# 3

# Lifting Geometry to Mapping Spaces I: Lie Groups

In this chapter, one aim is to study spaces of mappings taking their values in a Lie group. It will turn out that these spaces carry again a natural Lie group structure. However, before we prove this, let us recall the definition and basic properties of (infinite-dimensional) Lie groups.

## 3.1 (Infinite-Dimensional) Lie Groups

Our presentation of Lie groups modelled on infinite-dimensional spaces mostly follows Neeb (2006). There are many accounts in the literature for finite-dimensional Lie theory (see, e.g. Hilgert and Neeb, 2012), but infinite-dimensional Lie theory (beyond Banach spaces) is by comparison relatively young and in its modern form goes back to the seminal work of Milnor (1982, 1984).

**3.1 Definition** A *(locally convex) Lie group* $G$ is a manifold $G$ modelled on a locally convex space endowed with a group structure such that the multiplication map $m_G\colon G \times G \to G$ and the inversion map $\iota\colon G \to G$ are smooth. A morphism of Lie groups is a smooth group homomorphism. In the following we shall drop the adjective 'locally convex' and simply say 'Lie group'.

**Standard Notation** Let us fix some standard notation for objects occurring frequently in conjunction with Lie groups. Let $G$ be a Lie group, and we shall write

- $\mathbf{1}_G$ for the unit element (or shorter $\mathbf{1}$),
- $m_G$ for multiplication, $\iota_G$ for inversion,
- for $g \in G$ we let $\lambda_g : G \to G$, $h \mapsto gh$, and $\rho_g : G \to G$, $h \mapsto hg$, the *left-(right-)translation*. (Observe that $\lambda_g(\rho_h(x)) = gxh = \rho_h(\lambda_g(x))$.)



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**3.2 Example** A locally convex space $E$ is a Lie group with respect to vector addition and the usual manifold structure.

**3.3 Example** The following examples are the classical finite-dimensional examples encountered in a first course on Lie theory. We include them here for readers who are not familiar with Lie groups.

(a) Let $M_n(\mathbb{R})$ be the set of all $n{\times}n$ matrices and $\mathrm{Gl}_n(\mathbb{R}){:=}\{A{\in}M_n(\mathbb{R})|\det A{\neq}0\}$ be the set of $n \times n$ invertible matrices. Using the determinant, one sees that $\mathrm{Gl}_n(\mathbb{R}) \subseteq M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ is an open subset, hence a manifold. Since multiplication of matrices is given by polynomials in the entries of matrices, the multiplication is smooth with respect to the manifold structure. For invertible matrices, Cramer's rule shows that inversion is also polynomial in the entries of the matrix, hence smooth. In conclusion, matrix multiplication and inversion turns $\mathrm{Gl}_n(\mathbb{R})$ into a Lie group.[1]

(b) The *orthogonal group* $\mathrm{O}_n(\mathbb{R}) := \{A \in \mathrm{Gl}_n(\mathbb{R}) \mid AA^\top = \mathrm{id}_{R^n}\}$ is a closed submanifold of $\mathrm{Gl}_n(\mathbb{R})$ and this structure turns it into a Lie group. Further, the *special orthogonal group* $\mathrm{SO}_n(\mathbb{R}) := \{A \in \mathrm{O}_n(\mathbb{R}) \mid \det(A) = 1\}$ is an open subset of the orthogonal group and thus also a Lie group.

(c) The unit circle $\mathbb{S}^1 \subseteq \mathbb{R}^2$ is a submanifold (as the unit sphere of the Hilbert space $\mathbb{R}^2$). Identifying $\mathbb{R}^2 \cong \mathbb{C}$, complex multiplication induces a Lie group structure on $\mathbb{S}^1$ which is explicitly given by the formulae

$$(x,y) \cdot (a,b) := (xa - yb, xb + ay), \quad (x,y)^{-1} := (x,-y).$$

**3.4 Example** (Unit groups of continuous inverse algebras) To generalise the matrix group example to infinite dimensions, we recall the notion of a *continuous inverse algebra (CIA)*: Let $A$ be a locally convex space with a continuous bilinear map $\beta\colon A \times A \to A$ (we write shorter $xy := \beta(x,y)$) such that the associativity law $(xy)z = x(yz)$ holds. Furthermore, we assume that there exists an element $\mathbf{1} \in A$ such that $\mathbf{1}\,x = x = x\,\mathbf{1}$ for all $x \in A$, and define the set of all invertible elements

$$A^\times := \{x \in A \mid \text{ there exists } x^{-1} \in A \text{ such that } xx^{-1} = \mathbf{1} = x^{-1}x\}.$$

Then $A^\times$ is a group, under the multiplication, called the *unit group* of $A$. If $A^\times$ is open in $A$ and inversion $\iota\colon A^\times \to A^\times$, $x \mapsto x^{-1}$ is continuous, we call $A$ a *continuous inverse algebra (CIA)*. The unit group of a CIA is a Lie group. See Exercise 3.1.2.

[1] Alternatively, this example can be seen as a special case of the unit group of a continuous inverse algebra; see Example 3.4.

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CIAs generalise Banach algebras, for example, the algebra of continuous linear operators $L(E,E)$ of a Banach space $E$ with the operator norm $\|\cdot\|_{\mathrm{op}}$ is a CIA.

Before we continue with the general infinite-dimensional Lie theory, we will discuss now one of the most important example classes of such Lie groups: the diffeomorphism groups. These groups will return as a running example in the later sections to illustrate the concepts of Lie algebra and regularity.

**3.5 Example** Let $M$ be a compact manifold. Then $M$ possesses a local addition, whence $C^\infty(M,M)$ is a canonical manifold and by Proposition 2.23 the composition map $\mathrm{Comp}\colon C^\infty(M,M) \times C^\infty(M,M) \to C^\infty(M,M)$ is smooth. Recall from Corollary 2.8 that the *set of diffeomorphisms* $\mathrm{Diff}(M)$ is an open subset of $C^\infty(M,M)$ which forms a group under composition of smooth maps. Hence $\mathrm{Diff}(M)$ is an open submanifold of $C^\infty(M,M)$ and the group product is smooth with respect to this structure.

We will now prove that inversion $\iota\colon\ \mathrm{Diff}(M) \to \mathrm{Diff}(M)$ is smooth. Applying the exponential law for a canonical manifold, $\iota$ is smooth if and only if the mapping $\iota^\wedge\colon\ \mathrm{Diff}(M) \times M \to M$, $(\varphi,m) \mapsto \varphi^{-1}(m)$ is smooth. Consider the implicit equation

$$\mathrm{ev}(\phi,\iota^\wedge(\phi,m)) = \phi(\iota^\wedge(\phi,m)) = m, \quad \text{for all } m \in M, \tag{3.1}$$

which takes values in a finite-dimensional manifold, but has an infinite-dimensional parameter ($\phi \in \mathrm{Diff}(M)$). However, $\mathrm{ev}\colon\ \mathrm{Diff}(M) \times M \to M$ is smooth, and we can compute its partial differential as $T_{(\phi,x)}\,\mathrm{ev}(0,z) = T\phi(z)$ (cf. Exercise 2.3.4). Since $\phi$ is a diffeomorphism, we see that the partial derivative of ev is indeed invertible for every $\phi \in \mathrm{Diff}(M)$. Now smoothness of $\iota^\wedge$ follows from a suitable implicit function theorem. Observe that due to the infinite-dimensional parameter, the usual implicit function theorem Lang (1999, I. §5 Theorem 5.9) is not applicable to (3.1)! However, we invoke the generalised implicit function theorem (Glöckner, 2006b, Theorem 2.3) which can deal with parameters in locally convex spaces (as long as the target of the implicit equation is a Banach manifold). The usual application of the implicit function theorem to (3.1) shows then that $\iota^\wedge$ and thus $\iota$ is smooth. We conclude that $\mathrm{Diff}(M)$ is a Lie group.

**3.6 Remark** To establish smoothness of the inversion $\mathrm{Diff}(M)$, we needed the exponential law and a generalised implicit function theorem. Use of this machinery can be avoided: In Michor (1980, Theorem 11.11) differentiability of the inversion is directly verified (which is technical and requires the (non-trivial) verification of continuity first). Our approach is inspired by the proof

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in the convenient setting (Kriegl and Michor, 1997, Theorem 43.1). There the problem can be reduced to a finite-dimensional equation which circumvents the need for a generalised implicit function theorem.

The Lie group $\mathrm{Diff}(M)$ already comes with a canonical action on the manifold $M$ which we describe after recalling the notion of a Lie group action.

**3.7 Definition** (Lie group action) Let $M$ be a manifold and $G$ be a Lie group. Then a smooth map

$$\alpha\colon G \times M \to M, \quad (g,m) \mapsto \alpha(g,m) =: g \cdot m$$

is called a *(left) Lie group action* if it satisfies the following:

$$\alpha(\mathbf{1}_G, m) = m, \qquad \alpha(g_1, \alpha(g_2, m)) = \alpha(g_1 g_2, m), \quad \text{for all } g_1, g_2 \in G,\ m \in M.$$

A right action is a smooth map $\beta\colon M \times G \to M$, $(m,g) \mapsto \beta(g,m) =: m \cdot g$ such that

$$\beta(m, \mathbf{1}_G) = m, \qquad \beta(\beta(m,g_1), g_2) = \beta(m, g_1 g_2), \quad \text{for all } g_1, g_2 \in G,\ m \in M.$$

If $\beta$ is a right action, then $\alpha := \beta \circ (\mathrm{id}_M, \iota_G)$ is a left action. Similarly, we can obtain right actions from left actions and there is no essential difference between the notions.

**3.8 Example** Let $M$ be a compact manifold. Then, going through the construction in Example 3.5, it is immediately clear that the evaluation map

$$\alpha\colon\ \mathrm{Diff}(M) \times M \to M, \quad (\varphi, m) \mapsto \varphi(m)$$

induces a (left) Lie group action, called the *canonical action of the diffeomorphism group*. Furthermore, there is also the right action of $\mathrm{Diff}(M)$ on smooth functions

$$\beta\colon C^\infty(M,N) \times \mathrm{Diff}(M) \to C^\infty(M,N), \quad (f,\varphi) \mapsto \varphi^*(f) = f \circ \varphi.$$

If $C^\infty(M,N)$ is a canonical manifold of mappings, then Proposition 2.23 shows that $\beta$ is a (right) Lie group action. The right action $\beta$ is connected to several geometric structures such as the symplectic structure of the loop space $C^\infty(\mathbb{S}^1, N)$; see Wurzbacher (1995). We will encounter it again in the context of shape analysis in Chapter 5.

For a left Lie group action $\alpha$ the canonical map

$$\alpha^\vee\colon G \to \mathrm{Diff}(M), \quad g \mapsto \alpha(g, \cdot)$$

makes sense and yields a group morphism. If $M$ is finite dimensional, the exponential law shows that smoothness of the group action is equivalent to smoothness of $\alpha^\vee$. This breaks down for an infinite-dimensional manifold $M$ as there is no smooth structure on $\mathrm{Diff}(M)$. Similarly, if a Lie group $G$ acts

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by linear mappings on a vector space $E$ (this is called a representation of $G$) the literature considers the smoothness of $\alpha^\vee$ as a mapping to $\mathrm{Aut}(E)$ (the group of linear automorphisms of $E$). If $E$ is normable, the group $\mathrm{Aut}(E)$ inherits a canonical Lie group structure from the operator norm topology such that smoothness of $\alpha$ is equivalent to smoothness of $\alpha^\vee$. Again this equivalence breaks down for locally convex spaces which are not normable, as $\mathrm{Aut}(E)$ does, in general, not carry a Lie group structure. Note that this is not a serious problem, as one can still check smoothness with respect to the product $G \times M$ (and in infinite-dimensional representation theory of Lie groups even weaker concepts of smoothness of representations are more appropriate for the theory; see e.g. Neeb (2010) and also Neeb (2005, I.3.4)). However, we shall not discuss representation theory and the finer points of these problems in this book.

**3.9 Definition** Let $G$ be a Lie group. We call a submanifold $H \subseteq G$ a *Lie subgroup* if it is a subgroup of $G$.[2] If $H$ is, in addition, closed in $G$, we call $H$ a closed Lie subgroup.

**3.10 Example** The diffeomorphism group $\mathrm{Diff}(M)$ of a compact manifold $M$ contains many important subgroups of diffeomorphisms which preserve geometric structures. If $\omega$ is a differential form on $M$, we say *a diffeomorphism* $\phi \in \mathrm{Diff}(M)$ *preserves the differential form*, if $\phi^*\omega = \omega$ (where the pullback is as in Definition E.7). As the pullback commutes with function composition, we can consider the subgroup $\mathrm{Diff}_\omega(M)$ of diffeomorphisms preserving a given differential form $\omega$. The most important examples are the following subgroups:

(a) if $\omega = \mu$ is a volume form on $M$, we obtain the group $\mathrm{Diff}_\mu(M)$ of *volume-preserving diffeomorphisms*;
(b) for a symplectic form $\omega$, this yields the *group of symplectomorphism*;
(c) $\mathrm{Diff}_\theta(M) = \{\phi \in \mathrm{Diff}(M) \mid \phi^*\theta = f\theta$ for some $f \in C^\infty(M,\mathbb{R}\}$, the *group of contactomorphisms* for a contact form $\theta$.

