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CHROMATIC SOLUTIONS, II

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1. Introduction. This paper is a continuation of the Waterloo Research Report CORR 81-12, (see [**1**]) referred to in what follows as I. That Report is entitled "Chromatic Solutions". It is largely concerned with a power series h in a variable z^2, in which the coefficients are polynomials in a "colour number" λ. By definition the coefficient of z^{2r}, where $r > 0$, is the sum of the chromatic polynomials of the rooted planar triangulations of $2r$ faces. (Multiple joins are allowed in these triangulations.) Thus for a positive integral λ the coefficient is the number of λ-coloured rooted planar triangulations of $2r$ faces. The use of the symbol z^2 instead of a simple letter t is for the sake of continuity with earlier papers.

In I we consider the case

$$(1) \qquad \lambda = 2 + 2\cos(2\pi/n),$$

where n is an integer exceeding 4. For each n a set of parametric equations is exhibited. In principle, and sometimes in actuality, these permit the determination of the coefficients in h.

In the present paper we carry the theory further and obtain a differential equation for h. The proof applies to all values of λ of the form (1), with n at least 5. Because the coefficients in h are polynomials in λ it can be inferred that the equation holds for all values of λ except 4. The value 4 is excluded because it makes some of the coefficients in the equation infinite. The paper concludes with a consideration of the limiting case, leading to a special differential equation for h valid when $\lambda = 4$.

2. The functions β and γ. The theory of I is expressed in terms of a power series β in z^2. There are equations in the Report permitting us to relate β to h. From Equation I(50) with $r = 2$ we have

$$(2) \qquad \lambda p_3' = \beta p_2' - (4 - \theta) p_2.$$

Here, and hereafter in this Report, a prime denotes differentiation with respect to z^2. The functions p_2 and p_3 are defined in terms of other functions q_2 and q_3 by I(43). In the even case, when $\theta = 0$, these are given in terms of z^2 and h by I(21) and I(22). In the odd case ($\theta = 1$) they are given by I(23) and I(24). Hence we can, in both cases, find β in

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terms of z^2 and h. In both cases the result is

(3) $$\beta = \nu\{\lambda^{-1}z^4h + \nu z^2\}' + 3z^2.$$

(For ν, see I(5) and I(6).) We note that β has the integral

(4) $$\gamma = \lambda^{-1}\nu z^4 h + \nu^2 z^2 + 3z^4/2.$$

In working towards our differential equations we shall use γ rather than h.

3. A partial differential equation. Let us introduce a new independent indeterminate w and write also $u = w^2$. We make the definition

(5) $$U = \sum_{r=\theta}^{M} p_r w^{2r-\theta},$$

so that U is a function of two independent variables w and z^2. We can now rewrite Equation I(54) as

(6) $$(\alpha + \beta u)\frac{\partial U}{\partial (z^2)} = 2u^2 \frac{\partial U}{\partial u}.$$

The function α is evaluated in I and found to be a constant. In fact

(7) $$\alpha = -\lambda$$

by I(60).

Given appropriate boundary conditions we might hope to use the partial differential equation (6) to determine U as a power series in z^2 and u, in terms of the coefficients in β. Such boundary conditions are available. Let us write F_0 or $[F]_0$ to indicate the coefficient of z^0 in a power series F. Then

$$U_0 = \sum_{r=\theta}^{M} [p_r]_0 w^{2r-\theta}$$

$$= \sum_{r=\theta}^{M} \frac{n(-1)^r(M + r - \theta - 1)!\nu^{2r-\theta}w^{2r-\theta}}{2 \cdot (2r - \theta)!(M - r)!},$$

by I(43) and Theorem I(3.1),

$$= \sum_{r=0}^{M} \frac{n(-1)^{r+\theta}(M - r - 1)!(\nu w)^{2r+\theta}}{2 \cdot (2r + \theta)!(M - r - \theta)!}.$$

Hence

(8) $$U_0 = (-1)^M \cos\{n \cos^{-1}(\tfrac{1}{2}\nu w)\},$$

by I(11).

