STRONG REGULARITY IN ARBITRARY RINGS

TETSUO KANDÔ

An element a of a ring R is called regular, if there exists an element x of R such that $axa = a$, and a two-sided ideal $\mathfrak{a}$ in R is said to be regular if each of its elements is regular. B. Brown and N. H. McCoy [1] has recently proved that every ring R has a unique maximal regular two-sided ideal $M(R)$, and that $M(R)$ has the following radical-like property: (i) $M(R/M(R)) = 0$; (ii) if $\mathfrak{a}$ is a two-sided ideal of R, then $M(\mathfrak{a}) = \mathfrak{a} \cap M(R)$; (iii) $M(R_n) = (M(R))_n$, where R_n denotes a full matrix ring of order n over R. Arens and Kaplansky [2] has defined an element a of R to be strongly regular when there exists an element x of R such that $a^2x = a$. We shall prove in this note that replacing "regularity" by "strong regularity," we have also a unique maximal strongly regular ideal $N(R)$, and shall investigate some of its properties.

1. Existence and properties of $N(R)$.

The existence proof of $N(R)$ can be accomplished along the line of Brown and McCoy [1].

Definition 1. An element a of a ring R is called *strongly regular*, if and only if there exists an element x of R such that $a^2x = a$. A (two-sided) ideal $\mathfrak{a}$ in R is called a *strongly regular ideal*, if each of its element is strongly regular. Finally, we call an element $a \in R$ *properly strongly regular*, if the principal ideal (a) generated by a is strongly regular.

LEMMA 1. *If $a^2y - a$ is strongly regular, so is a too.*

Proof. By virtue of strong regularity of $a^2y - a$, there exists an element z such that $(a^2y - a)^2z = a^2y - a$. Setting $x = y - z + ayz + yaz - ya^2yz$, we have readily $a^2x = a$.

LEMMA 2. *The set $N(R)$ of all properly strongly regular elements of R is a strongly regular ideal.*

Proof. That $z \in N(R)$ and $t \in R$ implies $(zt) \subset N(R)$ whence $zt \in N(R)$; similarly, $tz \in N(R)$. On the other hand, let z_1 and z_2 be any elements of $N(R)$ and let $a \in (z_1 - z_2)$. Then we have $a = u_1 - u_2$, where $u_i \in (z_i)$. By strong regularity of (z_1), we have $u_1^2 r = u_1$ for some $r \in R$. Then $a^2r - a = (u_1 - u_2)^2 r - (u_1 - u_2) = u_2 + u_2^2 r - u_1u_2r - u_2u_1r \in (u_2) \subset (z_2)$, and $a^2r - a$ is strongly regular. Then, Lemma 1 implies that a is strongly regular, and the proof is complete.

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From Lemma 2, we have immediately

THEOREM 1. *Every ring R has a unique maximal strongly regular ideal $N(R)$.*

THEOREM 2. *For every ring R we have $N(R/N(R))=0$.*

Proof. Let us denote by $\bar{a}$ the residue class mod $N(R)$ containing a. Suppose that $\bar{b}\in N(R/N(R))$ and a is any element of (b). Then $\bar{a}$ is as an element of $(\bar{b})$ strongly regular in the ring $R/N(R)$: $\bar{a}^2\bar{x}=\bar{a}$, that is $a^2x-a\in N(R)$, and hence a^2x-a is strongly regular. It follows therefore from Lemma 1 that a is strongly regular. Since we have proved that every element of the ideal (b) is strongly regular, we have $b\in N(R)$, i.e. $\bar{b}=0$.

LEMMA 3. *Let $\mathfrak{a}$ be a two-sided ideal of R. Then, an element a of $\mathfrak{a}$ is properly strongly regular in the ring $\mathfrak{a}$ if and only if it is strongly regular in the ring R.*

Proof. Let a be properly strongly regular in $\mathfrak{a}$, and let b be any element of the ideal (a) generated by a in R. Then, we have $b=na+ua+av+\sum u_iav_i$, where n is an integer and u's and v's are elements of R. Since a is strongly regular, there exists an element $x\in\mathfrak{a}$ such that $a^2x=a$. Consequently, $b=na+(ua)ax+a(axv)+\sum(u_ia)a(xv_i)$, $ua, axv, u_ia, xv_i\in\mathfrak{a}$. Hence we have $b\in(a)'$, where $(a)'$ denotes an ideal generated by a in $\mathfrak{a}$. Therefore, b is strongly regular, and the element a is properly strongly regular in R. The converse part is clear.

From Lemma 3, we have immediately

THEOREM 3. *If $\mathfrak{a}$ is a two-sided ideal in R, then $N(\mathfrak{a})=\mathfrak{a}\cap N(R)$.*

2. Some relations between $N(R)$ and $M(R)$.

Let us consider some properties of elements in $N(R)$.

LEMMA 4. *$N(R)$ has no non-zero nilpotent element.*

Proof. Let $a\in N(R)$, and $a^n=0$. Then $a^2x=a$, and so $a=a^2x=\dots=a^nx^{n-1}=0$.

LEMMA 5. *Let $a\in N(R)$ and x be an element in R such that $a^2x=a$. Then, (i) $a^2x=axa=xa^2=a$, and a is regular. (ii) $ax=xa$, and ax is an idempotent. (iii) $e=ax$ belongs to the center of R.*

Proof. From $a^2x=a$, we have easily $(a-axa)^2=0$. Since $a-axa\in N(R)$. Lemma 4 implies $a=axa$, and similarly $axa=xa^2$, so we have (i). From $ax=(xa^2)x=x(a^2x)=xa$ we have (ii). As for (iii), let u be any element of R. By analogous argument as (i), we have $ue=eue$, $ue=eue$, and therefore $ue=eu$.

The above lemma shows that each element of $N(R)$ is regular, so we have

THEOREM 4. $N(R)\subset M(R)$.

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While $M(R)$ satisfies $M(R_n)=(M(R))_n$ (cf. [1]), $N(R)$ does not possess this property, which is shown by the following theorem:

THEOREM 5. *Let R_n be the full matrix ring of order $n>1$ over R. Then,* $N(R_n)=0$.

Proof. First, let us suppose that R has a unit element. Let $A\in N(R_n)$. Then there exists a matrix X such that $A^2X=A$, and AX belongs to the center of R_n. Hence, $AX=aE$, where a is an element belonging to the center of R, and E is the unit matrix of R_n. So, we have $A=aA$, and

$$B=\begin{pmatrix} 0 & a & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}=aE\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}\in (aE)\subset (A)\subset N(R_n).$$

Therefore B is strongly regular: $B=B^2Y$. But since $B^2=0$, we have $B=0$, $a=0$, and $A=0$.

When R does not possess a unit element, we can obtain a ring $\hat{R}$ in the usual way by adjoining a unit element to R. Then R_n *is* an ideal of $\hat{R}_n$, and $N(R_n)=R_n\cap N(\hat{R}_n)=0$.

The above theorem shows that there exists a ring R such that $M(R)\subsetneqq N(R)$.

THEOREM 6. *$M(R)=N(R)$ if and only if $M(R)$ has no non-zero nilpotent element.*

Proof. Suppose that $M(R)$ has no non-zero nilpotent element. Then, since for every $a\in M(R)$ there is an x such that $a=axa$ whence $(a-a^2x)^2=0$, we have $a=a^2x$, $a\in N(R)$. This means $M(R)=N(R)$. The converse follows from Lemma 4.

COROLLARY. *If R is either commutative or has no non-zero nilpotent element, then $M(R)=N(R)$.*

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Mathematical Institute,
Nagoya University