A CLASS OF ABELIAN GROUPS

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1. Introduction. If M is any finite set we define a *chain* on M as a mapping f of M into the set of ordinary integers. If $a \in M$ then $f(a)$ is the *coefficient* of a in the chain f. The set of all $a \in M$ such that $f(a) \neq 0$ is the *domain* $|f|$ of f. If $|f|$ is null, that is if $f(a) = 0$ for all a, then f is the *zero* chain on M. If M is null it is convenient to say that there is just one chain, a zero chain, on M.

The *sum* $f + g$ of two chains f and g on M is a chain on M defined by the following rule:

$$(1.1) \qquad (f+g)(a) = f(a) + g(a), \qquad a \in M.$$

If M is null we take this to mean that the sum of the zero chain on M with itself is again the zero chain on M.

With this definition of addition the chains on M are the elements of an additive Abelian group $A(M)$. The zero element of $A(M)$ is the zero chain on M and the negative in $A(M)$ of a chain f on M is obtained from f by multiplying each coefficient $f(a)$ by -1. We define a *chain-group* on M as any subgroup of $A(M)$.

Let N be any chain-group on M. A chain f of N is an *elementary* chain of N (written f elc N) if it is non-zero and there is no non-zero $g \in N$ such that $|g|$ is a proper subset of $|f|$. If in addition the coefficients of f are restricted to the values 0, 1 and -1 we say that f is a *primitive* chain of N. We note that the negative of a primitive chain of N is another primitive chain of N.

We call N *regular* if for each elementary chain f of N there exists a primitive chain g of N such that $|g| = |f|$.

In this paper we study the properties of regular chain-groups. We find in particular that any finite graph has two associated regular chain-groups, and we relate the structure of these chain-groups to that of the graph. In discussing graphs we use the definitions and notation laid down in the introduction to **(4)**.

2. Cycles and coboundaries on a graph. Let G be any finite graph. If $S \subseteq E(G)$ we denote by $G\,.\,S$ that subgraph of G whose edges are the members of S and whose vertices are the ends in G of the members of S. We denote by $G:S$ that subgraph of G whose edges are the members of S and whose vertices are all the vertices of G. Clearly $G\,.\,S$ may be derived from $G:S$ by suppressing its *isolated* vertices, that is the vertices not ends of edges of $G:S$.

We denote by G ctr S the graph whose vertices are the *components* of G: $(E(G) - S)$ and whose edges are the members of S, the ends in G ctr S of an

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edge A being those components of $G : (E(G) - S)$ which contain as vertices the ends of A in G. We may regard G ctr S as obtained from G by contracting each component of $G : (E(G) - S)$ to a single point. We denote by $G \times S$ the graph obtained from G ctr S by suppressing its isolated vertices. These vertices are clearly those components of G whose edges all belong to $E(G) - S$.

If S is the set of edges of a circular path P in G we denote the graph $G . S$ by $G(P)$ and call it a *circuit* of G.

We call a graph a *bond* if it has just two vertices, no loops, and at least one link. Each link of course has the two vertices as its ends. A *bond of G* is a graph of the form $G \times S$ which is a bond.

Now let an orientation of G be given and let it be described by a function $\eta(A, a)$ as in **(4)**. We refer to chains on $V(G)$ and $E(G)$ as 0-*chains* and 1-*chains* on G respectively. We define their boundaries and coboundaries in the usual way. Thus the *boundary* ∂f of a 1-chain f is given by

(2.1) $$(\partial f)(a) = \sum_{A \epsilon E(G)} \eta(A, a) f(A),$$

and the *coboundary* δg of a 0-chain g by

(2.2) $$(\delta g)(A) = \sum_{a \epsilon V(G)} \eta(A, a)\, g(a).$$

If $E(G)$ is null we take (2.1) to mean that ∂f is the zero chain on $V(G)$. Similarly if $V(G)$ is null δg is the zero chain on $E(G)$. A *cycle* on G is a 1-chain whose boundary is the zero chain on $V(G)$.

The set of all cycles on the oriented graph G is clearly a chain-group $\Gamma(G)$ on $E(G)$. Another chain-group on $E(G)$ is the set $\Delta(G)$ of the coboundaries of the 0-chains on G. We proceed to show that $\Gamma(G)$ and $\Delta(G)$ are regular.

(2.3) *Let $G . S$ be any circuit of G. Then there is a primitive chain g of $\Gamma(G)$ such that $|g| = S$.*

Proof. There is a circular path $P = (a_0, A_1, \ldots, A_r, a_0)$ in G such that $G(P) = G . S$. Let g be a 1-chain of G defined as follows:

(i) If $A \notin S$ then $g(A) = 0$,

(ii) $g(A_i) = 1$ or -1 according as a_{i-1} is or is not the positive end of $A_i (0 < i \leqslant r)$.

Applying (2.1) we find that ∂g is a zero chain. Hence $g \in \Gamma(G)$.

If g is not an elementary chain of $\Gamma(G)$ there exists $k \in \Gamma(G)$ such that $|k|$ is a non-null proper subset of S. Then S has at least two elements. Hence by the definition of a circuit the elements of $|k|$ are links of G and some vertex of $G . |k|$ is an end of only one of them. This vertex must have a non-zero coefficient in ∂k, which is impossible. Accordingly g is elementary, and therefore primitive since its coefficients are restricted to the values 0, 1, and -1.