From (8) we can find the values at $z = 0$ of the derivatives of U of all orders with respect to $u = w^2$. Using (6) we can express the derivatives of U, of all orders with respect to u and z^2, in terms of the derivatives

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with respect to u. Hence we can determine the values at $z = 0$ of all the derivatives, in terms of the coefficients in β, and therefore we can determine U as a power series in u and z^2. It is important to notice that this procedure determines the power series U uniquely.

We can treat similarly a power series W in u and z^2 defined as a solution of the partial differential equation

$$(9) \qquad (\alpha + \beta u)\frac{\partial W}{\partial (z^2)} = 2u^2 \frac{\partial W}{\partial u}$$

satisfying the boundary condition

$$(10) \qquad [W]_0 = n \cos^{-1} (\tfrac{1}{2}\nu w).$$

The inverse cosine is to be defined as a power series in w with initial term $\pi/2$. As with U we find that W is uniquely determined.

The two power series W and U are related as follows.

$$(11) \qquad U = (-1)^M \cos W.$$

To prove this we have only to observe that the function U determined by (11) satisfies both (6) and (8). In what follows we shall work with W rather than U.

4. The polynomial Y**.** We now study Equation I(44). Parameters ψ corresponding to even values of $M + m$ are roots of the equation

$$(12) \qquad \sum_{r=\theta}^{M} p_r w^{2r-\theta} = 1$$

in w. Moreover by I(45) they are repeated roots of this equation.

Similarly the parameters ψ corresponding to odd values of $M + m$ are repeated roots of the equation

$$(13) \qquad \sum_{r=\theta}^{M} p_r w^{2r-\theta} = -1.$$

Combining these observations we see that each of our $M - 2$ distinct parameters ψ is a repeated root of

$$(14) \qquad \left\{ \sum_{r=\theta}^{M} p_r w^{2r} \right\}^2 - w^{2\theta} = 0.$$

By the definitions of I each ψ has a positive initial term, so no one of them can be the negative of another. Since the left of (14) is a polynomial in w^2 we can now recognize the negatives of the parameters ψ as additional repeated roots of (14). Accordingly the squares of the parameters ψ are $M - 2$ distinct repeated roots for u of the equation

$$(15) \qquad \left\{ \sum_{r=\theta}^{M} p_r u^{r} \right\}^2 - u^{\theta} = 0.$$

But it is pointed out in Section 5 of I that these squares are the roots of

$$\sum_{r=0}^{M-2} p_{r+2}'u^r = 0.$$

We therefore have a polynomial identity

$$(16)\quad \left\{\sum_{r=\theta}^{M} p_r u^r\right\}^2 - u^\theta = Y\left\{\sum_{r=0}^{M-2} p_{r+2}'u^r\right\}^2,$$

where Y is a polynomial of degree 4 in u, whose coefficients are functions of z^2.

If $\theta = 0$ then $p_0 = 1$, by I(46). Since p_2' is non-zero, by I(58) and I(59), we deduce that Y always has u as a factor. Hence we can write

$$(17)\quad Y = u(A + Bu + Cu^2 + Du^3),$$

where A, B, C and D are functions of z^2.

Writing (16) in terms of U we find that

$$(18)\quad u^4(1 - U^2) = -Y\left\{\frac{\partial U}{\partial(z^2)}\right\}^2.$$

Hence, by (11),

$$(19)\quad u^4 = -Y\left\{\frac{\partial W}{\partial(z^2)}\right\}^2.$$

We now try to eliminate W from (19). First we write the equation in the form

$$(20)\quad \frac{\partial W}{\partial(z^2)} = u^2(-Y)^{-\frac{1}{2}}.$$

Hence, by (9),

$$(21)\quad \frac{\partial W}{\partial u} = \tfrac{1}{2}(\alpha + \beta u)(-Y)^{-\frac{1}{2}}.$$

Accordingly

$$(22)\quad \frac{\partial}{\partial u}\{u^2(-Y)^{-\frac{1}{2}}\} = \frac{\partial}{\partial(z^2)}\{\tfrac{1}{2}(\alpha + \beta u)(-Y)^{-\frac{1}{2}}\},$$

from which we can deduce

$$(23)\quad (8 - 2\beta')uY - 2u^2\frac{\partial Y}{\partial u} + (\alpha + \beta u)\frac{\partial Y}{\partial(z^2)} = 0.$$