In the three cases mentioned above, one can show that the subgroups are also submanifolds of $\mathrm{Diff}(M)$ and thus Lie subgroups of $\mathrm{Diff}(M)$. We refer to Smolentsev (2007, Section 3) for detailed proofs. See, however, E.18 for a sketch of the construction for volume-preserving diffeomorphisms.

In finite-dimensional Lie theory a useful result states that every closed subgroup of a finite-dimensional Lie group is a Lie subgroup (Hilgert and Neeb, 2012, Theorem 9.3.2). This is no longer true in infinite dimensions, as the next example shows (see also Neeb, 2005, Remark V.2.4 (c)).

[2] Note that thanks to Lemma 1.39 this structure turns $H$ into a Lie group.

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**3.11 Example** (Wockel, 2014) Consider the space $(\ell^2, +)$ of all real sequences which are square summable (Meise and Vogt, 1997, Example 12.11). This is a Hilbert space with respect to the inner product

$$\langle (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\rangle \coloneqq \sum_{n\in\mathbb{N}} x_n y_n \quad \left(\text{and norm } \|(x_n)_{n\in\mathbb{N}}\| = \sqrt{\sum_{n\in\mathbb{N}} x_n^2}\right).$$

We consider $\ell^2$ as an abelian Lie group and define the subgroup

$$H \coloneqq \left\{(x_n)_{n\in\mathbb{N}} \in \ell^2 \mid x_n \in \frac{1}{n}\mathbb{Z},\ n \in \mathbb{N}\right\}.$$

As the projections $\pi_j : \ell^2 \to \mathbb{R}$, $(x_n)_{n\in\mathbb{N}} \mapsto x_j$, $j \in \mathbb{N}$, are continuous linear, $H = \bigcap_{n\in\mathbb{N}} \pi_n^{-1}(\frac{1}{n}\mathbb{Z})$ is a closed subgroup. However, we shall see in Exercise 3.1.5 that $H$ is not a submanifold (it is not a manifold with respect to the subspace topology).

For Lie groups, the tangent bundle is again a Lie group and moreover, the tangent bundle is trivial (i.e. it splits as a product of a vector space and the base manifold).

**3.12 Lemma** *Let $G$ be a Lie group. Identify $T(G\times G) \cong TG \times TG$.*

(a) *The tangent map of the multiplication*

$$\begin{aligned} Tm_G\colon T(G\times G) &\cong TG\times TG \to TG, \\ T_gG\times T_hG \ni (v_g, w_h) &\mapsto Tm_G(v_g, w_h) = T_g\rho_h(v_g) + T_h\lambda_g(w_h) \end{aligned} \tag{3.2}$$

*induces a Lie group structure on $TG$ with identity element $0_{\mathbf{1}} \in T_{\mathbf{1}}G$ and inversion*

$$T\iota_G\colon TG\to TG, \qquad T_gG \ni v \mapsto -T\rho_{g^{-1}}T\lambda_{g^{-1}}(v) = -T\lambda_{g^{-1}}T\rho_{g^{-1}}(v). \tag{3.3}$$

*The projection $\pi_G\colon TG \to G$ becomes a morphism of Lie groups with kernel $(T_{\mathbf{1}}G, +)$ and the zero-section $\mathbf{0}\colon G\to TG$, $g\mapsto 0_g$ is a morphism of Lie groups with $\pi_G\circ \mathbf{0} = \mathrm{id}_G$.*

(b) *The map*

$$\Phi\colon G\times T_{\mathbf{1}}G \to TG, \qquad (g, v)\mapsto g\cdot v \coloneqq Tm_G(0_g, v)$$

*is a diffeomorphism.*

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*Proof* (a) Since $m_G$ and $\iota_G$ are smooth, the same holds for their tangent maps. The group axioms for $TG$ follow from the ones for $G$ by virtue of the chain rule (and have the claimed unit element). Linearity of the tangent map implies (3.2); we leave this formula and (3.3) as Exercise 3.1.3. From the definition of the zero section and the projection, the morphism properties follow. As a result of (3.2), $T_{(\mathbf{1},\mathbf{1})}m_G(v_\mathbf{1}, w_\mathbf{1}) = v_\mathbf{1} + w_\mathbf{1}$. Hence the multiplication on the normal subgroup $\ker \pi_G = T_\mathbf{1}G$ is the addition.

(b) Since $\Phi = Tm_G(0_g, v) = Tm_G(\mathbf{0}(g), v)$ and the zero-section $\mathbf{0}$ is smooth, smoothness of the multiplication shows that $\Phi$ is smooth. Now a computation shows that $\Phi^{-1}(v) = (\pi_G(v), T_{\pi_G(v)}\lambda_{\pi_G(v)^{-1}}(v))$, whence $\Phi$ is bijective and its inverse is smooth (as inversion and multiplication in $G$ are smooth and the projection $\pi_G$ is smooth). □

**3.13 Remark** Note that Lemma 3.12 shows that $T_\mathbf{1}G$ is a normal Lie subgroup of $TG$, and $TG$ is as a Lie group a semidirect product $T_\mathbf{1}G \rtimes G$.[3] Moreover, instead of left multiplication one can use right multiplication to identify the tangent bundle (the two different choices are related by the adjoint action; see Example 3.28).

The tangent space at the identity of a Lie group plays a special role. In the next section this tangent space will be endowed with an additional structure, the Lie bracket.

## Exercises

3.1.1 Verify that $\mathbb{S}^1$ is a Lie group with the structure described in Example 3.3(c).

3.1.2 In this exercise we verify that the unit group $A^\times$ of a CIA $(A, \beta)$ forms a Lie group. Note that the multiplication is smooth by Exercise 1.3.2. Hence it suffices to prove smoothness of inversion.

(a) Use the identity $b^{-1} - a^{-1} = b^{-1}(a - b)a^{-1}$ to deduce that the differential quotient $d\iota(x; y)$ exists and satisfies $d\iota(x; y) = -x^{-1}yx^{-1}$.

(b) Use the formula from (a) to prove that $\iota$ is $C^1$ and inductively is $C^k$ for all $k \in \mathbb{N}_0$.

[3] A Lie group $G$ with normal Lie subgroup $N$ and Lie subgroup $H$ such that $N \cap H = \{\mathbf{1}_G\}$ is a semidirect product $N \rtimes H$ if there exists a split exact sequence of Lie group homomorphisms

$$0 \to N \hookrightarrow G \xrightarrow{p} H \to \mathbf{1}$$

where splitting means that $p|_H = \mathrm{id}_H$. Equivalently, $G \cong N \times H$ (as manifolds) and the group product is given by $(n, h) \cdot (\tilde{n}, \tilde{h}) = (n\tilde{h}\tilde{n}\tilde{h}^{-1}, h\tilde{h})$. See Hilgert and Neeb (2012, 2.2.2) for further alternative characterisations.

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3.1.3 Fill in the missing details in the proof of Lemma 3.12.

(a) Prove (3.2) and verify that the tangent maps induce a group structure on $TG$.

(b) Establish (3.3), that is, $T_a\iota(v) = -T\lambda_{a^{-1}}T\rho_{a^{-1}}(v) = -T\rho_{a^{-1}}T\lambda_{a^{-1}}(v)$.
*Hint:* Let $\gamma\colon\, ]-\varepsilon,\varepsilon[ \to G$ be smooth with $\gamma(0) = a$. Differentiate the relation $\mathbf{1} = \gamma(t)(\gamma(t))^{-1}$.

(c) Show that one can obtain a diffeomorphism $TG \cong T_{\mathbf{1}}G \times G$ using right multiplication instead of left multiplication in part (b).

3.1.4 Let $(A,\beta)$ be a continuous inverse algebra (CIA) and let $C^\infty(K,A)$ be endowed with the compact open $C^\infty$-topology.

(a) Show that then $C^\infty(K,A)$ with the pointwise product is a CIA.

(b) Show that $C^\infty(K,A^\times) = C^\infty(K,A)^\times$ and thus the group $C^\infty(K,A^\times)$ with the pointwise product is a Lie group.

3.1.5 We supply the details for Example 3.11: Let $H = \{(x_n)_{n\in\mathbb{N}} \in \ell^2 \mid x_n \in \frac{1}{n}\mathbb{Z}, n \in \mathbb{N}\}$.

(a) Show that every 0-neighbourhood in the subspace topology of $H$ contains at least one non-zero element.
*Hint:* It suffices to consider norm balls.

(b) Show that there is no 0-neighbourhood in $H$ which contains a continuous path connecting 0 with a non-zero element. Deduce that $H$ is not locally homeomorphic to an open subset of a locally convex space and thus is not a (sub)manifold.

3.1.6 Let $G$ be a Lie group, and show that the map $L\colon G \times TG \to TG$, $(g, v_h) \mapsto T\lambda_g(v_h)$ is a left Lie group action. Dually right multiplication yields a right action $R\colon TG \times G \to TG$. Work out a formula relating $L(g, v_h)$ to $R(v_h, g)$.

## 3.2 The Lie Algebra of a Lie Group

We now associate a Lie algebra to a Lie group. This construction allows one to reformulate many problems in Lie theory in terms of linear algebra.

**3.14 Definition** A *Lie algebra* is a vector space $\mathfrak{g}$ together with a *Lie bracket*, that is, a bilinear map $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

(a) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$, for all $x, y, z \in \mathfrak{g}$ (Jacobi identity);
(b) $[x,x] = 0$, for all $x \in \mathfrak{g}$.

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If $\mathfrak{g}$ is a locally convex space and the Lie bracket is continuous, $\mathfrak{g}$ is a *locally convex Lie algebra*. A (continuous) linear map $h\colon \mathfrak{g} \to \mathfrak{h}$ between (locally convex) Lie algebras is a morphism of (locally convex) Lie algebras if $h([x,y]) = [h(x),h(y)]$, for all $x,y \in \mathfrak{g}$.

**3.15 Remark** It will be essential for us that the Lie bracket and Lie algebra morphisms are continuous (see e.g. the proof of Proposition E.14 for an example where continuity is needed). From now on we will mostly work with locally convex Lie algebras; hence we drop the phrase 'locally convex' and write only 'Lie algebra'.

**3.16 Example** If $A$ is a continuous inverse algebra (or more generally a locally convex algebra), then the commutator $[x,y] := xy - yx$ turns $A$ into a Lie algebra. Hence the Lie bracket measures commutativity of the algebra product.

**3.17 Example** Every locally convex space $E$ is a Lie algebra, called an *abelian Lie algebra*, with the *trivial bracket* $[x,y] := 0$.

**3.18** (The Lie algebra of vector fields) Let $M$ be a manifold, and let

$$\mathcal{V}(M) := \{X \in C^\infty(M,TM) \mid \pi_M \circ X = \mathrm{id}_M\}$$

be the locally convex space of all vector fields (see Appendix D). If $f \in C^\infty(M,E)$ is smooth with values in some locally convex space $E$ and $X \in \mathcal{V}(M)$, then we obtain a smooth function

$$X.f := df \circ X\colon M \to E \quad (\text{recall } df = \mathrm{pr}_2 \circ Tf).$$

For $X,Y \in \mathcal{V}(M)$, there exists a unique vector field $[X,Y] \in \mathcal{V}(M)$ determined by the property that on each $U \Subset M$ we have

$$[X,Y].f = X.(Y.f) - Y.(X.f) \quad \text{for all } f \in C^\infty(U,E). \tag{3.4}$$

Thus $\mathcal{V}(M)$ becomes a Lie algebra (the local case is verified in Appendix D and we discuss the general case in Exercise 3.2.3). If $M$ is finite dimensional, Corollary D.13 shows that $(\mathcal{V}(M),[\cdot,\cdot])$ is a locally convex Lie algebra.

**3.19** Let $G$ be a Lie group. A vector field $X \in \mathcal{V}(G)$ is called *(left) invariant* if $X$ is $\lambda_g$-related to itself for all $g \in G$ (i.e. $X \circ \lambda_g = T\lambda_g \circ X$; see Appendix D). We write $\mathcal{V}^\ell(G)$ for the set of left-invariant vector fields. Note that relatedness of vector fields is inherited by the Lie bracket thanks to Exercise 3.2.3, and so $\mathcal{V}^\ell(G)$ is a Lie subalgebra of $\mathcal{V}(G)$.

