by $S^{[C]}$, with C an arbitrary positive constant, where

$$S^{[C]} = S^{[C]}(x) := \left\{ \frac{D}{E} \in \mathbb{Q} : \frac{D}{E} \text{ is a best approximant of } x \right. \\ \left. \text{and } \Theta \left(\frac{D}{E}, x \right) \leq C \right\}.$$

We enumerate the elements of $S^{[C]}$ by increasing denominator and we denote the sequence thus obtained by

$$\left\{ \frac{D_k^{[C]}}{E_k^{[C]}} \right\}_{k=1}^{\#S^{[C]}}.$$

In Section 2 we will see that $\#S^{[C]} = \infty$ for almost all x and we will prove more interesting analogues of Theorem (1.4) and Lévy’s result for the elements of $S^{[C]}$.

(1.7) **REMARK.** Many quantities in this article are dependent on x . However, for the convenience of the exposition we will not make visible this dependence in the notation of these quantities. Further we assume that the fractions P/Q , L/M and D/E are in their lowest terms and that their denominators are always positive.

Finally we point out that if we write “for almost all x ”, this is meant to be in the Lebesgue sense.

2. Detailed study of the best approximants

(2.1) **DEFINITION.** Let $P_{-1} := 1$, $Q_{-1} := 0$ and let P_n and Q_n be as in (1.1) for $n \geq 0$. We call a rational number a *mediant* of x if it is of the

form

$$\frac{L_n^{(B)}}{M_n^{(B)}} = \frac{BP_{n-1} + P_{n-2}}{BQ_{n-1} + Q_{n-2}},$$

where n and B are integers with $n \geq 1$ and $0 < B < B_n$.

Write

$$(2.2) \quad \Theta_n^{(B)} = \Theta_n^{(B)}(x) := \Theta \left(\frac{L_n^{(B)}}{M_n^{(B)}}, x \right), \quad n \geq 1, \quad 0 < B < B_n.$$

The best approximants are directly connected with the convergents and the mediants. The following two theorems describe exactly which fractions are best approximants in terms of convergents and mediants. The proofs of these two theorems can be found in [8], zweites Kapitel, Satz 18, 21 and 22.

(2.3) THEOREM. *Every best approximant of x is a convergent or a mediant of x .*

(2.4) THEOREM. (a) *For $n \geq 1$ every convergent P_n/Q_n is a best approximant; the convergent P_0/Q_0 is a best approximant if and only if $B_1 \geq 2$.*

(b) *A mediant $L_n^{(B)}/M_n^{(B)}$ is a best approximant if and only if $2B > B_n$ or both $2B = B_n$ and*

$$[B_n; B_{n-1}, B_{n-2}, \dots, B_1] > [B_n; B_{n+1}, B_{n+2}, \dots].$$

It is a consequence of Theorem (1.4) and Theorem 2.4(a) that for almost all x the set $S^{[C]}$ contains infinitely many elements for all $C > 0$.

Now we will do some preparatory work for Theorem (2.16) and Theorem (2.18). We note that we will use methods from [2] which in their turn go back to [3].

(2.5) DEFINITION. Define the sequence $\{(T_n, V_n)\}_{n=0}^\infty$ by

$$T_n = T_n(x) := [0; B_{n+1}, B_{n+2}, \dots], \quad n \geq 0,$$

and

$$V_n = V_n(x) := [0; B_n, B_{n-1}, \dots, B_1], \quad n \geq 0.$$

All the metrical results we will find are based on ergodic theory by using the following theorem.

(2.6) THEOREM. *For every x and every n the pair (T_n, V_n) is an element of the unit square $[0, 1] \times [0, 1]$. Moreover, for almost all x the sequence $\{(T_n, V_n)\}_{n=0}^\infty$ is distributed over the unit square according to the density function*

$$\frac{1}{\log 2} \frac{1}{(1 + TV)^2}.$$

The first part of the theorem is trivial. The second part is a corollary of a fundamental ergodic theorem of Ito, Nakada and Tanaka. For that fundamental theorem the reader is referred to [7] and for the corollary to [5].