Using (17) we can write this as

$$\begin{aligned}(8 - 2\beta')&(Au^2 + Bu^3 + Cu^4 + Du^5)\\ &- 2(Au^2 + 2Bu^3 + 3Cu^4 + 4Du^5)\\ &+ (\alpha + \beta u)(A'u + B'u^2 + C'u^3 + D'u^4) = 0.\end{aligned}$$

By equating coefficients of like powers of u in this equation we obtain the following differential equations.

(24) $\alpha A' = 0,$

(25) $(6 - 2\beta')A + \alpha B' + \beta A' = 0,$

(26) $(4 - 2\beta')B + \alpha C' + \beta B' = 0,$

(27) $(2 - 2\beta')C + \alpha D' + \beta C' = 0,$

(28) $-2\beta' D + \beta D' = 0.$

In order to solve these equations we need to know A_0, B_0, C_0 and D_0, the coefficients of z^0 in A, B, C and D respectively. But from (9) and (19) we have

$$u^4(\alpha + \beta_0 u)^2 = -[Y]_0 \cdot 4u^4\left\{\frac{\partial W_0}{\partial u}\right\}^2.$$

We have $\beta_0 = \nu^2$, by (3), and of course $\alpha = -\lambda$. We can now deduce from (10) that

$$(29)\quad \begin{aligned} (\lambda - \nu^2 u)^2 &= -n^2\nu^2[Y]_0 u^{-1}(4 - \nu^2 u)^{-1}, \\ n^2\nu^2[Y]_0 &= -u(4 - \nu^2 u)(\lambda - \nu^2 u)^2. \end{aligned}$$

As consequences of (29) we have

$$(30)\quad \begin{aligned} n^2\nu^2 A_0 &= -4\lambda^2,\ n^2 B_0 = 8\lambda + \lambda^2, \\ n^2 C_0 &= -(4 + 2\lambda)\nu^2,\ n^2 D_0 = \nu^4. \end{aligned}$$

5. A differential equation for γ. From (24) we see that A is constant. Hence, by (30),

(31) $n^2\nu^2 A = -4\lambda^2.$

We can now integrate (25) as

$$6Az^2 - 2A\beta + \alpha B + c_1 = 0,$$

where c_1 is a constant. Taking coefficients of z^0 in this and applying (30) we find that $n^2 c_1 = \lambda^3$. We can deduce that

(32) $n^2\nu^2 B = \lambda\{\lambda\nu^2 - 24z^2 + 8\beta\},$

(33) $n^2\nu^2 B' = \lambda\{-24 + 8\beta'\}.$

Substituting from these equations in (26) we find

$$(34)\quad \begin{aligned} n^2\nu^2\lambda C' &= (4 - 2\beta')\lambda(\lambda\nu^2 - 24z^2 + 8\beta) + \beta\lambda(-24 + 8\beta'), \\ n^2\nu^2 C' &= 4\lambda\nu^2 - 96z^2 + 8\beta + 48z^2\beta' - 2\lambda\nu^2\beta' - 8\beta\beta'. \end{aligned}$$

Integrating this we obtain

$$n^2\nu^2 C = 4\lambda\nu^2 z^2 - 48z^4 + 8\gamma + 48z^2\beta - 48\gamma - 2\lambda\nu^2\beta - 4\beta^2 + c_2,$$

where c_2 is a constant. We now take coefficients of z^0, noting that γ_0 is zero by (4). By (30)

$$-(4+2\lambda)\nu^1 = -2\lambda\nu^4 - 4\nu^4 + c_2.$$

Hence $c_2 = 0$. We can write

(35) $n^2\nu^2 C = 4\lambda\nu^2 z^2 - 48z^4 - 40\gamma + 48z^2\beta - 2\lambda\nu^2\beta - 4\beta^2.$