**3.20 Proposition** *Let G be a Lie group; then the map*

$$\Theta\colon T_{\mathbf{1}}G \to \mathcal{V}^\ell(G), \quad v \mapsto (g \mapsto T_{\mathbf{1}}\lambda_g(v))$$

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*is an isomorphism of locally convex spaces with inverse* $\Theta^{-1}(X) = X(\mathbf{1})$. *Thus* $\mathbf{L}(G) := T_\mathbf{1}G$ *can be endowed with the Lie bracket*

$$[v, w] := \Theta^{-1}([\Theta(v), \Theta(w)]) = [\Theta(v), \Theta(w)](\mathbf{1}),$$

*turning it into a Lie algebra. We call* $(\mathbf{L}(G), [\cdot, \cdot])$ *the* Lie algebra associated to $G$.

*Proof* As $\Theta(v)(hg) = T_\mathbf{1}\lambda_{hg}(v) = T_g\lambda_h T_\mathbf{1}\lambda_g(v) = T_g\lambda_h\Theta(v)(g)$, the map $\Theta$ makes sense and its image consists of left-invariant vector fields (see Exercise 3.2.4). Linearity of $\Theta$ follows directly from the linearity of the tangent map. For $X \in \mathcal{V}^\ell(G)$ we have $X(g) = X \circ \lambda_g(\mathbf{1}) = T_\mathbf{1}\lambda_g X(\mathbf{1}) = \Theta(X(\mathbf{1}))(g)$, $\Theta$ is surjective. As the translations $\lambda_g$ are diffeomorphisms, it is clear that only $0 \in T_\mathbf{1}G$ gets mapped to the zero-vector field. We conclude that $\Theta$ is a vector space isomorphism (its inverse is obviously evaluation in $\mathbf{1}$). Note that $\mathcal{V}^\ell(G)$ carries the subspace topology induced by $\mathcal{V}(G)$ from Appendix D.4. This immediately shows that $\Theta^{-1}$ is continuous since point evaluations are continuous in this topology. The continuity of $\Theta$ is left as Exercise 3.2.5. That $[\cdot, \cdot]$ is a Lie bracket on $T_\mathbf{1}G$ follows directly by trivial computations since $\mathcal{V}^\ell(G)$ is a Lie algebra. □

If the Lie group $G$ is finite dimensional, the above discussion shows that $(T_\mathbf{1}G, [\cdot, \cdot])$ is a locally convex Lie algebra. Here only the continuity of the Lie bracket is unclear in the general case. We shall now prove that the Lie bracket on $T_\mathbf{1}G$ is always continuous; hence the Lie algebra $\mathbf{L}(G)$ associated to a Lie group is always a locally convex Lie algebra. To this end, we need a local model of the multiplication.

**3.21** Let $G$ be a Lie group. Since $T_\mathbf{1}G$ is isomorphic to the model space of $G$, we can pick a chart $\varphi\colon G \supseteq U_\varphi \to V_\varphi \Subset T_\mathbf{1}G$ such that $\mathbf{1} \in U_\varphi$ and $\varphi(\mathbf{1}) = 0$. Moreover, we may assume that $T_\mathbf{1}\varphi = \mathrm{id}_{T_\mathbf{1}G}$. Due to the continuity of the multiplication of $G$ there is an open $\mathbf{1}$-neighbourhood $W$ with $W \cdot W \subseteq U_\varphi$ (here $W \cdot W$ denotes the set of all products of two elements in $W$). Hence we can define a local multiplication

$$*\colon \varphi(W) \times \varphi(W) \to V_\varphi, \quad (x, y) \mapsto x * y := \varphi(\varphi^{-1}(x)\varphi^{-1}(y)).$$

By construction, the local multiplication is smooth and $*(0, x) = x = *(x, 0)$. Hence the construction gives rise to a so-called local Lie group (see Neeb, 2005, Remark III.1.14). As with the Lie group $G$ we can compute a Lie bracket using a local version of a left-invariant vector field. To distinguish the local operations from the Lie group operations, let us introduce a new symbol for

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left translation: $\ell_x : \varphi(W) \to T_{\mathbf{1}}G$, $y \mapsto x * y$, $x \in \varphi(W)$. For any $v \in T_{\mathbf{1}}G$ we can thus define a left-invariant vector field with respect to the local product

$$\Lambda^v : \varphi(W) \to T_{\mathbf{1}}G, \quad x \mapsto d\ell_x(0; v) = \left.\frac{d}{dt}\right|_{t=0} x * tv.$$

We will see in Exercise 3.2.6 that a left-invariant vector field $X$ with $X(\mathbf{1}) = v$ is $\varphi$-related to $\Lambda^v$. Together with the properties of the chart this yields the identity

$$\begin{aligned}[v,w] &= [\Theta(v),\Theta(w)](\mathbf{1}) = [\Lambda^v,\Lambda^w](0) = d\Lambda^w(0;\Lambda^v(0)) - d\Lambda^v(0;\Lambda^w(0)) \\ &= \left( \left.\frac{d^2}{dtds}\right|_{t,s=0} sv * tw - \left.\frac{d^2}{dtds}\right|_{t,s=0} tw * sv \right).\end{aligned}$$

This formula shows immediately that the Lie bracket $[\cdot,\cdot]$ is continuous on $T_{\mathbf{1}}G$.

**3.22 Corollary** *The Lie algebra* $(\mathbf{L}(G),[\cdot,\cdot])$ *associated to a Lie group is a locally convex Lie algebra.*

**3.23 Remark** (Left–right confusion) The reader may wonder now why one uses left-invariant vector fields to compute the Lie algebra. Instead one could as well use *right-invariant vector fields*, that is, $X(g) = T\rho_g X(\mathbf{1})$. This would also lead to a Lie algebra structure on $T_{\mathbf{1}}G$; however, the induced Lie bracket would have the opposite sign (see Exercise 3.2.7). Indeed there is no reason to prefer left-invariant vector fields over right-invariant ones: The choice of the former is historically motivated and customary.

**3.24 Remark** There are several alternative ways to introduce the Lie bracket on $T_{\mathbf{1}}G$. For example, one can use the adjoint action of the Lie group (which is briefly discussed in Example 3.28). In addition, the construction of the Lie bracket using the local multiplication in 3.21 can be interpreted (see Neeb, 2005, Lemma III.1.6) as a computation of the antisymmetric part of the second-order Taylor polynomial of the local multiplication at $(0,0)$. Moreover, the Lie bracket measures commutativity of the group multiplication (see Exercise 3.2.8).

**3.25 Example** (The Lie algebra of $\mathrm{Diff}(M)$) Let $M$ be a compact manifold. In Example 3.5 we have seen that by identifying $\mathrm{Diff}(M)$ as an open subset of $C^\infty(M,M)$, it becomes a Lie group under composition of maps. Further, Proposition 2.21 shows that $T_{\mathrm{id}}\,\mathrm{Diff}(M) = T_{\mathrm{id}}C^\infty(M,M) \cong \mathcal{V}(M)$. We shall now show that the Lie algebra is given as $(\mathcal{V}(M), -[\cdot,\cdot])$ with the *negative* of the usual bracket of vector fields (see 3.18). Extend $X \in \mathcal{V}(M) = T_{\mathrm{id}}\ \mathrm{Diff}\ (M)$ to the right-invariant vector field $R_X \in \mathcal{V}^\rho(\mathrm{Diff}(M)), R_X(\varphi) := X \circ \varphi$ and

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consider the product vector field $R_X \times \mathbf{0}_M \in \mathcal{V}(\mathrm{Diff}(M) \times M) = \mathcal{V}(\mathrm{Diff}(M)) \times \mathcal{V}(M)$. We now exploit the canonical action $\alpha\colon \mathrm{Diff}(M) \times M \to M$, $(\varphi, m) \mapsto \varphi(m)$, Example 3.8, and note that as a restriction of ev, a manifold version of (2.5) (see Exercise 2.3.4) yields the tangent map of $\alpha$:

$$T\alpha(R_X \times \mathbf{0}_M)(\varphi, m) = T_{(\varphi,m)}\alpha(X \circ \varphi, \mathbf{0}_M(m)) = X \circ \varphi(m) + T\varphi(0_m)$$
$$= X \circ \varphi(m) = X(\alpha(\varphi, m)).$$

Hence the product vector field $R_X \times \mathbf{0}_M$ is $\alpha$-related to $X$. Thus for $X, Y \in \mathcal{V}(M)$ the bracket $[X, Y]$ is $\alpha$-related to $[R_X \times \mathbf{0}_M, R_Y \times \mathbf{0}_M] = [R_X, R_Y] \times \mathbf{0}_M$, whence $[R_X, R_Y]$ is the negative of the usual bracket, see Remark 3.23.

If $M$ admits a volume form $\mu$, we have seen that the volume-preserving diffeomorphisms $\mathrm{Diff}_\mu(M)$ form a Lie subgroup of $\mathrm{Diff}(M)$. Thanks to E.18, the Lie algebra $\mathbf{L}(\mathrm{Diff}_\mu(M))$ of this subgroup can be identified as the Lie subalgebra of divergence-free vector fields $\mathcal{V}_\mu(M) = \{X \in \mathcal{V}(M) \mid \operatorname{div} X = 0 (\Leftrightarrow \mathcal{L}_X \mu = 0)\}$.

**3.26 Example** Consider the locally convex space $E$ as the abelian Lie group $(E, +)$. An easy computation shows that $\mathbf{L}(E) = E$ with the trivial Lie bracket. Thus the Lie algebra of this abelian Lie group is an abelian Lie algebra (see Exercise 3.2.8).

We can associate to every Lie group morphism a morphism of (locally convex) Lie algebras as the following lemma shows.

**3.27 Lemma** *If $f\colon G \to H$ is a Lie group morphism (i.e. a smooth group homomorphism) then the map $\mathbf{L}(f) := T_\mathbf{1} f\colon \mathbf{L}(G) \to \mathbf{L}(H)$ is a morphism of Lie algebras, that is, $\mathbf{L}(f)([v, w]) = [\mathbf{L}(f)(v), \mathbf{L}(f)(w)]$, for all $v, w \in \mathbf{L}(G)$.*

*Proof* Let $v \in T_\mathbf{1} G$ and $\tilde{v} := T_\mathbf{1} f(v) \in T_\mathbf{1} H$. Since $f$ is a group morphism we have

$$\Theta(\tilde{v})(f(g)) = T_\mathbf{1}\lambda_{f(g)} T_\mathbf{1} f(v) = T_g f(T_\mathbf{1}\lambda_g(v)) = T_g f \Theta(v).$$

Hence for every $v \in T_\mathbf{1} G$ the left-invariant vector field $\Theta(v)$ is $f$-related to $\Theta(\tilde{v})$. As $f$-relatedness is inherited by the Lie bracket (Exercise 3.2.4) we see that

$$T_\mathbf{1} f[\Theta(v), \Theta(w)] = [\Theta(\tilde{v}), \Theta(\tilde{w}] \circ f$$

evaluating in $\mathbf{1}$, we obtain the claimed formula since $f(\mathbf{1}_G) = \mathbf{1}_H$. □

**3.28 Example** Let $G$ be a Lie group and $g \in G$. Then the conjugation with $g$ is the Lie group morphism $c_g\colon G \to G$, $h \mapsto ghg^{-1}$. Hence for every $g$ we obtain a Lie algebra morphism $\mathrm{Ad}_g := \mathbf{L}(c_g)\colon \mathbf{L}(G) \to \mathbf{L}(G)$, called the *adjoint map* of $g$. This gives rise to a smooth mapping

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$$\mathrm{Ad}\colon G \times \mathbf{L}(G) \to \mathbf{L}(G), \quad (g,x) \mapsto \mathrm{Ad}_g(x),$$

called the *adjoint action* of $G$.

The upshot of this short repetition is that every Lie group comes with an associated Lie algebra, and every Lie group morphism gives rise to a Lie algebra morphism. In finite dimensions, the interplay between these objects leads to the classical Lie theorems.

**The Lie Theorems for Finite-Dimensional Lie Groups and Lie Algebras**

**Lie 2** *If $G, H$ are Lie groups and $G$ is simply connected, then for every Lie algebra homomorphism $f\colon \mathbf{L}(G) \to \mathbf{L}(H)$ there exists a Lie group morphism with $f = \mathbf{L}(\varphi)$.*

**Lie 3** *For every Lie algebra $\mathfrak{g}$ there exists a connected Lie group $G$ with $\mathbf{L}(G) = \mathfrak{g}$.*

We omitted the first Lie theorem as it is a purely local statement which does not admit a global formulation on the Lie group. It is well known that the third Lie theorem fails in infinite dimensions. For example, one can show the following.

**3.29** (A Lie algebra without an associated Lie group; Omori, 1981) Let $M$ be a connected *non-compact* finite-dimensional manifold. Then there exists no Lie group $G$ such that $\mathbf{L}(G) = \mathcal{V}(M)$.