(2.7) THEOREM. Let $A(n, B, z, x)$ be defined by

$$A(n, B, z, x) := \# \left\{ j \leq n : \frac{L_j^{(B)}}{M_j^{(B)}} \text{ exists, } \frac{L_j^{(B)}}{M_j^{(B)}} \text{ is a best approximant of } x \text{ and } \Theta_j^{(B)} \leq z \right\}.$$

Then for almost all x and for all integers $B \geq 1$ the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} A(n, B, z, x)$$

exists and equals $(1/\log 2)g^{(B)}$ where

$$g^{(1)}(z) = \begin{cases} 0 & \text{if } z \leq \frac{1}{2}, \\ -\frac{1}{2} + z - \frac{1}{2} \log(2z) & \text{if } \frac{1}{2} \leq z \leq \frac{2}{3}, \\ \frac{1}{2} - \frac{1}{2}z + \frac{1}{2} \log\left(\frac{9}{8}z\right) & \text{if } \frac{2}{3} \leq z \leq 1, \\ \frac{1}{2} \log \frac{9}{8} & \text{if } 1 \leq z, \end{cases}$$

and where for integers $B \geq 2$

$$g^{(B)}(z) = \begin{cases} g_0^{(B)}(z) = 0 & \text{if } z \leq \frac{B}{B+1}, \\ g_1^{(B)}(z) = -1 + \frac{B+1}{B}z - \log\left(\frac{B+1}{B}z\right) & \text{if } \frac{B}{B+1} \leq z \leq \frac{B+1}{B+2}, \\ g_2^{(B)}(z) = \frac{1}{B(B+1)}z + \log \frac{B(B+2)}{(B+1)^2} & \text{if } \frac{B+1}{B+2} \leq z \leq \frac{B}{2}, \\ g_3^{(B)}(z) = \frac{1}{2} - \frac{1}{B+1}z + \frac{1}{2} \log\left(\frac{2B(B+2)^2}{(B+1)^4}z\right) & \text{if } \frac{B}{2} \leq z \leq \frac{B+1}{2}, \\ g_4^{(B)}(z) = \frac{1}{2} \log \frac{B(B+2)^2}{(B+1)^3} & \text{if } \frac{B+1}{2} \leq z. \end{cases}$$

PROOF. If we define

$$\mathcal{A}^{(B)} := \left\{ (T, V) : \frac{1}{2B} < T \leq \frac{1}{B+1}, 0 \leq V \leq 1 \right\} \cup \left\{ (T, V) : \frac{1}{2B+1} < T \leq \frac{1}{2B}, V > \frac{1}{T} - 2B \right\},$$

it follows from Theorem (2.3) and Theorem (2.4) that $(T_{n-1}, V_{n-1}) \in \mathcal{A}^{(B)}$ if and only if $L_n^{(B)}/M_n^{(B)}$ exists and is a best approximant of x . From (1.3) and (2.2), from Definitions (2.1) and (2.5) and from the basic relations

$$\Theta_n = \frac{T_n}{1 + T_n V_n} \quad \text{and} \quad \Theta_{n-1} = \frac{V_n}{1 + T_n V_n},$$

using the alternating way in which the sequence of convergents approaches x , we can derive that

$$\Theta_n^{(B)} = \frac{(1 - BT_{n-1})(B + V_{n-1})}{1 + T_{n-1}V_{n-1}}, \quad n \geq 1, \quad 0 < B < B_n.$$

Defining

$$\mathcal{H}^{(B)}(z) := \{(T, V) : (1 - (B + z)T)V \leq B^2T + z - B\} \subset \mathbb{R} \times \mathbb{R},$$

we can conclude that

$$(T_{n-1}, V_{n-1}) \in \mathcal{H}^{(B)}(z) \Leftrightarrow \Theta_n^{(B)} \leq z, \quad (T_{n-1}, V_{n-1}) \in \mathcal{A}^{(B)}.$$

So together with Theorem (2.6) we obtain for almost all x

$$(2.8) \quad \lim_{n \rightarrow \infty} \frac{1}{n} A(n, B, z, x) = \frac{1}{\log 2} \iint_{\mathcal{A}^{(B)} \cap \mathcal{H}^{(B)}(z)} \frac{dV dT}{(1 + TV)^2} =: \frac{1}{\log 2} g^{(B)}(z).$$

Now we have to compute this double integral. By the definition of $\mathcal{H}^{(B)}(z)$, we see that $\mathcal{H}^{(B)}(z)$ is the area between or on the two branches of the hyperbola

$$V = \frac{B^2T + z - B}{1 - (B + z)T}.$$

With increasing z , the hyperbola moves to the left; the left-hand branch of the hyperbola moves through $\mathcal{A}^{(B)}$ while the right-hand branch intersects $\mathcal{A}^{(B)}$ for no value of z . The following figure gives schematically the possible positions of the left-hand branch of the hyperbola (indicated by a dotted line) with respect to $\mathcal{A}^{(B)}$.