(2.4) *Suppose $S \subseteq E(G)$. Then S is the domain of an elementary chain of $\Gamma(G)$ if and only if $G . S$ is a circuit of G.*

Proof. Suppose g elc $\Gamma(G)$. We show that there is a circuit $G\,.\,S$ of G such that $S \subseteq |g|$. If $G\,.\,|g|$ has a loop this result is trivial. If not, each vertex of $G\,.\,|g|$ is an end of at least two links of $G\,.\,|g|$, by (2.1). Hence, starting at an arbitrary vertex a_0 of $G\,.\,|g|$, we can construct a path

$$P = (a_0, A_1, a_1, A_2, a_2, \ldots)$$

of arbitrary length in $G\,.\,|g|$ such that A_i and A_{i+1} are distinct for each i such that both exist as terms of P. We continue the path until some vertex b is repeated. Then the part of P extending from the first to the second occurrence of b is a circular path in $G\,.\,|g|$ defining a circuit $G\,.\,S$ such that $S \subseteq |g|$.

By (2.3) there exists $k \in \Gamma(G)$ such that $|k| = S \subseteq |g|$. Since g elc $\Gamma(G)$ it follows that $|g| = S$. Thus $G\,.\,|g|$ is a circuit of G.

Since the converse result is contained in (2.3) the Theorem follows.

(2.5) *$\Gamma(G)$ is a regular chain-group.*

Proof. Suppose f elc $\Gamma(G)$. By (2.4) there is a circuit $G\,.\,S$ of G such that $|f| = S$. Hence by (2.3) there is a primitive chain g of $\Gamma(G)$ such that $|g| = S = |f|$.

(2.6) *Let $G \times S$ be any bond of G. Then there is a primitive chain g of $\Delta(G)$ such that $|g| = S$.*

Proof. There are two distinct components X and Y of $G : (E(G) - S)$ such that in G each edge of S has one end in X and one in Y. Let f be the 0-chain on G such that $f(a) = 1$ if a is a vertex of X and $f(a) = 0$ otherwise. Write $g = \delta f$. Then $|g| = S$ by (2.2). Further the coefficients of G are restricted to the values 0, 1, and -1.

If g is not an elementary chain of $\Delta(G)$ there exists $k \in \Delta(G)$ such that $|k|$ is a non-null proper subset of S. Then X and Y are subgraphs of the same component, Z say, of $G : (E(G) - |k|)$. There is a 0-chain f on G such that $k = \delta f$. Since each edge of $|k|$ has both its ends in Z there are two vertices of Z having different coefficients in f. Since Z is connected it must have a link B whose ends have different coefficients in f. But then

$$k(B) = (\delta f)(B) \neq 0$$

by (2.2), which is impossible. Accordingly g is elementary, and therefore primitive since its coefficients are restricted to the values 0, 1, and -1.

(2.7) *Suppose $S \subseteq E(G)$. Then S is the domain of an elementary chain of $\Delta(G)$ if and only if $G \times S$ is a bond of G.*

Proof. Suppose g elc $\Delta(G)$. There is a 0-chain f on G such that $g = \delta f$. Since g is non-zero there is, by (2.2), a link A of G with ends a and b such that $f(a) \neq f(b)$. Write $f(a) = x$. Let W be the set of all $c \in V(G)$ such that $f(c) = x$. Let $G[U]$ be that component of $G[W]$ which has a as a vertex. (Here we use the notation of **(4)**). Let S be the set of all links of G having just one end in $G[U]$. Then $A \in S$. Moreover $S \subseteq |g|$, by (2.2).

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Now $G[U]$ is one component of $G : (E(G) - S)$. Let Z be the component of $G : (E(G) - S)$ which has b as a vertex and let T be the set of all links of G having just one end in Z. Then $A \in T \subseteq S$. Let f' be that 0-chain on G in which the vertices of Z have coefficient 1 and all other vertices of G have coefficient 0. Then

$$A \in |\delta f'| = T \subseteq S \subseteq |g|,$$

by (2.2). Hence $|\delta f'| = |g|$, since g elc $\Delta(G)$, and therefore $T = S = |g|$.

We now see that each edge of S has one end in $G[U]$ and one in Z. Hence $G \times S$, that is $G \times |g|$, is a bond of G.

Since the converse result is contained in (2.6) the theorem follows.

(2.8) *$\Delta(G)$ is a regular chain-group.*

Proof. Suppose f elc $\Delta(G)$. By (2.7) there is a bond $G \times S$ of G such that $|f| = S$. Hence by (2.6) there is a primitive chain g of $\Delta(G)$ such that $|g| = S = |f|$.

3. Some operations on chain-groups. Let N be any chain-group on a set M. Let a subset S of M be chosen and let the coefficient of each member of S in each chain of N be multiplied by -1. The resulting chains are clearly the elements of a chain-group N' on M. We say that N' is obtained from N by *reorienting* the members of S.

Suppose M is the set of edges of an oriented graph G. By *reorienting* the members of S in G we mean interchanging positive and negative ends for each edge of G in S. By (2.1) and (2.2) the effect of this operation on the chain-groups $\Gamma(G)$ and $\Delta(G)$ is to reorient the members of S in each of them.

Properties of chain-groups which are invariant under reorientation are of special interest. Clearly one such property is that of regularity. We note also that the class of domains of elementary chain-groups is invariant under reorientation. In the case of $\Gamma(G)$ and $\Delta(G)$ the invariant properties correspond to properties of the underlying unoriented graph.

If $f \in N$ we define the *restriction* $f . S$ of f to S as that chain on S in which each $a \in S$ has the same coefficient as in f.