However, under certain assumptions one can salvage at least the second Lie theorem. The main issue here is that in infinite-dimensional spaces, even simple differential equations might not have solutions (see Appendix A.6 for explicit counterexamples). Thus in our general setting, a major task is to establish the existence of solutions to differential equations relevant to Lie theory. These *equations of Lie type* will be discussed next.

## Exercises

3.2.1 Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra. Show that $[x,x] = 0$ for all $x \in \mathfrak{g}$ is equivalent to skew-symmetricity of the Lie bracket, that is, $[x,y] = -[y,x]$, for all $x, y \in \mathfrak{g}$.

3.2.2 Show that the commutator bracket (Example 3.16) is a Lie bracket.

3.2.3 With the help of the material in Appendix D show that the bracket of vector fields from 3.18 turns $\mathcal{V}(M)$ into a Lie algebra.

(a) Let $X, Y \in \mathcal{V}(M)$ and $\mathcal{A}$ be an atlas of $M$. The brackets of the local representatives yield $([X_\phi, Y_\phi])_{\phi \in \mathcal{A}}$. Show that this induces a vector field $[X,Y]$ with the properties from 3.18.

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(b) Prove that $[X, X] = 0$ for all $X \in \mathcal{V}(M)$ and the Jacobi identity holds.

(c) Show that the bracket is continuous with respect to the topology from D.4.

*Hint:* All assertions can be localised in charts where Lemma D.11 holds.

3.2.4 Prove a global version of Lemma D.9, that is, if $(X_i, Y_i) \in \mathcal{V}(M) \times \mathcal{V}(N)$, $i = 1, 2$, are pairs of $f$-related vector fields, then $[X_1, X_2]$ and $[Y_1, Y_2]$ are $f$-related.

3.2.5 We check several details in the proof of Proposition 3.20. Show that:

(a) $X_v : G \to TG$, $g \mapsto T_{\mathbf{1}}\lambda_g(v)$ is a smooth left-invariant vector field for $v \in T_{\mathbf{1}}G$;

(b) $\Theta : T_{\mathbf{1}}G \to \mathcal{V}^{\ell}(M) \subseteq \mathcal{V}(M)$, $v \mapsto X_v$ is continuous.

*Hint:* Combine D.4 with Lemma 2.10.

3.2.6 Let $G$ be a Lie group and $\varphi$ be a chart with the properties from 3.21 with respect to which we define a local multiplication $*$. Let $X \in \mathcal{V}^{\ell}(G)$ be left invariant with $X(\mathbf{1}) = v$. Prove that $X_{\varphi}|_{\varphi(W)} = \Lambda^v$ and deduce that the (principal part of) left-invariant vector fields are thus locally related to the left-invariant vector fields with respect to the local multiplication.

3.2.7 Let $v \in \mathbf{L}(G)$ and denote by $L_v$, $R_v$ the left- (/right-)invariant vector field constructed from $v$. Show that $L_v$ and $-R_v$ are $\iota$-related and deduce $[L_v, L_w] = -[R_v, R_w]$.

*Hint:* Use Exercise 3.1.3 to show that $T_{\mathbf{1}}\iota(v) = -v$.

3.2.8 Let $G$ be an abelian Lie group. Show that the Lie bracket of $\mathbf{L}(G)$ is trivial, that is, the Lie bracket vanishes and the Lie algebra is abelian.

*Hint:* Since $G$ is abelian all left-invariant vector fields are also right-invariant.

3.2.9 Let $A$ be a CIA. Show that the Lie algebra of $A^{\times}$ is $A$ with Lie bracket $[a, b] = ab - ba$.

*Hint:* Note that since $A^{\times} \Subset A$, the equation for left invariance is $X_v(g) = gv (= \beta(g, v))$ in $A$.

3.2.10 Let $\mathfrak{g}, \mathfrak{h}$ be two Lie algebras. Show that there is a canonical way to turn the product $\mathfrak{g} \times \mathfrak{h}$ into a Lie algebra. Explain how this was exploited in Example 3.25.

3.2.11 Let $G$ be a Lie group with associated Lie algebra $(\mathbf{L}(G).[\cdot, \cdot])$. In this exercise, we consider the adjoint action from Example 3.28. Show that:

(a) $\mathrm{Ad} : G \times \mathbf{L}(G) \to \mathbf{L}(G)$ is a Lie group action;

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(b) for $\varphi\colon G \to H$ a Lie group morphism, $\mathrm{Ad}_{\varphi(g)}(\mathbf{L}(\varphi)(x)) = \mathbf{L}(\varphi)(\mathrm{Ad}_g(x))$;

(c) for $x, y \in \mathbf{L}(G)$ one has $\mathrm{ad}_x(y) := T_{\mathbf{1}_G, y}\,\mathrm{Ad}(x, 0_y) = [x, y]$.

## 3.3 Regular Lie Groups and the Exponential Map

In this section, we discuss differential equations needed for advanced tools in Lie theory.

**3.30 Definition** Let $G$ be a Lie group with Lie algebra $\mathbf{L}(G)$. We say $G$ is *semiregular* if for each smooth curve $\eta \in C^\infty([0,1], \mathbf{L}(G))$ the initial value problem

$$\begin{cases} \dot{\gamma}(t) &= \gamma(t).\eta(t) = T_{\mathbf{1}}\lambda_{\gamma(t)}(\eta(t)) \\ \gamma(0) &= \mathbf{1} \end{cases} \tag{3.5}$$

has a (unique) solution $\mathrm{Evol}(\eta) := \gamma\colon [0,1] \to G$. We also say that (3.5) is a *Lie type equation*. The group $G$ is *regular (in the sense of Milnor)* if $G$ is semiregular and the following *evolution map* is smooth: (cf. C.15)

$$\mathrm{evol}\colon C^\infty([0,1], \mathbf{L}(G)) \to G, \quad \eta \mapsto \mathrm{Evol}(\eta)(1)$$

**3.31** For a smooth curve $c\colon [a,b] \to G$, we can define the *left logarithmic derivative* $\delta^\ell(c)\colon [a,b] \to \mathbf{L}(G)$, $t \mapsto T\lambda_{c(t)}(\dot{c}(t))$. Note that the logarithmic derivative inverts the evolution, that is, $\delta^\ell(\mathrm{Evol}(\eta)) = \eta$. There are many identities relating $\delta^\ell$, Evol and their counterparts defined via right multiplication (see Kriegl and Michor (1997, Section 38) for an account, also Exercise E.2.3). The left- (right-)logarithmic derivative is closely connected to the left- (right-) Maurer–Cartan form on $G$; see Example E.10.

**3.32 Remark** (a) Solutions to (3.5) are automatically unique by Kriegl and Michor (1997, 38.3 Lemma).

(b) For regular Lie groups, Lie's second theorem holds. Since the proof can most conveniently be formulated within the framework of differential forms, we defer it to Appendix E.2.

(c) Instead of using the left multiplication in (3.5) one can also use right multiplication to define regularity. Similar to the definition of the Lie algebra, using inversion of the group shows that the two notions of regularity are the same.

**3.33 Remark** Thanks to the usual solution theory of ordinary differential equations, every Banach–Lie group (i.e. Lie group modelled on a Banach space) and thus every finite-dimensional Lie group is regular; see Neeb (2006).

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**3.34 Example** Consider the locally convex space $E$ as a Lie group $(E,+)$. Note that its Lie algebra is again $E$ with the zero-bracket (Exercise 3.3.3). For a smooth curve $\eta\colon [0,1] \to \mathbf{L}(E) = E$ we interpret the Lie type equation in $TE = E \times E$ and obtain $(\gamma(t),\gamma'(t)) = (\gamma(t),\eta(t))$. Hence a solution $\gamma$ of (3.5) satisfies $\gamma' = \eta$. Therefore, if $(E,+)$ is regular, then $E$ is Mackey complete; Definition 1.12. Conversely, if $E$ is Mackey complete, then $\mathrm{evol}_E(\eta) := \int_0^1 \eta(s)\mathrm{d}s$ defines a continuous linear (hence smooth) map $\mathrm{evol}_E\colon C^\infty([0,1],E) \to E$, whence $(E,+)$ is regular if and only if it is Mackey complete. Note that one can show that any Lie group which is regular is necessarily modelled on a Mackey complete space (Neeb, 2006, Remark II.5.3(b)).

**3.35 Example** Let $(A,\cdot)$ be a continuous inverse algebra (CIA) which is Mackey complete. If the topology of $A$ is generated by a family of seminorms which are submultiplicative, that is, $q(xy) \leq q(x)q(y)$, for all $x,y \in A$, then $(A^\times,\cdot)$ is regular (Glöckner and Neeb, 2012). In this case, the solutions to the evolution equation are given by the *Volterra series*

$$\gamma(t) = \mathbf{1} + \sum_{n\in\mathbb{N}} \int_0^t \int_0^{t_{n-1}} \cdots + \int_0^{t_2} \eta(t_1)\cdots\eta(t_n)\mathrm{d}t_1\cdots\mathrm{d}t_n. \tag{3.6}$$

This has interesting applications in physics, control theory and rough path theory, as for certain CIAs the above series models signatures of irregular paths; see Chapter 8.

**3.36 Example** (Diff$(M)$ is regular) In Example 3.5 we saw that for a compact manifold $M$ the group Diff$(M)$ is a Lie group with Lie algebra $\mathcal{V}(M)$; Example 3.25. If $c\colon [0,1] \to \mathcal{V}(M) \subseteq C^\infty(M,TM)$ is smooth, we use the exponential law to interpret $X := c^\wedge$ as a smooth time-dependent vector field on $M$. Following Remark 3.32(c) we can solve Lie type equations with respect to the right multiplication to establish regularity. As right multiplication $\rho_\phi$ in Diff$(M)$ is the pullback with $\phi$, we deduce from Exercise 2.3.1 $T_{\mathbf{1}}\rho_\phi(c(t)) = c(t)\circ\phi$. Hence the exponential law allows us to rewrite the Lie type equation (3.5) with respect to right multiplication on Diff$(M)$ as a differential equation for the vector field $X_t$ (subscript denoting time dependence):

$$\gamma'(t) = X_t(\gamma(t)), \qquad \gamma(0) = \mathrm{id}_M. \tag{3.7}$$

In other words, $\gamma$ solves the Lie type equation (3.5) (with respect to right multiplication) if and only if it satisfies (3.7), whence $\gamma$ is the flow $\mathrm{Fl}^X$ of the time-dependent vector field $X$ on $[0,1]\times M$ (see D.5). Since $[0,1]\times M$ is compact, the usual (finite-dimensional!) theory of ordinary differential equations (Lang, 1999, IV. §2) shows that the flow of such a vector field always exists. We deduce that Diff$(M)$ is semiregular and can consider the evolution operator $\mathrm{evol}\colon C^\infty([0,1],\mathcal{V}(M)) \to \mathrm{Diff}(M)$.

To see that the evolution is smooth, recall that $\mathrm{Diff}(M) \Subset C^\infty(M,M)$. Now we exploit that $C^\infty(M,M)$ is a canonical manifold of mappings to deduce that evol is smooth if

$$\mathrm{evol}^\wedge \colon C^\infty([0,1],\mathcal{V}(M)) \times M \to M, \quad (X,m) \mapsto \mathrm{Fl}_1^X(m)$$

is smooth. Here $\mathrm{Fl}_1^X$ denotes the time 1-flow of the time-dependent vector field $X$. We view (3.7) now as an ordinary differential equation (ODE) whose right-hand side $F(t,X,m) = \mathrm{ev}(X,t)(m) = X(t,m)$ depends smoothly on the parameter $X$. The theory of parameter-dependent ordinary differential equations shows that $\mathrm{Fl}^X$ depends smoothly on $X$; see, for example, Alzaareer and Schmeding (2015, Proposition 5.13), which permits the (infinite-dimensional!) space $C^\infty([0,1],\mathcal{V}(M))$ as a parameter space. Hence $\mathrm{evol}^\wedge$ is smooth and $\mathrm{Diff}(M)$ is regular.

As a consequence of the regularity of $\mathrm{Diff}(M)$, we can apply Lie's second theorem, Proposition E.14, to Lie algebra morphisms into the Lie algebra of vector fields on a compact manifold. Milnor (1982) used these observations to prove a restricted version of the Lie–Palais theorem; see Palais (1957). In its general form, the Lie–Palais theorem asserts that every finite-dimensional Lie algebra of vector fields $\mathfrak{g}$ of vector fields on a finite-dimensional smooth manifold $M$, which is generated by complete vector fields,[4] consists of complete vector fields and can be integrated to a global action of a Lie group $G$ on $M$. We discuss a proof for a severely limited version of this result as Exercise 3.3.7.

Not all Lie groups are regular, as one can construct some pathological examples modelled on *incomplete* spaces. However, regularity has been established for almost all naturally occurring classes of Lie groups. Nevertheless, the following conjecture is still open.