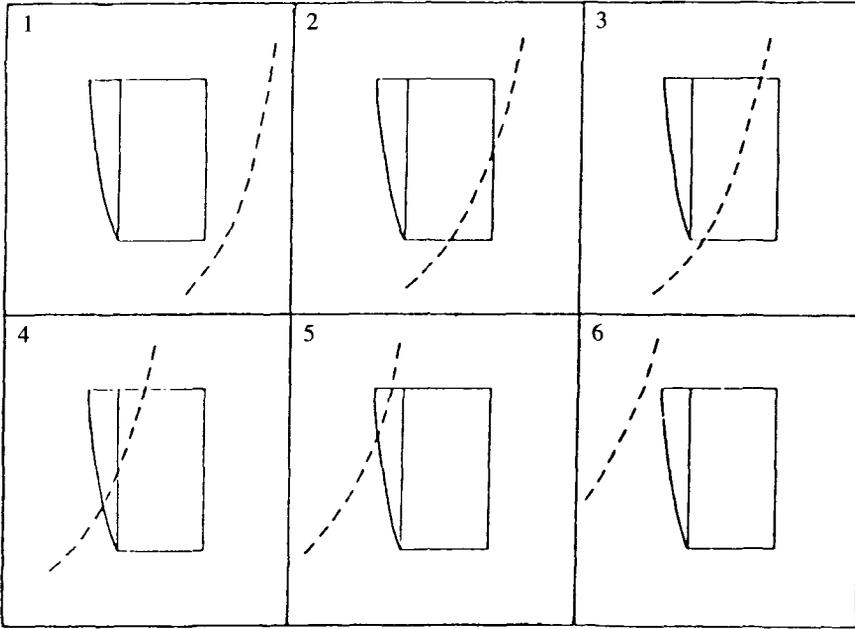


FIGURE 1.

With the help of Figure 1 we find, after some computations, for $B \geq 2$, that

$$z < \frac{B}{B+1} \Rightarrow \mathcal{A}^{(B)} \cap \mathcal{H}^{(B)}(z) = \emptyset,$$

$$\begin{aligned} \frac{B}{B+1} \leq z \leq \frac{B+1}{B+2} &\Rightarrow \mathcal{A}^{(B)} \cap \mathcal{H}^{(B)}(z) \\ &= \left\{ (T, V) : \frac{B-z}{B^2} \leq T \leq \frac{1}{B+1}, 0 \leq V \leq \frac{B^2 T + z - B}{1 - (B+z)T} \right\}, \end{aligned}$$

$$\begin{aligned} \frac{B+1}{B+2} \leq z < \frac{B}{2} &\Rightarrow \mathcal{A}^{(B)} \cap \mathcal{H}^{(B)}(z) \\ &= \left\{ (T, V) : \frac{B-z}{B^2} \leq T \leq \frac{B+1-z}{B^2+B+z}, 0 \leq V \leq \frac{B^2 T + z - B}{1 - (B+z)T} \right\} \\ &\cup \left\{ (T, V) : \frac{B+1-z}{B^2+B+z} < T \leq \frac{1}{B+1}, 0 \leq V \leq 1 \right\}, \end{aligned}$$