Solving (28) we have

$$D = c_3\beta^2,$$

for some constant c_3. Using (30) we find that $n^2 c_3 = 1$. Hence

(36) $n^2 D = \beta^2,$

(37) $n^2 D' = 2\beta\beta'.$

So far we have not used (27). Let us multiply that equation by $n^2\nu^2$ and substitute in it from (34), (35) and (37). We find that

$$(2-2\beta')(4\lambda\nu^2z^2 - 48z^4 - 40\gamma - 2\lambda\nu^2\beta + 48z^2\beta - 4\beta^2) - 2\lambda\nu^2\beta\beta' + \beta(4\lambda\nu^2 - 96z^2 + 8\beta + 48z^2\beta' - 2\lambda\nu^2\beta' - 8\beta\beta') = 0.$$

This simplifies to

$$(1-\beta')(\lambda\nu^2z^2 - 12z^4 - 10\gamma + 6z^2\beta) = 6z^2\beta,$$

that is

(38) $(1-\gamma'')(-\lambda\nu^2z^2 + 12z^4 + 10\gamma - 6z^2\gamma') + 6z^2\gamma' = 0.$

In view of (4) this differential equation for γ can be regarded as the promised differential equation for h. It can be written also as

(39) $(1-\gamma'')(-\lambda\nu^2z^2 + 12z^4 + 10\gamma) + 6z^2\gamma'\gamma'' = 0.$

It should be emphasized that our proof of Equations (38) and (39) applies only when λ is of the form (1), with $n > 4$. But the proof can be extended to other values of λ, and this is done in the following section.

6. The range of validity of the differential equation. Let us discuss Equation (38) without reference to any previous definition of γ. We do this for an arbitrary real or complex value of λ, excluding only the value $\lambda = 4$ which makes ν infinite. We consider the possibility of the equation having a solution in the form of a power series

(40) $\gamma = \sum_{r=0}^{\infty} \gamma_r z^{2r},$

where the γ_r depend only on λ. For any such solution we have also

(41) $\gamma' = \sum_{r=1}^{\infty} r\gamma_r z^{2(r-1)}$

(42) $\gamma' = \sum_{r=2}^{\infty} r(r-1)\gamma_r z^{2(r-2)}.$

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If this γ is to be identical with the γ of Equation (4) we must have

$$(43) \quad \gamma_0 = 0, \gamma_1 = \nu^2.$$

Accordingly we impose (43) as an extra condition on γ.

6.1. *There is one and only one power series γ of the form* (40) *which satisfies both* (38) *and* (43). *Moreover the coefficients in γ have the following property: each γ_j with $j > 1$ is of the form νP_j, where P_j is a polynomial in ν^{-1} whose coefficients are rational numbers.*

Proof. Let us begin by assuming a power series (40) to satisfy (38) and (43). Let us study the relations that must hold between its coefficients.

Equating coefficients of z^0 in (38) we find

$$(1 - 2\gamma_2)(10\gamma_0) = 0,$$

which is consistent with (43). For coefficients of z^2 we have

$$(1 - 2\gamma_2)(-\lambda\nu^2 + 10\gamma_1 - 6\gamma_1) + 6\gamma_1 = 0.$$

If we put $\gamma_1 = \nu^2$ as required by (43) we have

$$-\lambda\nu^2 + 10\gamma_1 - 6\gamma_1 = (4 - \lambda)\nu^2 = \nu.$$

It follows that

$$(44) \quad 2\gamma_2 = 6\nu + 1 = \nu(6 + \nu^{-1}).$$

Accordingly we say that P_2 is $3 + \frac{1}{2}\nu^{-1}$.

We next equate coefficients of z^4.

$$(1 - 2\gamma_2)(12 + 10\gamma_2 - 12\gamma_2) + (-6\gamma_3)(\nu) + 12\gamma_2 = 0.$$

Substituting from (44) we deduce that

$$(45) \quad \gamma_3 = \nu(6 - 5\nu^{-1} + \nu^{-2}).$$

We say therefore that P_3 is $6 - 5\nu^{-1} + \nu^{-2}$.

