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A NOTE ON WEAKLY SYMMETRIC RINGS

BY
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Introduction. T. Nakayama showed in [2, Theorem 13] that symmetric algebras have the property that the left and right annihilators of their two-sided ideals are equal. He also gave examples [2, p. 630] to show that QF algebras with this property are not necessarily symmetric, and that weakly symmetric algebras need not have this property.

In this note a result of K. R. Fuller [1] is used to show that weakly symmetric rings can be characterized in terms of these annihilator conditions.

Preliminaries. Throughout this paper R denotes a ring with identity and N the *Jacobson radical* of R. If ${}_RM$ is a left R-module, then the *left annihilator* of M is the ideal

$$\ell(M) = \{x \in R \mid xM = 0\}.$$

Similarly for a right module M_R we define the *right annihilator* $r(M_R)$. The *injective hull* and the *socle* of ${}_RM$ are denoted by $E({}_RM)$ and $\mathrm{Soc}({}_RM)$ respectively. If ${}_RS$ is a simple left R-module, then the *S-socle* of R is the ideal

$$R[{}_RS] = \sum \{S' \leq R \mid {}_RS' = {}_RS\}$$

If e and f are primitive idempotents, we say *Re is paired to fR* if $\mathrm{Soc}(Re) \simeq Rf/Nf$ and $\mathrm{Soc}(fR) \simeq eR/eN$.

LEMMA 1. *Let R be an artinian ring with primitive idempotents e and f.*

(a) *If* $\mathrm{Soc}(Re)$ *and* $\mathrm{Soc}(fR)$ *are both simple, then* $\ell(Re) = r(fR)$ *if and only if Re is paired to fR.*

(b) *If R is a QF ring, then Re is paired to fR if and only if* $R[Rf/Nf] = R[eR/eN]$.

Proof. (a) ($\Rightarrow$) Suppose that $\mathrm{Soc}(Re) \not\simeq Rf/Nf$. Then $fr(N)e = 0$ as $\mathrm{Soc}(Re)$ is simple. Thus $fRr(N)e = 0$ as $r(N)$ is an ideal. Now, since $r(N)e \subseteq r(fR) = \ell(Re)$, we have $r(N)eRe = 0$. Thus $\mathrm{Soc}(Re) = r(N)e = 0$ which contradicts the fact that R is artinian. Similarly we show $\mathrm{Soc}(fR) \simeq eR/eN$.

($\Leftarrow$) Now

$$\begin{aligned} r(fR) &= \ell(E(Rf/Nf) \quad \text{by [1. Lemma 1.1]}, \\ &= \ell(E(\text{Son } Re)) \quad \text{by hypothesis}, \\ &= \ell(Re) \quad \text{as } Re \text{ is injective by} \\ &\qquad\qquad [1, \text{Theorem } 3.1] \end{aligned}$$

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(b) (⇒) Now $\mathrm{Soc}(Re) \simeq Rf/Nf$ implies $fr(N)e \neq 0$. But

$$fr(N)e \subseteq r(N)e = \mathrm{Soc}(Re) = R[Rf/Nf],$$

and

$$fr(N)e \subseteq f\ell(N)e \subseteq f\ell(N) = \mathrm{Soc}(fR) \subseteq R[eR/eN].$$

Thus $R[Rf/Nf] \cap R[eR/eN] \neq \varnothing$. Consequently $R[Rf/Nf]=R[eR/eN]$, as both these ideals are simple.

(⇐) As R is a QF ring, assume $\mathrm{Soc}(Re) \simeq Rg/Ng$, where g is a primitive idempotent. Thus $R[eR/eN]=R[Rg/Ng]$. But $R[eR/eN]=R[Rf/Nf]$ by hypothesis. Thus $Rg/Ng \simeq Rf/Nf$, and consequently Re is paired to fR since R is a QF ring.

LEMMA 2. *If R is an artinian ring and e a primitive idempotent, then $\ell(Re/Ne) = r(eR/eN)$.*

Proof. Obvious.

Before proceeding to the theorem we recall that an artinian ring R has an orthogonal set of primitive idempotents $e_1, \ldots, e_n$ that is *basic* in the sense that $Re_1, \ldots, Re_n$ represent one copy of each of the indecomposable projective left R-modules; and that R is called a *weakly symmetric* ring in case R is QF and $\mathrm{Soc}(Re_i) \simeq Re_i/Ne_i$ for each $i=1, \ldots, n$ (i.e., in case each Re_i is paired to e_iR).

THEOREM. *Let R be a QF ring and $e_1, \ldots, e_n$ a basic set of primitive idempotents. Then the following are equivalent.*

(a) *R is weakly symmetric,*
(b) *$\ell(Re_i)=r(e_iR)$ for $i=1, 2, \ldots, n$,*
(c) *$\ell(Z)=r(Z)$ for every ideal $Z \supseteq N$,*
(d) *$\ell(Z)=r(Z)$ for every minimal ideal Z,*
(e) *$\ell(Z)=r(Z)$ for every maximal ideal Z.*

Proof. (a)⇔(b). This follows directly from part (a) of Lemma 1.

(a)⇒(c). If Z is an ideal containing N, then

$$\ell(Z) \subseteq \ell(N) = \mathrm{Soc}(R_R) = \mathrm{Soc}({}_RR).$$

But since $\ell(Z)$ is an ideal, we have

$$\ell(Z) = \textstyle\sum_{i=1}^{m} R[Re_i/Ne_i], \quad \text{where } m \leq n, \text{ with}$$

renumbering if necessary. Thus

$$Z \subseteq r(R[Re_i/Ne_i]) = \ell(R[Re_i/Ne_i]) \quad \text{for } i = 1, 2, \cdots, m,$$

with this last step following from part (b) of Lemma 1, Lemma 2 and the fact that $r(R[Re_i/Ne_i])=r(Re_i/Ne_i)$. Thus $\ell(Z) \subseteq r(Z)$. Similarly we show $r(Z) \subseteq \ell(Z)$.

(c)⇒(d). Suppose Z is a minimal ideal in R. Then $\ell(Z) \supseteq \ell r(N)=N$ follows since $Z \subseteq r(N)$ and R is a QF ring. Consequently by hypothesis we can write

$$Z = r(\ell(Z)) = \ell(\ell(Z)).$$

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That is $Z\ell(Z)=0$, which proves $\ell(Z)\subseteq r(Z)$. Similarly we show $\ell(Z)\supseteq r(Z)$.

(d)$\Rightarrow$(b). Consider the minimal ideal $Z=R[Re_i/Ne_i]$. Now

$$\begin{aligned} r(R[Re_i/Ne_i]) &= \ell(R[Re_i/Ne_i]) \quad \text{by hypothesis,} \\ &= r(R[e_iR/e_iN]) \quad \text{by Lemma 2.} \end{aligned}$$

Therefore

$$\begin{aligned} R[Re_i/Ne_i] &= \ell r(R[Re_i/Ne_i]) \\ &= \ell r(R[e_iR/e_iN]), \\ &= R[e_iR/e_iN] \quad \text{as } R \text{ is } QF. \end{aligned}$$

Thus

$$\ell(Re_i) = r(e_iR) \quad \text{by Lemma 1.}$$

The statement (e) of the theorem is obviously implied by (c), and assuming (e) one can readily prove statement (d), as in a QF ring any minimal ideal is the left annihilator of a maximal ideal.

References

1. K. R. Fuller, *On indecomposable injectives over artinian rings*, Pacific J. Math. **29** (1969), 115.
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