$$\begin{aligned}
 \frac{B}{2} \leq z \leq \frac{B+(B+1)}{2B+1} &\Rightarrow \mathcal{A}^{(B)} \cap \mathcal{N}^{(B)}(z) \\
 &= \left\{ (T, V) : \frac{1}{B+2z} < T \leq \frac{1}{2B}, \frac{1}{T} - 2B < V \leq \frac{B^2T+z-B}{1-(B+z)T} \right\} \\
 &\cup \left\{ (T, V) : \frac{1}{2B} < T \leq \frac{B+1-z}{B^2+B+z}, 0 \leq V \leq \frac{B^2T+z-B}{1-(B+z)T} \right\} \\
 &\cup \left\{ (T, V) : \frac{B+1-z}{B^2+B+z} < T \leq \frac{1}{B+1}, 0 \leq V \leq 1 \right\}, \\
 \frac{B(B+1)}{2B+1} \leq z \leq \frac{B+1}{2} &\Rightarrow \mathcal{A}^{(B)} \cap \mathcal{N}^{(B)}(z) \\
 &= \left\{ (T, V) : \frac{1}{B+2z} < T \leq \frac{B+1-z}{B^2+B+z}, \frac{1}{T} - 2B < V \leq \frac{B^2T+z-B}{1-(B+z)T} \right\} \\
 &\cup \left\{ (T, V) : \frac{B+1-z}{B^2+B+z} < T \leq \frac{1}{2B}, \frac{1}{T} - 2B < V \leq 1 \right\} \\
 &\cup \left\{ (T, V) : \frac{1}{2B} < T \leq \frac{1}{B+1}, 0 \leq V \leq 1 \right\}, \\
 z \geq \frac{B+1}{2} &\Rightarrow \mathcal{A}^{(B)} \cap \mathcal{N}^{(B)}(z) \\
 &= \left\{ (T, V) : \frac{1}{2B+1} < T \leq \frac{1}{2B}, \frac{1}{T} - 2B < V \leq 1 \right\} \\
 &\cup \left\{ (T, V) : \frac{1}{2B} < T \leq \frac{1}{B+1}, 0 \leq V \leq 1 \right\}.
 \end{aligned}$$

The case $B = 1$ has to be considered separately, because $\mathcal{A}^{(1)}$ has a different form. We leave this case to the reader, as it presents no further difficulties.

Now the double integral in (2.8) can be evaluated by irksome but straightforward calculations and so one finds the expressions for $g^{(B)}(z)$ from the theorem. This is also left to the reader. Remark that the case $B/2 \leq z \leq B(B+1)/(2B+1)$ and the case $B(B+1)/(2B+1) \leq z \leq (B+1)/2$ yield the same expression for $g^{(B)}(z)$, as is to be expected when one compares the cases 4 and 5 from Figure 1.

We immediately proceed with the following theorem.

(2.9) THEOREM. Let $g^{(B)}$ be as in Theorem (2.7) and define h by

$$h(z) := \sum_{B=1}^{\infty} g^{(B)}(z).$$

Then

$$h(z) = \begin{cases} 0 & \text{if } z \leq \frac{1}{2}, \\ -\frac{1}{2} + z - \frac{1}{2} \log(2z) & \text{if } \frac{1}{2} \leq z \leq 1, \\ \frac{1}{2} + \frac{1}{2} \log\left(\frac{1}{2}z\right) & \text{if } 1 \leq z. \end{cases}$$

For the graph of the function h , see Figure 2.

PROOF. For $z < 2/3$ the theorem is easily verified. Suppose that $2/3 \leq z < 1$. We define the integer k by $k/(k+1) \leq z < (k+1)/(k+2)$. Then $k \geq 2$ and we get

$$h(z) = g^{(1)}(z) + \sum_{B=2}^{k-1} g_2^{(B)}(z) + g_1^{(k)}(z) + \sum_{B=k+1}^{\infty} g_0^{(B)}(z) = -\frac{1}{2} + z - \frac{1}{2} \log(2z).$$

For $z \geq 1$ let the integer k be determined by $k/2 \leq z < (k+1)/2$. Then $k \geq 2$ and we find

$$h(z) = g^{(1)}(z) + \sum_{B=2}^{k-1} g_4^{(B)}(z) + g_3^{(k)}(z) + \sum_{B=k+1}^{\infty} g_2^{(B)}(z) = \frac{1}{2} + \frac{1}{2} \log\left(\frac{1}{2}z\right).$$

(2.10) **DEFINITION.** For all $C > 0$ and for all integers $n \geq 0$ we define $k_n^{[C]} = k_n^{[C]}(x)$ by

$$k_n^{[C]} := \# \left\{ k \in \mathbb{N} : E_k \leq Q_n \text{ and } \Theta\left(\frac{D_k}{E_k}, x\right) \leq C \right\}.$$