**Conjecture** (Milnor, 1983) *Every Lie group modelled on a Mackey-complete space is regular.*

For regular Lie groups, one can solve the important class of Lie type differential equations. This allows us to discuss the Lie group exponential function (which can be viewed as an abstraction of the matrix exponential function to Lie groups).

**3.37 Lemma** *Let $G$ be a regular Lie group and $X \in \mathcal{V}^\ell(G)$. Then there exists a unique curve $\gamma_X \colon \mathbb{R} \to G$ with $\dot\gamma_X(t) = X(\gamma_X(t))$, for all $t$ and $\gamma_X(0) = \mathbf{1}$. This implies that every left-invariant vector field is complete.*

*Proof* Consider the smooth (constant) curve $\eta_X \colon \mathbb{R} \to T_{\mathbf{1}}G$, $t \mapsto X(\mathbf{1})$ (in the following we will frequently consider the restriction of $\eta_X$ to $[0,1]$ without

[4] A vector field $X \in \mathcal{V}(M)$ is complete if its integral curves $\dot\varphi_x(t) = X(\varphi_x(t))$, $\varphi_x(0) = x$ exist for all $t \in \mathbb{R}$ and $x \in M$.

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further notice). By regularity, we obtain a unique solution $\gamma\colon [0,1] \to G$ such that $\gamma(0) = \mathbf{1}$ and $\dot\gamma(t) = T\lambda_{\gamma(t)}(\eta_X(t)) = X(\gamma(t))$, $t \in [0,1]$. Thus this curve is the flow of the invariant vector field. Since the (constant) curve $\eta_X$ makes sense for all $t \in \mathbb{R}$, we can consider the equation (3.5) for every $t \in \mathbb{R}$. We now extend the flow $\gamma$ to all of $\mathbb{R}$.

*Step 1: Smooth extension to* $[-1,1]$.
Using right-regularity, we can construct a curve $\gamma_R\colon [0,1] \to G$ which satisfies $\gamma(0) = \mathbf{1}$ and $\dot\gamma(t) = T\rho_{\gamma(t)}(X(1))$. Let us show that the formula

$$\gamma_X(t) := \begin{cases} \gamma_R^{-1}(-t), & t \in [-1,0], \\ \gamma(t), & t \in [0,1] \end{cases}$$

yields a smooth extension of $\gamma$ whose derivative is $X(\gamma_X(t))$ at every $t$. We compute with the formula for the derivative of the inverse (3.3), the derivative of $\gamma_X$ on $[-1,0]$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\gamma_R^{-1}(-t) &= -T\lambda_{\gamma_R^{-1}(-t)}T\rho_{\gamma_R^{-1}(-t)}\dot\gamma_R(-t) \\ &= T\lambda_{\gamma_R^{-1}(-t)}T\rho_{\gamma_R^{-1}(-t)}T\rho_{\gamma_R(-t)}(v) = X(\gamma_R^{-1}(-t)). \end{aligned}$$

Hence $\gamma_X$ is a smooth integral curve of the left-invariant field $X$.

*Step 2: $\gamma_X$ extends to all of* $\mathbb{R}$.
Pick $0 < t_0 < 1$ and define $\gamma_{t_0}\colon [-1+t_0, 1+t_0] \to G$, $t \mapsto \gamma(t_0)\gamma_X(t-t_0)$. Note that $\gamma_{t_0}(t_0) = \gamma_X(t_0)$ and furthermore,

$$\begin{aligned} \dot\gamma_{t_0}(t_0+t) &= T\lambda_{\gamma(t_0)}(\dot\gamma_X(t)) = T\lambda_{\gamma(t_0)}(X(\gamma_X(t))) = X(\gamma(t_0)\gamma_X(t)) \\ &= X(\gamma_{t_0}(t+t_0)), \end{aligned}$$

where we have used the left invariance of $X$. Uniqueness of the solution implies now that on their common domain of definition $\gamma_X$ and $\gamma_{t_0}$ coincide. Thus we can extend $\gamma_X$ to $[-1, 1+t_0]$. Repeating the argument, the domain of $\gamma_X$ is not bounded from above. Choosing $-1 < t_0 < 0$ a similar argument shows that the domain cannot be bounded from below. Thus $\gamma_X$ can be continued for all of $\mathbb{R}$. □

**3.38 Definition** Let $G$ be a regular Lie group. Then we define the *Lie group exponential*

$$\exp_G\colon \mathbf{L}(G) \to G, \quad v \mapsto \gamma_v(1),$$

where $\gamma_v$ is the unique integral curve starting from $\mathbf{1}$ of the vector field $L_v(g) := T\lambda_g(v)$.

**3.39 Remark** Note that $C\colon \mathbf{L}(G) \to C^\infty([0,1], \mathbf{L}(G))$, $v \mapsto (t \mapsto v)$ is continuous linear, hence smooth. Thus $\exp_G = \mathrm{evol} \circ C$ is smooth for every regular Lie group $G$.

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**3.40 Lemma** *Let G be a regular Lie group. Define for* $\eta \in C^\infty([0,1], \mathbf{L}(G))$ *and* $s \in [0,1]$ *the curve* $\eta_s(t) := \eta(st)$*. Then*

$$\operatorname{Evol}(s\eta_s)(t) = \operatorname{Evol}(\eta)(st) \text{ for all } t \in [0,1] \quad \text{and} \quad \operatorname{Evol}(\eta)(s) = \operatorname{evol}(s\eta_s).$$

*In particular, this implies* $T_0 \exp = \mathrm{id}_{\mathbf{L}(G)}$.

*Proof* The curve $t \mapsto \operatorname{Evol}(\eta)(st)$ takes 0 to the identity in $G$, and its derivative is $\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Evol}(\eta)(st) = s \operatorname{Evol}(\eta)(st) \cdot \eta(st)$. Here we abuse the multiplication to denote the (derivative) of left translation. Thus it solves the Lie type equation for the curve $s\eta_s$. This proves the first identity, while we obtain the second for $t = 1$. Now let $\eta(t) := v$ be constant for $v \in \mathbf{L}(G)$. Then $\eta_s(t) = \eta(t)$ and we see that $\exp(sv) = \operatorname{evol}(s\eta_s) = \operatorname{Evol}(\eta)(s)$. Derivating at $s = 0$ yields

$$T_0 \exp(v) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp(sv) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \operatorname{Evol}(\eta)(s) = \underbrace{\operatorname{Evol}(\eta)(0)}_{=\mathbf{1}_G} \cdot \eta(0) = v.$$

□

Unfortunately, the observation that $T_0 \exp = \mathrm{id}_{\mathbf{L}(G)}$ is not as useful as in the Banach setting, where the inverse function theorem implies that the Lie group exponential is a local diffeomorphism onto a neighbourhood of the unit of the group (then the Lie group exponential yields a canonical chart called exponential coordinates). In general, the Lie group exponential need be *neither* locally injective *nor* locally surjective. Most prominently, this happens for the diffeomorphism group $\operatorname{Diff}(M)$ from Example 3.5. We discuss the special case for $M = \mathbb{S}^1$ in Example 3.42. Lie groups for which the exponential function is well behaved thus deserve a special name.

**3.41 Definition** A regular Lie group $G$ is called *locally exponential* if the exponential function $\exp\colon \mathbf{L}(G) \to G$ restricts to a local diffeomorphism between a neighbourhood of $0 \in \mathbf{L}(G)$ and $\mathbf{1}_G \in G$.

Unfortunately, diffeomorphism groups are not locally exponential, as the following classical example shows.

**3.42 Example** Consider the unit circle $\mathbb{S}^1$. We will show that the image of the Lie group exponential of the diffeomorphism group $\operatorname{Diff}(\mathbb{S}^1)$ contains no identity neighbourhood. Recall from Example 3.36 that the Lie group exponential is the map

$$\exp\colon \mathbf{L}(\operatorname{Diff}(\mathbb{S}^1)) = \mathcal{V}(\mathbb{S}^1) \to \operatorname{Diff}(\mathbb{S}^1), \quad V \mapsto \mathrm{Fl}_1^V$$

assigning to a (time-independent) vector field its time 1-flow.

Recall that $\theta\colon \mathbb{R} \to \mathbb{R}/2\pi \cong \mathbb{S}^1$, $\theta \mapsto e^{i\theta}$ is a submersion. Composing the submersion with a vector field of the circle, we identify vector fields of the

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circle with $2\pi$-periodic maps $\mathbb{R} \to \mathbb{R}$. For a constant vector field $X^c(\theta) \equiv c$, a quick computation shows that its flow, $\exp(tX^c(\theta)) = e^{i(\theta+tc)}$, is for fixed $t$ a rotation of the circle.

*Step 1: A diffeomorphism* $\eta \in \mathrm{Diff}(\mathbb{S}^1)$ *without fixed points is an exponential of* $V \in \mathcal{V}(\mathbb{S}^1)$ *if and only if* $\eta$ *is conjugate to a rotation.*
If $V \in \mathcal{V}(\mathbb{S}^1)$ has a zero, its exponential has a fixed point. Thus if $\eta = \exp(X)$, we must have $X(\theta) \neq 0$. We will now construct $\varphi \in \mathrm{Diff}(\mathbb{S}^1)$ and a constant vector field $X^c$ such that $\varphi \circ \eta = \exp(X^c) \circ \varphi$. Assume for a moment that for all $t \in [0,1]$ we have the identity $\varphi \circ \exp(tX) = \exp(tX^c) \circ \varphi$. Differentiating at $t = 0$ yields with Lemma 3.40 the identity $T_\theta\varphi(X(\theta)) = X^c(\varphi(\theta))$ for all $\theta \in \mathbb{S}^1$. For ease of computation identify now $\varphi$ and $X$ with periodic mappings $\mathbb{R} \to \mathbb{R}$. Then the equation reads $\varphi'(\theta)X(\theta) = c$ for all $\theta \in \mathbb{R}$. Integrating, we obtain, with $\varphi(0) = 0$ and the fact that $X$ vanishes nowhere, that $\varphi(\theta) = \int_0^\theta c/X(s)\mathrm{d}s$. We will now choose $c$ such that $\varphi(\theta + 2\pi) - \varphi(\theta) = \int_t^{t+2\pi} c/X(s)\mathrm{d}s = 2\pi$ (because then $\varphi$ descends to a diffeomorphism of $\mathbb{S}^1$). Since $X$ is $2\pi$-periodic, we can simply choose $c = 2\pi \int_0^{2\pi} X(s)\mathrm{d}s$. Since the flows of the $\varphi$-related vector fields $X$ and $T\varphi^{-1} \circ X \circ \varphi$ are conjugate by $\varphi$, we obtain $\varphi \circ \eta = \varphi \circ \exp(X) = \exp(X^c) \circ \varphi = R_c \circ \varphi$. Thus if a fixed-point-free diffeomorphism of $\mathbb{S}^1$ is the exponential of a vector field, it is conjugate to a rotation.

*Step 2: Diffeomorphisms near the identity which are not exponentials.*
We claim that there are diffeomorphisms $\varphi$, arbitrarily near the identity $\mathrm{id}_{\mathbb{S}^1}$, such that:

(a) $\varphi$ has no fixed points;
(b) there exists $\theta_0 \in \mathbb{S}^1$ and $1 < n \in \mathbb{N}$ such that $\varphi^n(\theta_0) = \theta_0$, but $\varphi^n \neq \mathrm{id}_{\mathbb{S}^1}$.

If this were true, then we note that if $\varphi$ is an exponential, (a) and Step 1 imply that it must be conjugate to rotation. However, this is impossible by (b), as a rotation which has a point of period $n$ must itself be periodic with period $n$. Thus the image of the exponential does not contain $\varphi$ and $\mathrm{Diff}(\mathbb{S}^1)$ cannot be locally exponential.

To construct a diffeomorphism with properties (a) and (b) consider for $n \in \mathbb{N}$ large enough and $0 < \varepsilon < 1/n$ the maps

$$f_{n,\varepsilon}(\theta) = \theta + \frac{\pi}{n} + \varepsilon \sin^2(n\theta)$$

descend to diffeomorphisms of the circle which satisfy (a)–(b) and can be made arbitrarily close to the identity (i.e. every identity neighbourhood in the compact open $C^\infty$-topology contains an $f_{n,\varepsilon}$ for $n$ large enough). We leave the details as Exercise 3.3.6.

More generally one can prove that the diffeomorphism group of a compact manifold is not locally exponential.

**3.43 Example** (a) As a consequence of Lemma 3.40 and the inverse function theorem, every Banach–Lie group and thus, in particular, every finite-dimensional Lie group is locally exponential.

(b) The unit group of a Mackey-complete CIA $A$ (see Example 3.35) is locally exponential (Glöckner and Neeb, 2012).

(c) If $G$ is a locally exponential Lie group, we shall see in Exercise 3.4.6 that the current groups $C^\infty(K, G)$ (see §3.4) are locally exponential. Thanks to Wockel (2007) this generalises even to the group of gauge transformations of a principal bundle with locally exponential gauge group.