The restrictions to S of the chains of N are clearly the elements of a chain-group on S. We denote this chain-group by $N . S$. Another chain-group on S is the set of restrictions to S of those chains f of N for which $|f| \subseteq S$. We denote this by $N \times S$. If $T \subseteq S \subseteq M$ the following identities hold:

(3.1) $$(N . S) . T = N . T,$$

(3.2) $$(N \times S) \times T = N \times T,$$

(3.3) $$(N . S) \times T = (N \times (M - (S - T))) . T,$$

(3.4) $$(N \times S) . T = (N . (M - (S - T))) \times T.$$

Formulae (3.1) and (3.2) follow at once from the definitions. To prove (3.3) we observe that each side is the set of restrictions to T of those chains f of N

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for which $|f| \cap (S - T)$ is null. We obtain (3.4) by writing $M - (S - T)$ for S in (3.3).

(3.5) *If N is regular then $N \,.\, S$ and $N \times S$ are regular.*

Proof. It is clear that the elementary and primitive chains of $N \times S$ are the restrictions to S of those elementary and primitive chains respectively of N whose domains are subsets of S. Hence $N \times S$ is regular.

Now suppose f elc $(N \,.\, S)$. There exists $g \in N$ such that $f = g \,.\, S$. Choose such a g so that $|g|$ has the least possible number of elements. Since N is regular it has a primitive chain h such that $|h| \subseteq |g|$. If $|h|$ does not meet S we can by adding h or $-h$ to g a sufficient number of times obtain $g' \in N$ such that $g' \,.\, S = f$ and $|g'|$ is a proper subset of $|g|$, contrary to the definition of g. We deduce that there is a non-zero chain $k = h \,.\, S$ of $N \,.\, S$ whose coefficients are restricted to the values 0, 1, and -1 and which satisfies $|k| \subseteq |h|$. Since f elc $(N \,.\, S)$ the chain k satisfies $|k| = |f|$ and is primitive. Thus $N \,.\, S$ satisfies the definition of a regular chain-group.

4. Dendroids and representative matrices.

If f is a chain on a finite set M and n is an integer we denote by nf the chain obtained from f by multiplying each coefficient by n. It is clear that any chain-group containing f as an element contains also nf.

Let N be any chain-group on a finite set M.

We define a *dendroid* of N as a subset D of M such that D, but no proper subset of D, meets the domain of every non-zero chain of N. If the only element of N is the zero chain then the null subset of M is the only dendroid of N. In every other case M meets the domain of every non-zero chain of N and therefore some subset of M is a dendroid of N.

Suppose that D is a dendroid of N and that $a \in D$. There exists $f \in N$ such that $|f|$ is non-null and $|f| \cap (D - \{a\})$ is null. It follows that $|f| \cap D = \{a\}$ and hence that $f(a) \neq 0$. We can clearly choose f so that $f(a)$ is positive. We denote a choice of f for which $f(a)$ has the least possible positive value by ${J^D}_a$. There is only one such chain ${J^D}_a$, for the difference of two distinct ones would be a non-zero chain of N with a domain not meeting D.

(4.1) *${J^D}_a$ is an elementary chain of N.*

Proof. Suppose k is a non-zero chain of N such that $|k|$ is a proper subset of $|{J^D}_a|$. Write $J^D\,(a) = m$ and $k(a) = n$. Since $D \cap |k|$ is non-null we have $n \neq 0$. The chain $n{J^D}_a - mk$ of N is zero since its domain does not meet D. Hence $|k| = |{J^D}_a|$, contrary to the definition of k.

(4.2) *If N is regular ${J^D}_a$ is primitive*

Proof. By (4.1) and the regularity of N there is a primitive chain g of N such that $|g| = |{J^D}_a|$. Replacing g by its negative if necessary we can arrange that $g(a) = 1$. Then by the definition and uniqueness of ${J^D}_a$ we have $g = {J^D}_a$.

(4.3) *Suppose N is regular and has a non-null dendroid D. Then for each chain J of N we have*

$$J = \sum_{a \epsilon D} J(a)\ J^D a.$$

Proof. Write

$$J' = J - \sum_{a \epsilon D} J(a)\ J^D{}_a.$$

It is clear, by (4.2), that $|J'|$ does not meet D. Hence J' is a zero chain.

In the rest of this section we suppose that the set M is non-null. We enumerate its elements as $a_1, \ldots, a_n$. If f is any chain on M we refer to the row-vector $\{f(a_1), \ldots, f(a_n)\}$ as the *representative vector* of f with respect to the chosen enumeration. Suppose R is a matrix of r rows and n columns whose elements are integers and whose rows are linearly independent. Then the set of chains on M whose representative vectors are the linear combinations of the rows of R with integral coefficients are the elements of a chain-group on M. If this chain-group is N we say that R is a *representative matrix* of N with respect to the chosen enumeration of the elements of M.

By the general theory of Abelian groups every chain-group on M having at least one non-zero element has a representative matrix. If N is a regular chain-group of this kind we may form a representative matrix R as follows. We select a dendroid D, necessarily non-null, and take as the rows of R the representative vectors of the corresponding chains $J^D{}_a$. It is easily seen that these vectors are linearly independent. It then follows from (4.3) that R is a representative matrix of N. We say that the representative matrix R thus constructed is *associated* with the dendroid D.

Suppose we have a representative matrix R of N, where N is not necessarily regular. Then if $S \subseteq M$ we denote by $R(S)$ the submatrix of R constituted by those columns of R which correspond to members of S. If $R(S)$ is square we denote its determinant by det $R(S)$.

(4.4) *Let R be an r-rowed representative matrix of N. Then a subset S of M is a dendroid of N if and only if it has just r elements and is such that* det $R(S) \neq 0$.

Proof. If the rank of $R(S)$ is less than r some linear combination of the rows of R with integral coefficients not all zero has only zeros in the columns corresponding to members of S. The corresponding chain of N is non-zero and has a domain not meeting S. Hence S is not a dendroid of N. In particular no dendroid of N has fewer than r elements.