(2.11) **REMARK.** $\{E_k^{[C]}\}_{k=1}^{\#S^{[C]}}$ is an increasing sequence, so we have

$$\left\{ E_k \leq Q_n : \Theta\left(\frac{D_k}{E_k}, x\right) \leq C \right\} = \{E_k^{[C]} : k = 1, 2, \dots, k_n^{[C]}\}.$$

Therefore, if $k_n^{[C]} \neq 0$ and $k_n^{[C]} + 1 \leq \#S^{[C]}$ then

$$E_{k_n^{[C]}}^{[C]} \leq Q_n < E_{k_n^{[C]}+1}^{[C]}.$$

Now we have the tools to prove the following important theorem.

(2.12) **THEOREM.** For almost all x

$$\lim_{n \rightarrow \infty} \frac{k_n^{[C]}}{n} = \frac{1}{\log 2} (f(C) + h(C)) = \begin{cases} \frac{1}{\log 2} C & \text{if } 0 < C \leq \frac{1}{2}, \\ \frac{1}{2 \log 2} (1 + \log(2C)) & \text{if } \frac{1}{2} \leq C \end{cases}$$

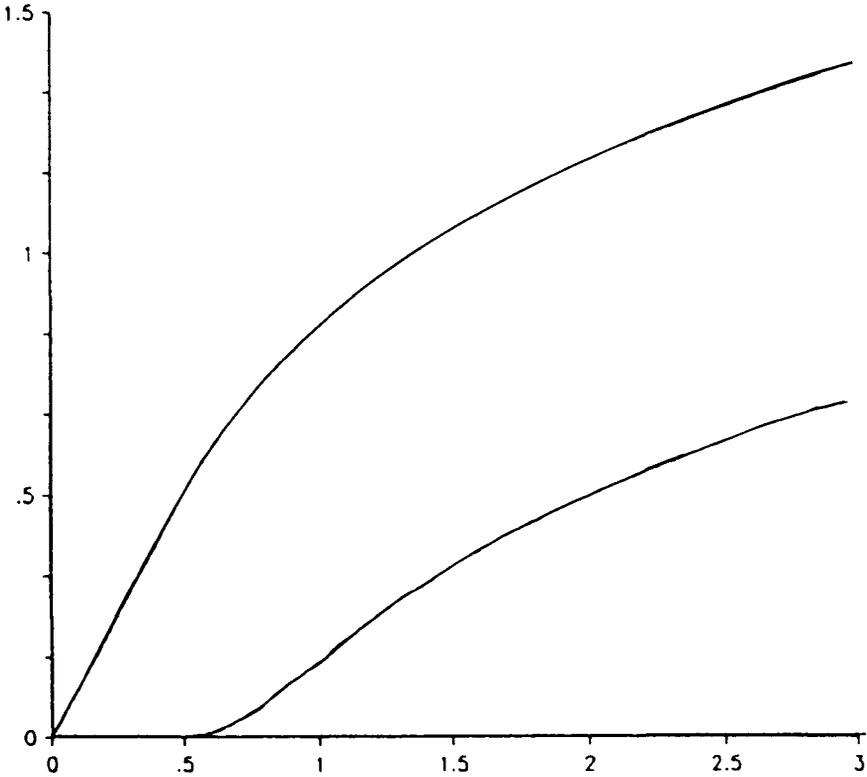


FIGURE 2. Graphs of the functions h and $f + h$

for all $C > 0$, with the f from Theorem (1.4) and the h from Theorem (2.9).

PROOF. We take a value of x for which the sequence $\{(T_n, V_n)\}_{n=0}^\infty$ is distributed according to the density function denoted in Theorem (2.6). This implies that for this x Theorem (1.4) and Theorem (2.7) hold (see the proofs of these theorems). Let C be an arbitrary positive constant. Since

$$Q_{n-1} < M_n^{(1)} < M_n^{(2)} < \dots < M_n^{(B_n-1)} < Q_n, \quad n \geq 2,$$

we get for all integers $b \geq 1$:

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{k_n^{[C]}}{n} &\geq \liminf_{n \rightarrow \infty} \frac{1}{n} \#\{j \leq n : \Theta_j \leq C\} + \liminf_{n \rightarrow \infty} \sum_{B=1}^b \frac{1}{n} A(n, B, z, x) \\ &= \frac{1}{\log 2} \left(f(C) + \sum_{B=1}^b g^{(B)}(C) \right) \end{aligned}$$

and when we let $b \rightarrow \infty$ we find

$$\liminf_{n \rightarrow \infty} \frac{k_n^{[C]}}{n} \geq \frac{1}{\log 2} (f(C) + h(C)).$$

The estimation of $\limsup_{n \rightarrow \infty} k_n^{[C]}/n$ is somewhat more complicated. For all integers $b \geq 1$ one has

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{k_n^{[C]}}{n} &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \#\{j \leq n : \Theta_j \leq C\} + \limsup_{n \rightarrow \infty} \sum_{B=1}^{b-1} \frac{1}{n} A(n, B, z, x) \\ (2.13) \quad &+ \limsup_{n \rightarrow \infty} \sum_{B=b}^{\infty} \frac{1}{n} \#\{j \leq n : (T_{j-1}, V_{j-1}) \in \mathcal{A}^{(B)} \cap \mathcal{H}^{(B)}(C)\} \\ &\leq \frac{1}{\log 2} \left(f(C) + \sum_{B=1}^{b-1} g^{(B)}(C) \right) \\ &+ \limsup_{n \rightarrow \infty} \sum_{B=b}^{\infty} \frac{1}{n} \#\left\{ j \leq n : \frac{B-C}{B^2} \leq T_{j-1} \leq \frac{1}{B+1} \right\}. \end{aligned}$$

It is easy to verify that

$$\frac{B-C}{B^2} \geq \frac{1}{(B+C)+1} > \frac{1}{(B+C+1)+1}, \quad B \geq C^2 + C.$$

This implies that for all $a \in \mathbb{R}$

$$\#\left\{ B \geq C^2 + C : a \in \left[\frac{B-C}{B^2}, \frac{1}{B+1} \right] \right\} \leq C + 1.$$

So we can conclude that

$$\begin{aligned} (2.14) \quad &\sum_{B=b}^{\infty} \frac{1}{n} \#\left\{ j \leq n : \frac{B-C}{B^2} \leq T_{j-1} \leq \frac{1}{B+1} \right\} \\ &\leq (C+1) \frac{1}{n} \#\left\{ j \leq n : 0 \leq T_{j-1} \leq \frac{1}{b+1} \right\}, \quad b \geq C^2 + C. \end{aligned}$$

When we combine (2.13) and (2.14) and let $b \rightarrow \infty$ this yields

$$\limsup_{n \rightarrow \infty} \frac{k_n^{[C]}}{n} \leq \frac{1}{\log 2} (f(C) + h(C)).$$

Let us take an x for which the limit relation of Theorem (2.12) holds and let $C > 0$. We fix $k \geq k_0^{[C]} + 1$ and we define $n = n(k, x)$ by

$$(2.15) \quad k_n^{[C]} + 1 \leq k \leq k_{n+1}^{[C]}.$$

By Theorem (2.12) the existence and uniqueness of this n is guaranteed. For $0 < z \leq C$ we have

$$\begin{aligned} \frac{1}{k} \# \left\{ j \leq k : \Theta \left(\frac{D_j^{[C]}}{E_j^{[C]}}, x \right) \leq z \right\} &\leq \frac{1}{k_n^{[C]}} \# \left\{ j \leq k_{n+1}^{[C]} : \Theta \left(\frac{D_j^{[C]}}{E_j^{[C]}}, x \right) \leq z \right\} \\ &= \frac{1}{k_n^{[C]}} \# \left\{ j \in \mathbb{N} : E_j \leq Q_{n+1} \text{ and } \Theta \left(\frac{D_j}{E_j}, x \right) \leq z \right\} \\ &= \frac{1}{k_n^{[C]}} k_{n+1}^{[z]} \end{aligned}$$

and similarly

$$\frac{1}{k} \# \left\{ j \leq k : \Theta \left(\frac{D_j^{[C]}}{E_j^{[C]}}, x \right) \leq z \right\} \geq \frac{1}{k_{n+1}^{[C]}} k_n^{[z]}.$$

When we now let $k \rightarrow \infty$ (and so $n \rightarrow \infty$), Theorem (2.12) yields the non-trivial part of the next theorem about the distribution of the approximation constants of the elements of $S^{[C]}$.