**3.44 Remark** While in infinite dimensions not every closed subgroup of a (locally exponential) Lie group is again a Lie group (see Example 3.11), there are conditions which ensure that a closed subgroup of a locally exponential Lie group is a Lie subgroup. We refer to Neeb (2006, IV) for more information.

## Exercises

3.3.1 A smooth map $\alpha\colon \mathbb{R} \to G$ to a Lie group is called a 1-*parameter subgroup* of (a regular Lie group) $G$ if it is a group homomorphism, that is, $\alpha(s+t) = \alpha(s)\alpha(t)$ for all $s, t \in \mathbb{R}$.

(a) Show that a 1-parameter subgroup $\alpha$ is an integral curve of the left-invariant vector field $L_v$ and the right-invariant field $R_v$, generated by $v := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\alpha(t)$, i.e. $\dot\alpha(t) = L_v(\alpha(t))$ and $\dot\alpha(t) = R_v(\alpha(t))$.

(b) Show that if $\alpha, \beta\colon \mathbb{R} \to G$ are smooth, $\alpha(0) = \mathbf{1} = \beta(0)$ with $\dot\alpha(t) = L_v(\alpha(t))$ and $\dot\beta(t) = R_v(\beta(t))$ (the associated right-invariant vector field) then $\alpha = \beta$ and these curves are a 1-parameter subgroup.

*Hint:* To show equality differentiate $\alpha(t)\beta(-t)$; for the second statement take the derivative of $\beta(t-s)\beta(s)$.

3.3.2 Consider the multiplicative group $(\mathbb{R}^\times, \cdot)$ as a Lie group. Show that $\delta^\ell(f)(t) = f'(t)/f(t)$ and explain the name logarithmic derivative.

3.3.3 Let $\gamma_i\colon [0,1] \to G, i = 1,2$ be smooth curves. Show that $\delta^\ell(\gamma_1) = \delta^\ell(\gamma_2)$ if and only if $\gamma_1(t) = g\gamma_2(t)$, for all $t \in [0,1]$ and some fixed $g \in G$.

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3.3.4 Consider a locally convex space $(E, +)$ as a Lie group. Show that the Lie bracket on $\mathbf{L}(E) = E$ vanishes and the Lie group exponential is $\exp_E = \mathrm{id}_E$.

3.3.5 Let $\alpha\colon G \to H$ be a Lie group morphism between regular Lie groups. Show that:

(a) for $\gamma \in C^1([0,1], G)$ we have $\delta^\ell(\alpha \circ \gamma) = \mathbf{L}(\alpha)(\delta^\ell(\gamma))$;

(b) for $\eta \in C^\infty([0,1], \mathbf{L}(G))$ one has $\mathrm{Evol}(\mathbf{L}(\alpha) \circ \eta) = \alpha \circ \mathrm{Evol}(\eta)$ (a similar formula holds for evol). Further, prove the naturality of the Lie group exponential, that is,

$$\exp_H \circ \mathbf{L}(\alpha) = \alpha \circ \exp_G . \tag{3.8}$$

3.3.6 Consider for $n \in \mathbb{N}$ and $0 < \varepsilon < 1/n$ the maps $f_{n,\varepsilon}\colon \mathbb{R} \to \mathbb{R}$, $\theta \mapsto \theta + \pi/n + \varepsilon \sin^2(n\theta)$.

(a) Show that $f_{n,\varepsilon}$ descends via the submersion $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$ to a diffeomorphism $\varphi_{n,\varepsilon}$ of $\mathbb{S}^1$. Moreover, prove that there are $\varphi_{n,\varepsilon}$ arbitrarily near the identity (where $\mathrm{Diff}(\mathbb{S}^1) \Subset C^\infty(\mathbb{S}^1, \mathbb{S}^2)$ carries the compact open $C^\infty$-topology).
*Hint:* Control $f_{n,\varepsilon}$ in the compact open $C^\infty$-topology on $\mathbb{R}$.

(b) Show that $\varphi_{n,\varepsilon}$ does not possess a fixed point, but since $f^{2n}_{n,\varepsilon}(0) = 0$ (modulo $2\pi$) it has a periodic point of period $2n$.

(c) Show that the $2n$-periodic orbit $\varphi^k_{n,\varepsilon}(0)$, $k = 1, \ldots, 2n-1$ is unique, that is, if $\theta$ is not contained in the orbit, then $\varphi^{2n}_{n,\varepsilon}(\theta) \neq \theta$. We deduce that $\varphi^{2n}_{n,\varepsilon} \neq \mathrm{id}_{\mathbb{S}^1}$.

3.3.7 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. By Lie's third theorem (Hilgert and Neeb, 2012, Theorem 9.4.11) there exists a connected, simply connected Lie group $G$ such that $\mathbf{L}(G) = \mathfrak{g}$. Assume that $M$ is a compact manifold such that $\phi\colon \mathfrak{g} \to \mathcal{V}(M)$ is a Lie algebra morphism (with respect to the negative of the usual bracket of vector fields). Prove that $\phi$ induces a smooth action of $G$ on $M$ (this is a very restricted version of the Lie–Palais theorem (Palais, 1957)).

## 3.4 The Current Groups

In this section, we construct a class of infinite-dimensional Lie groups which occur naturally in theoretical physics: loop groups and current groups. As a first step in the construction we prove a useful local characterisation of Lie groups.

**Notation** (Multiplicative Notation for Sets) In the statement of Proposition 3.45 we use multiplicative notation for sets. Recall that $A \cdot B$ means the set of all elements which can be written as a product of elements in $A$ and $B$ (in that order).

**3.45 Proposition** (Bourbaki, 1998, Ch. III, §1, No. 9 Proposition 18) *Let $G$ be a group and $U, V \subseteq G$ such that $\mathbf{1} \in V = V^{-1} := \{g \in G \mid g^{-1} \in V\}$ and $V \cdot V \subseteq U$. Assume that $U$ is equipped with a (smooth) manifold structure such that $V \Subset U$ and the mappings $\iota|_V^V : V \to V$ and $m_G|_{V \times V} : V \times V \to U$ are smooth. Then the following holds:*

(a) *There is a unique manifold structure on the subgroup*

$$G_0 := \langle V \rangle := \{v_1 \cdots v_k \mid v_i \in V, k \in \mathbb{N}\}$$

*such that $G_0$ becomes a Lie group, $V \Subset G_0$, and $G_0$ and $U$ induce the same manifold structure on $V$.*

(b) *Assume that for each $g$ in a generating set of $G$, there is $\mathbf{1} \in W_g \Subset U$ such that $gW_g g^{-1} \subseteq U$ and $c_g : W_g \to U$, $h \mapsto ghg^{-1}$ is smooth. Then there is a unique manifold structure on $G$ turning $G$ into a Lie group such that $V \Subset G$ and both $G$ and $U$ induce the same manifold structure on $V$.*

*Proof* *Step 1: Shifted open sets*. If $A \Subset V$ and $v_0 \in V$ with $v_0 A \subseteq V$, then $v_0 A$ is open in $V$ as the preimage of $A$ under the smooth map $\delta_{v_0} : V \to U$, $g \mapsto v_0^{-1} g$.

*Step 2: Shifted charts*. Pick $W \Subset V$ such that $W = W^{-1}$, $W^3 = W \cdot W \cdot W \subseteq U$ and there exists a manifold chart $(\varphi, W)$ of $U$. For $g \in G$ we consider the map $\varphi_g : gW \to \varphi(W)$, $h \mapsto \varphi(g^{-1}h)$. The idea is now to construct an atlas from these shifted charts which is compatible with the manifold structure on $V$.

*Step 3: Manifold induced by shifted charts*. Observe that for $g_1 W \cap g_2 W \neq \emptyset$ we have $g_2^{-1} g_1 \in W^2$ (this entails $g_1^{-1} g_2 \in W^2$). Then Step 1 yields $W \cap g_2^{-1} g_1 W \Subset W$, whence $D_{g_1, g_2} := \varphi_{g_1}(g_1 W \cap g_2 W) = \varphi(W \cap g_2^{-1} g_1 W)$ is open in the model space. For $d \in D_{g_1, g_2}$ we have now

$$\varphi_{g_2} \circ \varphi_{g_1}^{-1}(d) = \varphi(g_2^{-1} g_1 \varphi^{-1}(d)),$$

which is a smooth mapping by choice of $W$, and so the change of charts is smooth. Endow $G$ with the final topology with respect to the atlas $\mathcal{S} := (\varphi_g)_{g \in G}$ (this topology is Hausdorff since $W$ is Hausdorff). We conclude that $\mathcal{S}$ is a manifold atlas turning $G$ into a smooth manifold.

Moreover, for each $g_0 \in G$ we have $\varphi_{gg_0} \circ \lambda_{g_0}|_{gW} = \varphi_g$, whence the left translation with elements in $G$ is smooth for this manifold structure.

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*Step 4: The manifold structures coincide on $V$.* By our assumptions on multiplication and inversion, we can find for every $v_0 \in V$ a set $\mathbf{1} \in A \Subset W$ such that $v_0 A \subseteq V$. Step 3 shows that $v_0 A$ is open in $G$ and $A \to v_0 A$, $h \mapsto v_0 h$ is a diffeomorphism with respect to the structure induced by $\mathcal{S}$: in particular, $V \Subset G$. Now Step 1 implies that this holds for the structure induced by $U$, whence both coincide on $V$.

We are now in a position to prove both claims of the proposition:

(a) By definition we have $G_0 = \langle V \rangle = \bigcup_{k \in \mathbb{N}} V^k$. We can write $V^2 = \bigcup_{g \in V} gV \Subset G$ and thus, inductively, $V^k \Subset G$ and $G_0 \Subset G$. Denote by $(G, \mathcal{S})$ the manifold induced by the atlas $\mathcal{S}$. Then Step 4 shows that $\delta \colon V \times V \to G_0 \Subset (G, \mathcal{S})$, $(g,h) \mapsto gh^{-1}$ is smooth and this is equivalent to the smoothness of multiplication and inversion on the set $V$. For $g_0, h_0, g, h \in G$ we let $c_{h_0}(a) := h_0 a h_0^{-1}$ and obtain the identity

$$\begin{aligned}\delta(g,h) &= (h_0 g_0^{-1})^{-1} h_0 (g_0^{-1} g)(h_0^{-1} h)^{-1} h_0^{-1} \\ &= \lambda_{g_0^{-1} h_0} \circ c_{h_0} \circ \delta(\lambda_{g_0^{-1}}(g), \lambda_{h_0^{-1}}(h)). \end{aligned} \tag{3.9}$$

We proceed now by induction on $k$ and show first that $\delta$ is smooth on $V^k \times V$. For $k = 1$ we know that $\delta$ is smooth. However, note that also $c_{h_0} \colon V \to G_0$ is smooth since

$$c_{h_0}(g) = \lambda_{h_0} \circ \delta(g, h_0) \tag{3.10}$$

and $\delta$ is smooth on $V \times V$. Moreover, if $g \in V^k$ we can pick $g_0 \in V^{k-1}$ such that for all $x$ in a $g$-neighbourhood $g_0^{-1} x \in V$. Following again (3.10), we see that

$$c_{h_0}(x) = \lambda_{h_0 g_0} \delta(g_0^{-1} x, h_0) \text{ is smooth for all } h_0 \in H \text{ and } x \text{ near } g.$$

We conclude that $c_{h_0} \colon G_0 \to G_0$ is smooth for each $h_0 \in V$.

Thus for the induction step from $k$ to $k+1$ we see now that $\delta \colon V^{k+1} \times V \to G_0$ is given in a neighbourhood of $(g,h)$ by the composition of smooth maps (3.9) for $h_0 := h$ and $g_0 \in V$ with $g_0^{-1} g \in V^k$. It follows that $\delta \colon G_0 \times V \to G_0$ is smooth. Now a trivial induction together with (3.9) shows that $\delta \colon G_0 \times G_0 \to G_0$ is smooth. We conclude that $G_0$ is a Lie group.

(b) We prove inductively that $\delta$ is smooth of an element $(g,h)$ where $g$ is arbitrary and $h = s_1 \cdots s_n$ for $s_i$, $i = 1, \ldots, n$, in the generating set $\mathcal{G}$. Starting with $n = 1$ we choose $h_0 := s_1^{-1}$ and $g_0 := g^{-1}$. By (a) the map $\delta$ is smooth on $G_0$, whence there is an open $(g,h)$-neighbourhood which gets mapped by $\delta \circ (\lambda_{g_0}, \lambda_{h_0})$ into $W_g$. Since $h_0 = s_1 \in \mathcal{G}$ by the assumption, (3.9) implies that $\delta$ is smooth in a neighbourhood of $(g,h)$. Now by (3.10), conjugation with $h_0 \in \mathcal{G}$ is smooth in a $g$-neighbourhood, hence on all

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of $G$. If $\delta$ is smooth on an open neighbourhood of $G \times \bigcup_{1 \leq k \leq n} \mathcal{G}^k$ and $h = s_1 \cdots s_{n+1}$, we argue as above: Taking $h_0 = s_1^{-1}$ shows that $\delta$ is smooth in a neighbourhood of $G \times \bigcup_{1 \leq k \leq n+1} \mathcal{G}^k$ for all $n \in \mathbb{N}$. As smoothness of $\delta$ is equivalent to the smoothness of the group operations, we deduce that $G$ is a Lie group.