If the rank $R(S)$ is r there is a subset T of S of just r elements such that det $R(T) \neq 0$. Then the rows of $R(T)$ are linearly independent. Consequently T meets the domain of each non-zero chain of N and so some subset of T is a dendroid of N. This subset must be T itself since a dendroid of N has at least r elements. We conclude that S is a dendroid of N if it has r elements but not if it has more than r.

It follows from (4.4) that all the dendroids of N have the same number r of elements, and that each representative matrix of N has r rows. We call r the *rank* of N and denote it also by $r(N)$. The Theorem does not of course apply to the case in which N consists solely of the zero chain. In that case we write $r(N) = 0$. Then the only dendroid of N has $r(N)$ elements.

(4.5) *Let R be a matrix of r rows and n columns whose elements are integers and whose rows are linearly independent. Let M be a finite set of n elements. Then R is a representative matrix of a regular chain-group on M if and only if the determinants of its square submatrices of order r are restricted to the values* 0, 1, *and* -1.

Proof. Suppose first that R is a representative matrix of a regular chain-group N on M. Let $R(S)$ be any square submatrix of R of order r.

If S is not a dendroid of N then $\det R(S) = 0$, by (4.4). If S is a dendroid of N let R' be a representative matrix of N associated with S, and corresponding to the same enumeration of M as R. The rows of R' must be linear combinations of the rows of R with integral coefficients. Hence there is a square matrix P of order r whose elements are integers and which satisfies $R' = PR$. This implies $R'(S) = P \times R(S)$ and hence

$$\det R'(S) = \det P \,.\, \det R(S).$$

Now $\det R'(S) = \pm 1$, by the definition of R'. Since P and $R(S)$ are matrices of integers it follows that $\det R(S) = \pm 1$.

Conversely, suppose that the square submatrices of R of order r have determinants restricted to the values 0, 1, and -1. We fix an enumeration of the elements of M. There is a chain-group N on M whose representative matrix with respect to this enumeration is R.

Let f be any elementary chain of N. Let a be any member of $|f|$ and E any dendroid of $N \,.\, (M - |f|)$. Then if a chain h of N has a domain not meeting $E \cup \{a\}$ its domain must be a subset of $|f| - \{a\}$. Since f is elementary this is possible only if h is zero. We conclude that some subset D of $E \cup \{a\}$ is a dendroid of N. Since D must meet $|f|$ we have $D \cap |f| = \{a\}$.

By (4.4) and the restriction imposed on R we have $\det R(D) = \pm 1$. Hence the reciprocal of $R(D)$ is a matrix of integers. We write $R' = (R(D))^{-1}R$. The rows of R' are linear combinations, with integral coefficients, of the rows of R and are therefore representative vectors of chains of N. But $R'(D)$ is a unit matrix. Hence there is a chain g of N such that $g(a) = 1$ and $|g| \cap D = \{a\}$. Then $f - f(a)g$ is a zero chain since its domain does not meet D. Accordingly $f = f(a)g$.

Keeping $|f|$ fixed we may select f so that the highest common factor of its non-zero coefficients is as small as possible. With this choice of f the result just obtained requires $f(a) = \pm 1$. Since this is true for each $a \in |f|$ the chain f is then primitive. Thus N satisfies the definition of a regular chain-group.

(4.6) *Let R be a matrix of r rows and $n > r$ columns, whose elements are integers and in which the square submatrix A constituted by some r columns is unit matrix. Let the submatrix of R constituted by the remaining $n - r$ columns be B. Let M be any set of n elements. Then R is a representative matrix of a regular chain-group on M if and only if the determinants of the square submatrices of B are restricted to the values* 0, 1, *and* -1.

Proof. There is a 1–1 correspondence, q say, between the square submatrices of B and those square submatrices other than A of R which are of order r. If C is a square submatrix of B the corresponding submatrix qC of R is made up of those columns of B which contain elements of C and those columns of A which have only zeros in the rows of R meeting C. It is clear from this definition that $\det qC = \pm \det C$. Since the rows of A, and therefore the rows of R, are linearly independent the Theorem now follows from (4.5).

If R is a representative matrix of a regular chain-group N and R' is the transpose of R then the number $C(N)$ of dendroids of N is given by the formula

$$C(N) = \det (RR'). \tag{4.7}$$

This follows from (4.4) and (4.5), with the help of the well-known formula for the determinant of the product of two matrices of types (r, n) and (n, r).

5. Dual regular chain-groups. Two chains f and g on a finite set M are *orthogonal* if

$$\sum_{a \in M} f(a)\, g(a) = 0.$$

If M is null we take this to mean that the zero chain on M is self-orthogonal.

If N is a chain-group on M then these chains on M which are orthogonal to all the chains of N evidently constitute a chain-group on N. We denote this chain-group by N^* and call it the *dual* of N.

The *zero* chain-group on M includes only the zero chain. The *complete* chain-group on M includes all the chains on M. It is clear that these chain-groups are regular and that each is the dual of the other.

If N is a regular chain-group on M which is neither zero nor complete we may construct N^* as follows. We choose arbitrarily a dendroid D of N and denote by R a representative matrix of N associated with D. If $r(N) = r$ we may adjust the notation so that $R(D)$ is a unit matrix occupying the first r columns of R. We denote by B the matrix constituted by the remaining columns of R, which we suppose s in number. Now let T be the matrix of s rows and $r + s$ columns such that the submatrix formed by the first r columns is the negative of the transpose of B and the remaining s columns constitute a unit matrix. Let N_1 be the chain-group on M which has T as a representative matrix with respect to the chosen enumeration of M. By (4.6) N_1 is regular.