(2.16) **THEOREM.** *let $C \geq 0$. For almost all x the limit*

$$\lim_{k \rightarrow \infty} \frac{1}{k} \# \left\{ j \leq k : \Theta \left(\frac{D_j^{[C]}}{E_j^{[C]}}, x \right) \leq z \right\}$$

exists and equals $h^{[C]}(z)$ where

$$h^{[C]}(z) = \begin{cases} 0 & \text{if } z \leq 0, \\ \frac{f(z) + h(z)}{f(C) + h(C)} & \text{if } 0 \leq z \leq C, \\ 1 & \text{if } C \leq z \end{cases}$$

and

$$\frac{f(z) + h(z)}{f(C) + h(C)} = \begin{cases} \frac{1}{C} z & \text{if } 0 \leq z \leq C, \ 0 < C \leq \frac{1}{2}, \\ \frac{1}{1 + \log(2C)} 2z & \text{if } 0 \leq z \leq \frac{1}{2}, \ \frac{1}{2} \leq C, \\ \frac{1}{1 + \log(2C)} (1 + \log(2z)) & \text{if } \frac{1}{2} \leq z \leq C, \ \frac{1}{2} \leq C. \end{cases}$$

Since

$$\left\{ j \leq k : \Theta \left(\frac{D_j}{E_j}, x \right) \leq z \right\} \subset \left\{ j \leq k : \Theta \left(\frac{D_j^{[C]}}{E_j^{[C]}}, x \right) \leq z \right\}, \quad z \leq C,$$

we obtain the following corollary.

(2.17) COROLLARY. For almost all x

$$\lim_{k \rightarrow \infty} \frac{1}{k} \# \left\{ j \leq k : \Theta \left(\frac{D_j}{E_j}, x \right) \leq z \right\} = 0.$$

Now let x have a value for which (1.5) as well as the limit relation of Theorem (2.12) hold. Again, let $C > 0$ and let k and n be as in (2.15). Using Remark (2.11) we obtain

$$\frac{1}{k_{n+1}^{[C]}} \log Q_n < \frac{1}{k} \log E_k^{[C]} \leq \frac{1}{k_n^{[C]} + 1} \log Q_{n+1}$$

and this, together with (1.5) and Theorem (2.12), yields an extended analogue of Lévy’s result of the denominators for the best approximants.

(2.18) THEOREM. For almost all x

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log E_k^{[C]} = \frac{\pi^2}{12(f(C) + h(C))} = \begin{cases} \frac{\pi^2}{12C} & \text{if } 0 < C \leq \frac{1}{2}, \\ \frac{\pi^2}{6(1 + \log(2C))} & \text{if } \frac{1}{2} \leq C \end{cases}$$

for all $C > 0$.

(2.19) COROLLARY. For almost all x

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log E_k = 0.$$

(2.20) REMARK. It was proved by P. Erdős [4] (see also W. M. Schmidt [9]), that for almost all x

$$\lim_{n \rightarrow \infty} \frac{U(x, z, n)}{\log n} = \frac{12}{\pi^2} z, \quad 0 \leq z,$$

where

$$U(x, z, n) = \#\{(p, q) \in \mathbb{Z} \times \mathbb{N} : (p, q) = 1, \Theta(p/q, x) \leq z, q \leq n\}.$$

If we define

$$U_1(x, z, n) = \#\{(p, q) \in \mathbb{Z} \times \mathbb{N} : \Theta(p/q, x) \leq z, q \leq n, \\ p/q \text{ is a convergent of } x\}$$

and

$$U_2(x, z, n) = \#\{(p, q) \in \mathbb{Z} \times \mathbb{N} : \Theta(p/q, x) \leq z, q \leq n, \\ p/q \text{ is a best approximant of } x\},$$

then we find as a corollary of Theorems (1.4), (1.5) and (2.12) that for almost all x

$$\lim_{n \rightarrow \infty} \frac{U_1(x, z, n)}{\log n} = \frac{12}{\pi^2} f(z)$$

and

$$\lim_{n \rightarrow \infty} \frac{U_2(x, z, n)}{\log n} = \frac{12}{\pi^2} (f(z) + h(z)),$$

with the f from Theorem (1.4) and the h from Theorem (2.9).

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