The uniqueness of the manifold structure of the Lie groups $G_0$ and $G$ in (a) and (b) follows from the fact that any other manifold structure with these properties induces the same manifold structure on the open $\mathbf{1}$-neighbourhood $V$. □

We will now use Proposition 3.45 to construct the Lie group structure for the current group $C^\infty(K,G)$. The crucial insight here is that for a Lie group, an atlas of charts can be constructed by left translating a chart at the identity. We will see in the construction of the current group that this allows us to construct a canonical manifold of mappings by left translating the pushforward of a suitable chart at the identity.

**General Assumption** For the rest of this section we let $K$ be a compact manifold, $G$ a (possibly infinite-dimensional) Lie group with Lie algebra $\mathbf{L}(G)$. Then we choose and fix $\mathbf{1} \in U_1 \Subset G$ together with a chart $\varphi\colon U_1 \to U \subseteq \mathbf{L}(G)$ such that $\varphi(\mathbf{1}) = 0$ and $T_{\mathbf{1}}\varphi = \mathrm{id}_{\mathbf{L}(G)}$. Further, we pick $\mathbf{1} \in V_1 \Subset U_1$ such that $V_1 \cdot V_1 \subseteq U_1, V_1^{-1} = V_1$ and set $V := \varphi(V_1)$. Mapping spaces will as always be endowed with the compact open $C^\infty$-topology.

Let us now exploit again the local formulation of the Lie group structure of $G$ in a chart around the identity (as in the identification of the Lie algebra, 3.21). In order to clearly distinguish the local group operations from the (globally defined) group operations and the pushforwards appearing on the function spaces, we introduce new symbols for the local operations.

**3.46** The mappings

$$\mu\colon V \times V \to U, \mu(x,y) := \varphi(\varphi^{-1}(x)\varphi^{-1}(y)), \quad \iota\colon V \to V, \iota(x) := \varphi(\varphi^{-1}(x)^{-1})$$

are smooth. Moreover, $C^\infty(K,U_1)$ is an open subset of $C^\infty(K,G)$ and we equip it with the smooth manifold structure, making the bijection $C^\infty(M,U_1) \to C^\infty(K,U), \gamma \mapsto \varphi \circ \gamma$ a diffeomorphism of smooth manifolds. Note that thanks to Exercise 2.3.2, the manifold $C^\infty(K,U)$ is canonical. Then the pushforwards $\mu_*\colon C^\infty(K,V \times V) = C^\infty(K,V) \times C^\infty(K,V) \to C^\infty(K,U)$ and $\iota_*\colon C^\infty(K,V) \to C^\infty(K,V)$ are smooth by Corollary 2.19.

**3.47 Theorem** *The pointwise operations turn $C^\infty(K,G)$ into a Lie group, a* current group. *Its Lie algebra is $C^\infty(K,\mathbf{L}(G))$ with the pointwise Lie bracket (a* current algebra*).*

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*Proof* We check that the assumptions of Proposition 3.45 are satisfied: The identity element in $C^\infty(K,G)$ is the constant map $\mathbf{1}_{C^\infty(K,G)}(k) = \mathbf{1}_G$. Thus 3.46 shows that $C^\infty(K,U) \Subset C^\infty(K,G)$ is an open identity neighbourhood such that multiplication and inversion are smooth on the smaller neighbourhood $C^\infty(K,V)$. Now let $\gamma \in C^\infty(K,G)$; by compactness of $\gamma(K) \subseteq G$ there are $\mathbf{1} \in W_1 \Subset V_1$ and $\gamma(K) \subseteq P \Subset G$ such that $pW_1p^{-1} \subseteq U_1$ for all $p \in P$. Set $W := \varphi(W_1)$ and note that since $C^\infty(K,W) \Subset C^\infty(K,V)$ so is $C^\infty(K,W_1) \Subset C^\infty(K,V_1)$. Since $c\colon P \times W_1 \to U_1, h(p,w) := pwp^{-1}$ is smooth, so is

$$h_\gamma = \varphi \circ h \circ (\gamma \times \varphi^{-1})\colon K \times W \to U, h_\gamma(x,y) = \varphi(\gamma(x)\varphi^{-1}(y)\gamma(x)^{-1}).$$

Since the manifolds $C^\infty(K,W)$ and $C^\infty(K,U)$ are canonical (see 3.46), Proposition 2.17 yields a smooth map $(h_\gamma)_\star\colon C^\infty(K,W) \to C^\infty(K,U)$, $\eta \mapsto h_\gamma \circ (\mathrm{id}_W \times \eta)$. Now conjugation by $\gamma$ coincides on $C^\infty(K,W)$ with $(h_\gamma)_\star$ and is thus smooth on $C^\infty(K,W) \Subset C^\infty(K,G)$. Now Proposition 3.45 provides a unique smooth Lie group structure for $C^\infty(K,G)$.

To identify the Lie algebra, note that $\varphi_*\colon C^\infty(K,U_1) \to C^\infty(K,U)$, $\gamma \mapsto \varphi \circ \gamma$ is a chart around the identity of the Lie group $C^\infty(K,G)$. Exploiting that $T_{\mathbf{1}}\varphi = \mathrm{id}_{\mathbf{L}(G)}$, we use 2.22 to identify $T_{\mathbf{1}_{C^\infty(K,G)}}\varphi_* = (T_{\mathbf{1}}\varphi)_* = (\mathrm{id}_{\mathbf{L}(G)})_*$ whence $\mathbf{L}(C^\infty(K,G) \cong C^\infty(K,\mathbf{L}(G))$. Now the point evaluations $\mathrm{ev}_x\colon C^\infty(K,G) \to G$, $\gamma \mapsto \gamma(x)$ are Lie group morphisms (Exercise 3.4.2), and so $\mathbf{L}(\mathrm{ev}_x) = \mathrm{ev}_x\colon C^\infty(K,\mathbf{L}(G))$ is a Lie algebra morphism by Lemma 3.27. This implies $[\gamma,\eta](x) = [\gamma(x),\eta(x)]$ and thus the bracket is given by the pointwise bracket. □

**3.48 Remark** For non-compact manifolds $M$, the group $C^\infty(M,G)$ can, in general, not be made a Lie group. To see this, consider $\mathbb{N}$ a 0-dimensional manifold. Then $C^\infty(\mathbb{N},\mathbb{S}^1) \cong (\mathbb{S}^1)^{\mathbb{N}} := \prod_{n\in\mathbb{N}} \mathbb{S}^1$ is a compact topological group. However, since it is not locally contractible (see Exercise 3.4.5) it cannot be a manifold and thus it cannot be a Lie group.

We shall now prove that regularity of the target Lie group $G$ is inherited by the current group $C^\infty(K,G)$. The idea is simple: The Lie type equation on the current group is just a parametric version of the Lie-type equation on the target Lie group. In other words, if $\eta\colon [0,1] \to C^\infty(K,\mathbf{L}(G))$ is a smooth curve, then we obtain for each $k \in K$ a Lie type equation

$$\frac{\partial}{\partial t}\gamma^\wedge(t,k) = (\dot\gamma(t))(k) = (\gamma(t).\eta(t))(k) = \gamma^\wedge(t,k).\eta^\wedge(t,k).$$

Thanks to the regularity of the target Lie group $G$, we can solve the Lie type equation (uniquely) for each fixed $k$. Then the solutions to these equations

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glue back together to a solution on the current group (because of a suitable exponential law).

**3.49 Proposition** *If $G$ is a regular Lie group, then $C^\infty(K,G)$ is a regular Lie group.*

*Proof* In view of the preliminary considerations, we see that the pointwise solutions to the Lie type equation must be the solution to the equation on the current group. It remains to apply the exponential law[5] to show that these solutions depend smoothly on the initial data. We obtain a canonical isomorphism $\Theta$:

$$C^\infty([0,1],C^\infty(K,\mathbf{L}(G)) \cong C^\infty([0,1]\times K,\mathbf{L}(G)) \cong C^\infty(K,C^\infty([0,1],\mathbf{L}(G)).$$

The map $h := (\operatorname{evol}_G)_* \circ \Theta \colon C^\infty([0,1],C^\infty(K,\mathbf{L}(G)) \to C^\infty(K,G)$ is smooth by Corollary 2.19 as $\operatorname{evol}_G\colon C^\infty([0,1],\mathbf{L}(G)) \to G$ is smooth. Let $\operatorname{ev}_x\colon C^\infty(K,G) \to G$ and $e_x\colon C^\infty(K,\mathbf{L}(G)) \to \mathbf{L}(G)$ be the point evaluations in $x$. We note that $\mathbf{L}(\operatorname{ev}_x) = e_x$. Hence Exercise 3.3.5 yields $\operatorname{ev}_x \circ h = \operatorname{evol}_G \circ (e_x)_* \circ \Theta$ for all $x \in K$. Since the evaluations separate the points on $C^\infty(K,G)$, this implies $h = \operatorname{evol}_{C^\infty(K,G)}$. □

Similar to regularity being hereditary, current groups inherit the property of being locally exponential from their target Lie group. We leave this as Exercise 3.4.6. In the following sections, we discuss two special cases of current groups and their subgroups: loop groups and groups of gauge transformations (associated to a principal bundle).

## Loop Groups

If $K = \mathbb{S}^1$ the current group $\mathrm{LG} := C^\infty(\mathbb{S}^1,G)$ is better known as a *loop group* (see Pressley and Segal, 1986). Much is known about loop groups and their representation theory. We mention that they are, in particular, connected to the representation theory of Kac–Moody Lie algebras and to quantum field theory; see Schmid (2010).

**3.50 Remark** In the literature the group $C(\mathbb{S}^1,G)$ of continuous loops is also often called the loop group LG of $G$. For us the loop group will nevertheless consist always of smooth loops.

[5] Since [0, 1] has boundary, we cannot use the exponential law, Theorem 2.1.2. Instead, we have to appeal to the stronger version (Alzaareer and Schmeding, 2015, Theorem A). Note that the result is the same and, in particular, the (partial) derivative of the adjoint map corresponds to the derivative.

We shall discuss two canonical subgroups of $LG$ for finite-dimensional $G$.

**3.51** The canonical group morphism $I\colon G \to C^\infty(\mathbb{S}^1,G)$, $g \mapsto (k \mapsto g)$ identifies $G$ with the subgroup of constant loops. As $C^\infty(\mathbb{S}^1,G)$ is a canonical manifold by Exercise 3.4.3, smoothness of $I$ is equivalent to smoothness of $I^\wedge\colon G\times\mathbb{S}^1 \to G$, $I^\wedge(g,k) = g$ (which is obvious). Further, a local argument in charts around $(x,k) \in G \times \mathbb{S}^1$ shows that the partial derivative of $I^\wedge$ identifies the derivative of $I$. Thus by C.12 we obtain

$$TI(v_g) = (k \mapsto v_g \in T_gG) \in C^\infty(\mathbb{S}^1,TG) \cong TC^\infty(\mathbb{S}^1,G), \quad g \in G,\ v_g \in T_gG$$

and this map is clearly injective for every $g$. Thus $I$ is infinitesimally injective and since $G$ is finite dimensional, it is an immersion. Moreover, $I$ is a Lie group morphism with smooth inverse: Thanks to Exercise 3.4.2(c), the evaluation maps $\mathrm{ev}_k : LG \to G$, $k \in \mathbb{S}^1$ are Lie group morphisms. Choosing, for example, $h := \mathrm{ev}_{\mathbf{1}_{\mathbb{S}^1}}\big|_{I(G)}$, we obviously have $h \circ I = \mathrm{id}_G$, whence $I$ is a topological embedding onto its image. Hence $I$ is a smooth embedding and if we identify $G$ with $I(G)$, we can think of $G$ as a Lie subgroup of $LG$ (see Glöckner, 2016, Lemma 1.13).