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If $b \in M - D$ we denote by K_b that chain of N_1 which has a row of T as a representative vector and which satisfies $K_b(b) = 1$. It is clear that K_b is orthogonal to $J^D{}_a$ for each $a \in D$ and each $b \in M - D$. Hence by (4.3) the chains K_b are orthogonal to all the chains of N and therefore belong to N^*. It follows that $N_1 \subseteq N^*$. Now suppose N^* has a chain J not belonging to N_1. Write

$$J' = J - \sum_{b \in (M-D)} J(b)\, K_b.$$

The chain J' of N is orthogonal to each of the chains $J^D{}_a$ and its domain is a subset of D. It is therefore a zero chain. It follows that J belongs to N_1, contrary to supposition. We have thus proved that $N_1 = N^*$.

A similar argument in which the roles of the $J^D{}_a$ and the K_b are interchanged shows that R is a representative matrix of $(N^*)^*$ and hence that $(N^*)^* = N$. We now have

(5.1) *If N is a regular chain-group then N^* is regular and $(N^*)^* = N$.*

(5.2) *If N is a regular chain-group on a set M then the dendroids of N^* are the complements in M of the dendroids of N.*

Proof. Let D be any dendroid of N. If N is zero or complete it is clear that $M - D$ is a dendroid of N^*. Otherwise we form the matrix T as in the above construction. Since T is a representative matrix of N^* and $\det T(M - D) = 1$ it follows from (4.4) that $M - D$ is a dendroid of N^*. Replacing N by N^* in this result, and using (5.1), we find also that if $M - D$ is a dendroid of N^* then D is a dendroid of N.

Suppose N is a regular chain-group on a set M and that S is a subset of M. Then a chain g on S is orthogonal to every chain of $N . S$ if and only if it is of the form $f . S$, where $f \in N^*$ and $|f| \subseteq S$. We thus have

(5.3) $$(N . S)^* = N^* \times S.$$

By writing N^* for N in (5.3) and using (5.1) we obtain also

(5.4) $$(N \times S)^* = N^* . S.$$

(5.5) *Let G be a finite graph and let $\Gamma(G)$ and $\Delta(G)$ be defined in terms of the same orientation of G. Then $(\Delta(G))^* = \Gamma(G)$.*

Proof. If G has no edge the result is trivial. In the remaining case a 1-chain g on G is orthogonal to all the chains of $\Delta(G)$ if and only if

$$\sum_{A \in E(G)} \left\{ g(A) \sum_{a \in V(G)} \eta(A, a)\, f(a) \right\} = 0$$

for arbitrary integers $f(a)$. This is so if and only if

$$\sum_{A \in E(G)} \eta(A, a)\, g(A) = 0$$

for each $a \in V(G)$, that is, if and only if $g \in \Gamma(G)$.

The dendroids of a chain-group depend only on the domains of the chains of the group and are therefore invariant under reorientation. Hence if G is a finite graph and $\Delta(G)$ is its group of coboundaries with respect to some fixed orientation we may expect the dendroids of $\Delta(G)$ to be interpretable in terms of the structure of G only.

If H and K are two subgraphs of G we define their *intersection* $H \cap K$ as that subgraph of G whose edges and vertices are the common edges and vertices respectively of H and K. A *forest* is a graph which has no circuit. A *tree* is a connected forest. A *spanning forest* of G is a subgraph of G of the form $G : S$ whose intersection with each component of G is a tree.

(5.6) *Let G be a finite graph with a given orientation and let S be a subset of $E(G)$. Then S is a dendroid of $\Delta(G)$ if and only if $G : S$ is a spanning forest of G.*

Proof. Suppose $G : S$ is not a spanning forest of G. If $G : S$ has a circuit then $E(G) - S$ is not a dendroid of $\Gamma(G)$, by (2.4), and therefore S is not a dendroid of $\Delta(G)$, by (5.2) and (5.5). If $G : S$ has no circuit its intersection with each component of G is a forest. Hence there must be a component H of G such that $H \cap (G : S)$ is not connected. Let K be any component of $H \cap (G : S)$. Let f be the 0-chain on G such that $f(a) = 1$ if a is a vertex of K and $f(a) = 0$ otherwise. Then the chain δf is non-zero and its domain does not meet S. Again we find that S is not a dendroid of $\Delta(G)$.

Conversely suppose S is not a dendroid of $\Delta(G)$. Assume that $G : S$ is a spanning forest of G. Let f be any 0-chain on G such that δf is non-zero. Then some component H of G has two vertices a and b such that $f(a) \neq f(b)$. Since $H \cap (G : S)$ is a tree there are two vertices c and d of $H \cap (G : S)$, joined by an edge of S, such that $f(c) \neq f(d)$. Hence S meets $|\delta f|$. We deduce that some proper subset T of S is a dendroid of $\Delta(G)$. Choose $e \in S - T$ and write $Q = E(G) - T$. Now Q is a dendroid of $\Gamma(G)$, by (5.2). The non-zero element $J^Q{}_e$ of $\Gamma(G)$ satisfies $|J^Q{}_e| \subseteq S$. Hence, by (2.4), $G : S$ has a circuit, contrary to our assumption. We deduce that in fact $G : S$ is not a spanning forest of G. The Theorem follows.

6. Conformity.

Let f and g be chains on a finite set M. We say that f *conforms* to g if the following condition is satisfied: if $f(a) \neq 0$ then $g(a)$ is non-zero and has the same sign as $f(a)$. Conformity is clearly a transitive relation.