For an integer m exceeding 2 we can equate coefficients of $z^{2(m-1)}$ and have

$$(46) \quad \{-(m + 1)m\gamma_{m+1}\}\{\nu\} + \{-m(m - 1)\gamma_m\}\{12 - 2\gamma_2\} + \sum_{r=2}^{m-2} \{-(m + 1 - r)(m - r)\gamma_{m+1-r}\}\{10\gamma_{r+1} - 6(r + 1)\gamma_{r+1}\} + \{1 - 2\gamma_2\}\{10\gamma_m - 6m\gamma_m\} + 6m\gamma_m = 0.$$

Starting with the known values of γ_0, γ_1, γ_2 and γ_3 we can use this equation to determine γ_4, γ_5 and so on. The coefficients γ_j are thus uniquely determined by (46), in a recursive manner. We infer that there is at most one power series γ satisfying both (38) and (43). On the other hand the coefficients γ_j determined by (43), (44), (45) and (46) do specify a formal power series γ that satisfies (38). For when this power series is substituted on the left of (38) the coefficients of all powers of z^2 vanish.

To complete the proof we have to show that when $j > 1$ the coefficient γ_j is of the required form νP_j. We have already verified this in the cases $j = 2$ and $j = 3$. Assume as an inductive hypothesis that it is true up to $j = m$, where m is some integer exceeding 2, and consider the case $j = m + 1$. Then by (46) and the inductive hypothesis

$$\nu\gamma_{m+1} = \nu^2 P_{m+1}$$

where P_{m+1} is a polynomial in ν^{-1} with rational numbers as coefficients. Hence $\gamma_{m+1} = \nu P_{m+1}$ and the induction succeeds.

We have $\nu^{-1} = 4 - \lambda$. Hence we can assert the following variant of 6.1.

6.2. *There is one and only one power series γ of the form* (40) *which satisfies both* (38) *and* (43). *Moreover the coefficients in γ have the following property: each γ_j with $j > 1$ is of the form νP_j, where P_j is a polynomial in λ whose coefficients are rational numbers.*

Consider the power series γ defined in terms of h by Equation (4). Like h it is defined for all values of λ other than 4. (h is defined for the value 4 also.) The coefficients in this γ have the following property: If $j > 1$ the coefficient of z^{2j} is of the form νR_j, where R_j is a polynomial in λ whose coefficients are rational numbers. Here we use the fact that each chromatic polynomial appearing in h divides by λ. (The rational numbers mentioned are integers if $j > 2$.) For each such j the effect of the foregoing theory is to show that $P_j = R_j$ for infinitely many values of λ, of the form (1). Since P_j and R_j are polynomials in λ it follows that they are identical. We conclude that the power series γ defined in terms of h by Equation (4) is identical with the power series γ exhibited in the proof of 6.1, for all values of λ other than 4. Let us state this result as a theorem.

6.3. *The power series γ defined in terms of h by Equation* (4) *satisfies the differential equations* (38) *and* (39) *for all values of λ other than* 4.

7. The case $\lambda = 4$. Let us write

(47) $H = z^4 h.$

Then

(48) $\gamma = \lambda^{-1}\nu H + \nu^2 z^2 + 3z^4/2,$

(49) $\gamma' = \lambda^{-1}\nu H' + \nu^2 + 3z^2,$

(50) $\gamma'' = \lambda^{-1}\nu H'' + 3.$

We can substitute from these equations in (38), thereby obtaining a differential equation for the power series H. The author found that

$$(51) \quad \lambda^{-1}H''\{z^2 + 9\nu^{-1}z^4 + 10\lambda^{-1}H - 6\lambda^{-1}z^2H'\}$$
$$= -2\nu^{-1}z^2 + 6z^2 - 20\lambda^{-1}\nu^{-1}H + 18\lambda^{-1}\nu^{-1}z^2H'.$$

Considering the limiting case $\lambda \rightarrow 4$, $\nu^{-1} \rightarrow 0$, we arrive at the following theorem.

7.1. *In the case* $\lambda = 4$ *the power series* H *satisfies the differential equation*

$$(52) \quad H''\{2z^2 + 5H - 3z^2H'\} = 48z^2.$$

Reference

1. W. T. Tutte, *Chromatic solutions*, Can. J. Math. *34* (1982), 741–758.

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