Now define the group of all loops starting at $\mathbf{1}_G$ as $\Omega G := \{f \in LG \mid f(\mathbf{1}_{\mathbb{S}^1}) = \mathbf{1}_G\}$. Since $\mathrm{ev}_{\mathbf{1}_{\mathbb{S}^1}} : C^\infty(\mathbb{S}^1,G) \to G$ is a Lie group morphism and a submersion with $\Omega G = \mathrm{ev}_{\mathbf{1}_{\mathbb{S}^1}}^{-1}(\mathbf{1}_G)$, we see that $\Omega G$ is a split submanifold of $LG$, hence a Lie subgroup. Summarising,

$$\mathbf{1} \longrightarrow \Omega G \hookrightarrow \mathrm{LG} \underset{I}{\overset{\mathrm{ev}_{\mathbf{1}_{\mathbb{S}^1}}}{\rightleftarrows}} G \longrightarrow \mathbf{1}$$

is a split sequence of Lie groups, whence $\mathrm{LG} \cong \Omega G \rtimes G$. We finally note that this allows one to define the *fundamental homogeneous space*

$$\mathrm{LG}/G = \{[h] \mid f \in [h] \text{ if } f = g \cdot h \text{ for some } g \in G\} \ (\cong \Omega G),$$

which plays an important role in the theory of loop groups (see Pressley and Segal, 1986).

## Groups of Gauge Transformations

We briefly discuss an important class of infinite-dimensional Lie groups also arising in physics. In extremely broad strokes the setup is as follows: Fix a manifold $M$. In physics this would correspond to spacetime. Physical fields in the spacetime $M$ are described as sections of certain bundles $P \to M$ called principal bundles (see Definition 3.52). A premier example here is the electromagnetic field. In this formulation, the famous Maxwell equations – see

Example 3.54 – can be interpreted as differential equations for a certain connection on the principal bundle. We will not describe connections on principal bundles here, but point out that the geometric data of the connection corresponds to the potential of the field. However, the symmetry group of the space of connections on the bundle is the so-called group of gauge transformations. We shall describe it as an infinite-dimensional Lie group and identify it as a subgroup of the current groups studied earlier. Let us begin by recalling the notion of a principal bundle. Much more information on principal bundles can be found in the usual finite-dimensional literature (Baum, 2014; Husemoller, 1993).

**3.52 Definition** Let $G$ be a Lie group with a smooth right action $\rho\colon E\times G \to E$. Assume that the quotient $p\colon E \to M := E/G$ is a smooth locally trivial fibre bundle with typical fibre $F$, that is, $p$ is a smooth map such that there is an open cover $(U_i)_{i\in I}$ of $M$ and $\rho$-equivariant diffeomorphisms (*bundle trivialisations*) $\kappa_i : p^{-1}(U_i) \to U \times F$ conjugating $p$ to the projection. The quadruple $(E,p,M,F)$ is a *principal G-bundle* if the action $\rho$ is simply transitive, that is, for each $f_0 \in F \cong p^{-1}(x)$, we have $G \to F$, $g \mapsto f_0 g$ is a diffeomorphism.

The group $G$ in the definition of the principal bundle is also called the *structure group* of the principal bundle. In the physics literature it is customary to call the structure group the *gauge group* of the principal bundle. We will follow the physics terminology and warn the reader that in the literature, the term 'gauge group' is also used for the group of gauge transformations we are about to define.

**3.53 Remark** Note that for a principal $G$-bundle the fibre $F$ is diffeomorphic to the gauge group $G$, but since $F$ lacks a preferred choice of unit element, there is no preferred group structure on the fibre $F$ (one also says that $F$ is a *G-torsor*[6]).

**3.54 Example** If $M$ is a smooth manifold and $G$ a Lie group, then the trivial bundle $\mathrm{pr}_M : M \times G \to M$ is a principal bundle.

While on first sight nothing interesting is happening here, trivial principal bundles appear in interesting physics applications. As an example, we would like to mention Maxwell's equations from electromagnetics. Fix a contractible $U \Subset \mathbb{R}^3$. The electrical field $E$, the magnetic field $H$ and the current $J$ in $U$ can be described by time-dependent vector fields on $U$. Denote by $\rho$ the electrical charge and by $c$ the speed of light. Then Maxwell's equations can be expressed in differential operator notation as

[6] See https://math.ucr.edu/home/baez/torsors.html for more examples and explanations.

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$$\operatorname{curl} E = -\frac{1}{c}\frac{\partial H}{\partial t}, \quad \operatorname{div} H = 0, \quad \operatorname{curl} H = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c} J, \quad \operatorname{div} E = 4\pi\rho.$$

Now one can show that the Maxwell equations can be interpreted as differential equations for the curvature of a $\mathbb{S}^1$-connection on the trivial $\mathbb{S}^1$-bundle $(U\times\mathbb{R})\times\mathbb{S}^1 \to U\times\mathbb{R}$. We refer to Baum (2014, Section 7.1) for the derivation and more information.

Interpreting the Maxwell equations in the language of principal bundles turns out to be a fruitful idea. Replacing the trivial bundle by a bundle with a (non-abelian) structure group, one arrives at gauge theories such as the Yang–Mills theory or the Chern–Simons theory.

**3.55 Example** Let $K$ be a compact Lie group and $G$ a closed Lie subgroup of $K$; then one can show that the quotient $M := K/G$ is again a manifold such that the canonical mapping $q\colon K \to K/G = M$ becomes a submersion (this is a so-called *homogeneous space*). Then $(K,q,M,G)$ is a principal $G$-bundle (Bröcker and Tom Dieck, 1995, Chapter I, Theorem 4.3).

As a more concrete example, consider the compact Lie group $\mathrm{SO}_3(\mathbb{R})$ (see Example 3.3(b)). Recall from Duistermaat and Kolk (2000, 1.2.A) that $\mathrm{SO}_3(\mathbb{R})$ can be identified with the 3-dimensional unit sphere $\mathbb{S}^3$ in $\mathbb{R}^4$. Moreover, the spherical group $\mathbb{S}^1$ embeds into $\mathrm{SO}_3(\mathbb{R})$ as the rotation subgroup $\left\{\begin{bmatrix}\cos(\alpha) & -\sin(\alpha) & 0\\ \sin(\alpha) & \cos(\alpha) & 0\\ 0 & 0 & 1\end{bmatrix}, \alpha \in \mathbb{R}\right\}$. This identification allows us to interpret multiplication with $\mathbb{S}^1$-elements as rotations on $\mathbb{S}^3$ and it is not hard to see that the quotient, $\mathbb{S}^3/\mathbb{S}^1 = \mathrm{SO}_3(\mathbb{R})/\mathbb{S}^1$ is diffeomorphic to the 2-dimensional sphere $\mathbb{S}^2$. The resulting homogeneous space

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \xrightarrow{\;q\;} \mathbb{S}^2$$

is known in differential topology as the Hopf fibration; see Sharpe (1997, Example 3.14). Note that the Hopf fibration is not isomorphic (as an $\mathbb{S}^1$-principal bundle) to the trivial $\mathbb{S}^1$-bundle over $\mathbb{S}^2$ (Duistermaat and Kolk, 2000, p. 56).

**3.56 Definition** Let $(E,p,M,F)$ be a principal $G$-bundle. Then define the *group of gauge transformations* of the bundle as follows:

$$\mathrm{Gau}(E) := \{\varphi \in \mathrm{Diff}(E) \mid p\circ\varphi = p, \text{ for all } g \in G,\ \varphi\circ\rho(\cdot,g) = \rho(\cdot,g)\circ\varphi\}.$$

**3.57** If $\varphi$ is a gauge transformation we can identify $\varphi(e) = \rho(e,f(e))$ for some smooth function $f\colon E \to G$, and the relation $\varphi\circ\rho(\cdot,g) = \rho(\cdot,g)\circ\varphi$ then yields

$$f(\rho(e,g)) = g^{-1}f(e)g, \quad \text{for all } e \in E,\ g \in G. \tag{3.11}$$

Conversely, every smooth function $f\colon E \to G$ satisfying (3.11) defines a gauge transformation via $\varphi_f(e) := \rho(e, f(e))$. We will show in Exercise 3.4.7 that for

$$C^\infty(E,G)^G := \{f \in C^\infty(E,G) \mid \text{for all } e \in E,\ g \in G, f(\rho(e,g)) = g^{-1} f(e) g\},$$

the map

$$C^\infty(E,G)^G \to \mathrm{Gau}(E), \quad f \mapsto \varphi_f, \quad \varphi_f(e) = \rho(e, f(e)) \tag{3.12}$$

is a group isomorphism. Hence the group of gauge transformations can naturally be identified as a subgroup of the current group.

**3.58 Example** If $\mathrm{pr}_M\colon M \times G \to M$ is a trivial principal $G$-bundle, it is easy to see that $\mathrm{Gau}(M \times G) \cong C^\infty(M,G)$. So if $M$ is compact, the group of gauge transformations inherits a Lie group structure from the current group in this case.

For the trivial $\mathbb{S}^1$-principal bundle connected to Maxwell's equations, Example 3.54, the group of gauge transformations (aka the current group in this case) acts on $\mathbb{S}^1$-connections on the trivial bundle (connections in the same orbit are called gauge equivalent). As already mentioned, the Maxwell equations can be interpreted as a differential equation for the curvature of a $\mathbb{S}^1$-connection. See Baum (2014, Section 7.1).

**3.59 Remark** If $G$ is a locally exponential Lie group and $M$ a compact manifold, then the group of gauge transformations $\mathrm{Gau}(E)$ of a smooth principal $G$-bundle $(E, p, M, F)$ carries a natural Lie group structure, turning it into a locally exponential Lie group (Wockel, 2006). Indeed one can show that the group of gauge transformations then becomes a Lie subgroup of a finite product of suitable current groups (Wockel, 2007).

Alternatively, the Lie group structure of the group of gauge transformations can be derived by identifying the group of gauge transformations with the vertical bisections of a gauge groupoid; see Remark 6.17 and Example 6.13.

## Exercises

3.4.1 We establish some auxiliary results for the proof of Proposition 3.45.

(a) Let $G$ be a group which is also a manifold. Show that $G$ is a Lie group if and only if the mapping $\delta\colon G \times G \to G$, $(g,h) \mapsto gh^{-1}$ is smooth.

(b) Let $G$ be a Lie group (actually a topological group would be enough for the following) and $V \Subset G$ with $\mathbf{1} \in V$. Show that there is $\mathbf{1} \in W \Subset V$ with $W^{-1} = W$ and $W \cdot W \subseteq V$. (Can you generalise this to $W^n \subseteq V, n \in \mathbb{N}$?)

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(c) Let $G, H$ be Lie groups and $f\colon G \to H$ be a group homomorphism. Prove that $f$ is a Lie group morphism if and only if there exists an open $\mathbf{1}_G$-neighbourhood $U$ such that $f|_U$ is smooth.

(d) Let $G$ be a group with two manifold structures $\mathcal{S}$ and $\mathcal{T}$ turning $G$ into a Lie group. Show that $(G, \mathcal{S})$ and $(G, \mathcal{T})$ are diffeomorphic (as manifolds and then also as Lie groups) if and only if there is an open $\mathbf{1}$-neighbourhood on which the two manifold structures coincide.

3.4.2 Verify several details from the proof of Theorem 3.47:

(a) Let $K \subseteq G$ be a compact subset of a Lie group $G$ and $\mathbf{1} \in U \Subset G$. Show that there is an open $\mathbf{1}$-neighbourhood $W$ and an open $K$-neighbourhood $P$ such that $pWp^{-1} \subseteq U$ for all $p \in P$.

(b) Verify that the final topology, with respect to the charts $(\varphi_g)_{g\in G}$ from Step 3 of the proof, is Hausdorff.

(c) Show that the point evaluations $\mathrm{ev}_x\colon C^\infty(K, G) \to G, \gamma \mapsto \gamma(x)$ are Lie group morphisms.

*Hint:* For group homomorphisms it suffices to check smoothness in an identity neighbourhood.

3.4.3 Prove that the current group $C^\infty(K, G)$ is a canonical manifold.
*Hint:* For a Lie group, it suffices to construct an open identity neighbourhood whose manifold structure is canonical.

3.4.4 By C.2 every Lie group $G$ admits a local addition. Show that the manifold structure on $C^\infty(K, G)$ constructed in C.10 coincides with the one from Theorem 3.47.

3.4.5 Show that $C^\infty(\mathbb{N}, \mathbb{S}^1) = \prod_{n\in\mathbb{N}} \mathbb{S}^1$ is not locally contractible, that is, show that if $N$ is a neighbourhood of a point $x \in \prod_{n\in\mathbb{N}} \mathbb{S}^1$ then there is no continuous mapping $h\colon [0,1] \times N \to N$ such that $h(0, \cdot) = \mathrm{id}_N$ and $h(0, y) = x$, for all $y \in N$.

3.4.6 Let $G$ be a locally exponential Lie group. Show that then also $C^\infty(K, G)$ is locally exponential.
*Hint:* Study Proposition 3.49 and note, in addition, that $\exp_G$ induces a chart for $G$.

3.4.7 Show that the map $C^\infty(E, G)^G \to \mathrm{Gau}(E)$, $f \mapsto \varphi_f$ from 3.57 is a group isomorphism.