(6.1) *If N is a regular chain-group and f is a non-zero chain of N then there exists a primitive chain of N conforming to f.*

Proof. If possible choose f so that the Theorem fails and $|f|$ has the least number of elements consistent with this condition. Since N is regular it has a primitive chain h such that $|h| \subseteq |f|$. Choose $a \in |h|$ so that $f(a)$ has the least possible absolute value. Replacing h by its negative if necessary, we arrange that $h(a) = 1$. Write $k = f - f(a)h$. Clearly k conforms to f. If k is a zero chain then either h or $-h$ conforms to f. If k is non-zero there is a primitive chain g of N conforming to k, and therefore to f, since $|k|$ is a proper subset of $|f|$. In each case the definition of f is contradicted.

(6.2) *If N is a regular chain-group then each non-zero chain of N can be represented as a sum of primitive chains of N each conforming to it.*

Proof. If $f \in N$ let $Z(f)$ be the sum of the absolute values of the coefficients of f. If possible choose a non-zero $f \in N$ for which the Theorem fails and $Z(f)$ has the least value consistent with this condition. By (6.1) there is a primitive chain g of N conforming to f. Clearly $f - g$ conforms to f and $Z(f - g) < Z(f)$. By the latter result $f - g$ is either a zero chain or a sum of primitive chains of N conforming to it. But chains conforming to $f - g$ conform also to f. Hence the Theorem is true for f and we have a contradiction.

Let f and g be chains on a finite set M and let q be an integer >1. We say that g is a *q-representative* of f if the following conditions are satisfied:

(i) $g(a) \equiv f(a) \pmod q$ for each $a \in M$,

(ii) $|g(a)| < q$ for each $a \in M$.

(6.3) *If N is a regular chain-group on a set M and $f \in N$ then for each integer $q > 1$ some q-representative of f is a chain of N.*

Proof. Let f be any chain of N and q any integer >1. There is at least one $g \in N$ satisfying (i). For any such g we denote by $Y(g)$ the number of elements a of M for which $|g(a)| \geqslant q$. We choose a particular g satisfying (i) so that $Y(g)$ has the least possible value.

If $Y(g) > 0$ choose $b \in M$ such that $|g(b)| \geqslant q$. By (6.2) there is a primitive chain h of N conforming to g and such that $h(b) = \pm 1$. Write $g' = g - qh$. Clearly g' satisfies (i). Moreover we have

(1) $$|g'(b)| < |g(b)|,$$

(2) if $$|g(a)| < q \text{ then } |g'(a)| < q.$$

If $|g'(b)| \geqslant q$ we repeat the process with g' replacing g and with the same choice of b. Proceeding in this way we eventually obtain a chain g_1 of N which satisfies (i) and is such that $Y(g_1) < Y(g)$. This contradicts the definition of g. We conclude that $Y(g) = 0$, that is, g is a q-representative of f.

This Theorem is proved for the cycle-group of an oriented graph in **(3)**. For applications of it to the theory of graphs see **(3)** and **(4**, pp. 83–84**)**.

7. Homomorphisms. Let N be a regular chain-group on a set M. A *homomorphism* of N (into I) is a mapping ϕ of N into the set I of integers such that

(7.1) $$\phi(f + g) = \phi(f) + \phi(g)$$

for arbitrary chains f and g of N. This implies that $\phi(f) = 0$ if f is the zero chain. Hence $\phi(-f) = -\phi(f)$ for each $f \in N$.

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For arbitrary chains f and g on M we write

$$(7.2) \qquad (f \, . \, g) = \sum_{a \in M} f(a)\, g(a).$$

If M is null we take this to mean $(f \, . \, g) = 0$.

A *solution* of ϕ is a chain g on M such that $(f \, . \, g) = \phi(f)$ for each $f \in N$. In this section we study the solutions of the homomorphisms of N. We need the following definitions.

If $f \in N$ we define $P(f)$ as the set of all $a \in M$ such that $f(a) > 0$. We then write

$$(7.3) \qquad \beta(f) = \sum_{a \in P(f)} f(a).$$

If $P(f)$ is null we take $\beta(f)$ to be 0. We call f a *positive* chain of N if $P(f) = |f|$.

(7.4) *Let ϕ be any homomorphism of N and a any element of M. Then either $\{a\}$ is the domain of a chain of N or there is a homomorphism ϕ_a of $N \, . \, (M - \{a\})$ such that*

$$\phi_a(f \, . \, (M - \{a\})) = \phi(f)$$

for each $f \in N$.

Proof. Suppose $\{a\}$ is not the domain of a chain of N. Then no two distinct chains of N have the same restriction to $M - \{a\}$, for otherwise the domain of their difference would be $\{a\}$. Hence there is a unique mapping ϕ_a of $N \, . \, (M - \{a\})$ into I such that

$$\phi_a(f \, . \, (M - \{a\})) = \phi(f)$$

for each $f \in N$. It is easily verified that ϕ_a is a homomorphism.

(7.5) *If ϕ is any homomorphism of N and f is a chain of N such that $\phi(f) > \beta(f)$ then there is a primitive chain g of N conforming to f such that $\phi(g) > \beta(g)$.*

Proof. The chain f is necessarily non-zero. Hence by (6.2) it is a sum $f_1 + f_2 + \ldots + f_s$ of primitive chains f_i of N conforming to f. If the Theorem is false, $\phi(f_i) \leqslant \beta(f_i)$ for each of these. Then by addition we have $\phi(f) \leqslant \beta(f)$, contrary to hypothesis.

(7.6) *If ϕ is any homomorphism of N, a an element of M, and f a chain of N such that $f(a) \neq 0$ and*

$$\phi(f) - \beta(f) + \epsilon f(a) > 0,$$

where ϵ is 1 or -1, then there is a primitive chain g of N conforming to f such that either $\phi(g) > \beta(g)$ or g satisfies the equations $\phi(g) = \beta(g)$ and $g(a) = \epsilon$.

Proof. By (6.2) f is a sum $f_1 + f_2 + \ldots + f_s$ of primitive chains f_i of N each conforming to f. There must be just $|f(a)|$ of these such that $|f_i(a)| = 1$.

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If one of the f_i satisfies $\phi(f_i) > \beta(f_i)$ the Theorem is true. In the remaining case we have by addition $\phi(f) - \beta(f) \leqslant 0$. But

$$\phi(f) - \beta(f) + \epsilon f(a) > 0.$$

One consequence of this is that $f(a)$ has the same sign as ϵ, whence it follows that the $|f(a)|$ chains f_i satisfying $|f_i(a)| = 1$ satisfy also $f_i(a) = \epsilon$. Another consequence is that at most $|f(a)| - 1$ of the chains f_i satisfy

$$\phi(f_i) - \beta(f_i) < 0.$$

Combining these results we see that one of the chains f_i satisfies both $\phi(f_i) = \beta(f_i)$ and $f_i(a) = \epsilon$.

(7.7) *Let ϕ be any homomorphism of N. Then in order that ϕ shall have a solution whose coefficients are restricted to the values* 0 *and* 1 *it is necessary and sufficient that $\phi(g) \leqslant \beta(g)$ for each primitive chain g of N.*

Proof. Let us call a solution of a homomorphism *limited* if its coefficients are restricted to 0 and 1.

The theorem is trivially true if M is null. Assume as an inductive hypothesis that it is true whenever the number $\alpha(M)$ of elements of M is less than some positive integer q. Consider the case $\alpha(M) = q$.

Suppose there is a primitive chain g of N such that $\phi(g) > \beta(g)$. Then any chain h on M with coefficients restricted to the values 0 and 1 satisfies

$$(g \,.\, h) \leqslant \beta(g) < \phi(g).$$

Hence no limited solution of ϕ exists.

Conversely suppose ϕ has no limited solution. Assume there is no primitive chain g of N such that $\phi(g) > \beta(g)$. It may happen that each $a \in M$ constitutes the domain of a chain of N. Then, since N is regular, there is for each $a \in M$ a chain f_a of N such that $f_a(a) = 1$ and $f_a(b) = 0$ if $b \neq a$. We define a chain h on M, with coefficients restricted to the values 0 and 1, by writing $h(a) = \phi(f_a)$ for each $a \in M$. Then for each $f \in N$ we have

$$\begin{aligned}(f \,.\, h) &= ((\sum_{a \in M} f(a) f_a) \,.\, h) = \sum_{a \in M} f(a)(f_a \,.\, h) \\ &= \sum_{a \in M} f(a)\ \phi(f_a) = \phi(f).\end{aligned}$$

Thus h is a limited solution of ϕ. But this is impossible.

We deduce that there exists $a \in M$ such that $\{a\}$ is not the domain of a chain of N. We define ϕ_a as in (7.4). There is no limited solution of ϕ_a, for such a solution would be the restriction to $M - \{a\}$ of a limited solution d of ϕ satisfying $d(a) = 0$. Hence, by the inductive hypothesis and (3.5) there exists $f \in N$ such that

$$\phi_a(f \,.\, (M - \{a\})) - \beta(f \,.\, (M - \{a\})) > 0.$$

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If $f(a) \leqslant 0$ it follows that $\phi(f) - \beta(f) > 0$. By (7.5) this is contrary to our assumptions. Hence $f(a) > 0$ and we have

$$\phi(f) - \beta(f) + f(a) > 0.$$

By (7.6) and our assumptions it follows that there is a chain j of N such that

(i) $$\phi(j) - \beta(j) = 0 \text{ and } j(a) = 1.$$

Now let ψ be the homomorphism of N defined by $\psi(f) = \phi(f) - f(a)$ for each $f \in N$. The homomorphism ψ_a of $N . (M - \{a\})$ has no limited solution, for such a solution would be a restriction to $M - \{a\}$ of a limited solution d of ϕ such that $d(a) = 1$. Hence by the inductive hypothesis and (3.5) there exists $f \in N$ such that

$$\phi(f) - f(a) - \beta(f . (M - \{a\})) > 0.$$

If $f(a) \geqslant 0$ this gives $\phi(f) - \beta(f) > 0$. By (7.5) this is contrary to our assumptions. Hence $f(a) < 0$ and

$$\phi(f) - \beta(f) - f(a) > 0.$$

By (7.6) and our assumptions it follows that there exists $k \in N$ such that

(ii) $$\phi(k) - \beta(k) = 0 \text{ and } k(a) = -1.$$

It follows from (i) and (ii) that $\phi(j + k) - \beta(j + k) > 0$. This is contrary to our assumptions, by (7.5). This completes the proof for the case $\alpha(M) = q$.
The general theorem follows by induction.

(7.8) *Let ϕ be any homomorphism of N. Then ϕ has a solution whose coefficients are all non-negative if and only if $\phi(f) \geqslant 0$ for each positive primitive chain f of N.*

Proof. N has only a finite number of primitive chains. Hence we can find an integer $q > 0$ such that $\phi(f) < q$ for each primitive chain f of N.

Choose a set U, the union of $\alpha(M)$ disjoint sets U_a, one for each $a \in M$. Each U_a is to have just q elements. If $k \in N$ we denote by k' the chain on U in which the coefficient of each element of U_a is $k(a)$, for each $a \in M$. The chains k' constitute a chain-group N' on U. Elementary and primitive chains of N' correspond respectively to elementary and primitive chains of N. Hence N' is regular. There is a homomorphism ϕ' of N' such that $\phi'(k') = \phi(k)$ for each $k \in N$.

If $\phi(f) < 0$ for some positive primitive chain f of N it is clear that ϕ has no solution whose coefficients are all non-negative.

In the remaining case we have $\phi'(g') \leqslant \beta(g')$ for each primitive chain g' of N'. This follows from the definition of N' if $\beta(g') > 0$. In the remaining case $-g'$ corresponds to a positive chain $-g$ of N, and so $\phi'(g') = -\phi(-g) \leqslant 0 =$

$\beta(g')$. Hence by (7.7) ϕ' has a limited solution h'. There is a corresponding solution h of ϕ defined by

$$h(a) = \sum_{c \in U_a} h'(c), \quad a \in M.$$

The coefficients in h are all non-negative.

(7.9) *If $a \in M$ then either N or N^* has a positive primitive chain f such that $a \in |f|$.*

Proof. By (6.2) it is sufficient to show that either N or N^* has a positive chain f such that $a \in |f|$.

Let ϕ be the homomorphism of N such that $\phi(f) = -f(a)$ for each $f \in N$. If $\{a\}$ is the domain of a chain of N the Theorem is clearly true. If not we define ϕ_a as in (7.4). Then, if ϕ_a has a solution h' with coefficients all non-negative, let h be the chain on M such that $h(a) = 1$ and $h \,.\, (M - \{a\}) = h'$. Then $(f \,.\, h) = 0$ for each $f \in N$ and so h is a positive chain of N^*. If no such solution h' exists, then by (7.8) there exists $f \in N$ such that $f \,.\, (M - \{a\})$ is positive and $-f(a) = \phi(a) < 0$. Then f is a positive chain of N such that $a \in |f|$. In either case the Theorem is true.

8. Some applications to graph theory. Let G be a graph taken with a fixed orientation.

A *directed bond* of G is a bond $G \times S$ of G such that the positive ends of the edges of S all belong to the same component of $G : (E(G) - S)$. A *directed circuit* of G is a circuit $G \,.\, S$ of G defined by a circular path in which each edge is immediately succeeded by its positive end. Using (2.4) and (2.7) we may verify that the subsets S of $E(G)$ such that $G \times S$ is a directed bond or $G \,.\, S$ a directed circuit of G, are the domains of the positive primitive chains of $\Delta(G)$ and $\Gamma(G)$ respectively. If we apply this to (5.5) and (7.9) we obtain the following graph-theoretical result.

(8.1) *Any edge of G is an edge of some directed bond or of some directed circuit of G.*

In conclusion we show how (7.8) may be applied to obtain a known theorem concerning the 1-factors of even graphs **(1; 2)**.

We suppose henceforth that G is *even*, that is, the set $V(G)$ falls into two disjoint subsets V_1 and V_2 such that each edge of G has one end in V_1 and the other in V_2. We fix an orientation by taking the positive end of each edge in V_2. If $a \in V(G)$ we write $\sigma(a) = 1$ or -1 according as a is in V_2 or V_1. We call G *balanced* if each component has the same number of vertices in V_1 as in V_2. The decomposition $\{V_1, V_2\}$ of $V(G)$ is unique within each component of G, apart from the order of V_1 and V_2. Hence if G is balanced for one such decomposition it is balanced for all of them.

A 1-*factor* of G is a subgraph $G : F$ of G such that each vertex of G is an end of just one edge of F. It is clear that a graph which is not balanced has no 1-factor. For balanced graphs we prove the following theorem.

(8.2) *Suppose G balanced. Then G has a* 1*-factor if and only if there is no subset* U *of* V_1 *such that the set of all vertices of* V_2 *joined by edges of G to vertices of U has fewer members than U.*

Proof. If such a subset U of V_1 exists it is clear that G has no 1-factor.

Conversely suppose G has no 1-factor. Then for each $g \in \Delta(G)$ we write

$$\text{(i)} \qquad \phi(g) = \sum_{a \in V(G)} \sigma(a) f(a),$$

where f is any 0-chain on G such that $\delta f = g$. If f_1 and f_2 are two such 0-chains and $\phi_1(g)$ and $\phi_2(g)$ are the corresponding values of $\phi(G)$ we have

$$\text{(ii)} \qquad \phi_1(g) - \phi_2(g) = \sum_{a \in V(G)} \sigma(a)(f_1(a) - f_2(a)).$$

Now $\delta(f_1 - f_2) = \delta(f_1) - \delta(f_2)$, which is the zero 1-chain on G. Hence by (2.2) $f_1(a) - f_2(a)$ is the same for all vertices a of any one component of G. Since G is balanced it follows from (ii) that $\phi_1(g) = \phi_2(g)$. Hence $\phi(g)$ is uniquely defined for each $g \in \Delta(G)$. It is now clear that ϕ is a homomorphism of $\Delta(G)$.

Suppose ϕ has a solution h whose coefficients are all non-negative. By considering the coboundaries $\delta(f)$ such that f has only one non-zero coefficient we find that

$$\text{(iii)} \qquad \sum_{A \in E(G)} \eta(A, a)\, h(A) = \sigma(a)$$

for each $a \in V(G)$. But $\eta(A, a)\, \sigma(a) \geqslant 0$ for each a, A. It follows that $h(A)$ is 0 or 1 for each A and that the edges for which $h(A) = 1$ define a 1-factor of G. This contradicts our supposition. Hence by (7.8) there is a positive primitive chain k of $\Delta(G)$ such that $\phi(k) < 0$.

Now $G \times |k|$ is a directed bond of G. Let C be the component of $G : (E(G) - |k|)$ which includes the positive ends of the members of $|k|$. Let f be the 0-chain on G such that $f(a) = 1$ or 0 according as a is or is not a vertex of C. By (i) we have

$$\sum \sigma(a) < 0,$$

where the summation is over the vertices of C. If U is the set of all vertices of C in V_1 it follows that U is a subset of V_1 of the kind specified in the enunciation.

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