An Integral Formula for $Q_n(\cos\theta)$

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§ 1. *Introduction.*

The function

$$\int_{-\pi}^{\pi} \frac{f(u)\, du}{z + ix\cos u + iy\sin u} \tag{1.1}$$

is, as is well-known, a general solution of Laplace's equation of degree -1 in (x, y, z). In 1926* I proved that the particular solution $r^{-1}Q_0(z/r)$ cannot be represented in this form whereas the solution $r^{-1}Q_0(y/r)$ can. In the present note I find a very simple expression for the latter solution in the form (1.1), and I deduce from it an apparently new integral formula for $Q_n(\cos\theta)$.

§ 2. *The formula for* $r^{-1}Q_0(y/r)$.

Our first result is contained in

Theorem 1. *If* $z \neq 0$,

$$\frac{1}{r} Q_0\left(\frac{y}{r}\right) = \frac{i}{4}\int_{-\pi}^{\pi} \frac{f(u)\, du}{z + ix\cos u + iy\sin u}$$

where $f(u) = +1$ *when* $u > 0$, $f(u) = -1$ *when* $u < 0$.

For, if we put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, we have

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{f(u)\, du}{z + ix\cos u + iy\sin u}
&= \frac{1}{r}\left\{\int_0^{\pi} - \int_{-\pi}^{0} \frac{du}{\cos\theta + i\sin\theta\cos(\phi - u)}\right\} \\
&= \frac{1}{r}\int_{-\phi}^{\pi-\phi}\left\{\frac{1}{\cos\theta + i\sin\theta\cos v} - \frac{1}{\cos\theta - i\sin\theta\cos v}\right\} dv \\
&= -\frac{2i}{r}\int_{-\phi}^{\pi-\phi} \frac{\sin\theta\cos v}{1 - \sin^2\theta\sin^2 v}\, dv \\
&= -\frac{2i}{r}\log\frac{1 + \sin\theta\sin\phi}{1 - \sin\theta\sin\phi} = \frac{4}{ir} Q_0\left(\frac{y}{r}\right).
\end{aligned}$$

I owe this simple proof of Theorem 1 to a referee.

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* *Proc. Edin. Math. Soc.* (1), **44** (1926), 22-25. A slight correction is needed in that paper. I quoted an asymptotic formula for $Q_0^m(\cos\alpha)$, due to Watson; but as in that paper m is a positive integer, the expression given is the actual value of $Q_0^m(\cos\alpha)$ without the order term.

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§ 3. *An integral formula for* Q_n (cos θ).

It follows from Theorem 1 that, when $x \neq 0$,

$$\frac{1}{2r} \log \frac{r+z}{r-z} = \frac{i}{4} \int_{-\pi}^{\pi} \frac{f(u)\, du}{x + iy \cos u + iz \sin u}.$$

But we also have*

$$Q_n (\cos \theta) = \frac{(-1)^n\, r^{n+1}}{n!} \frac{d^n}{dz^n} \left\{ \frac{1}{2r} \log \frac{r+z}{r-z} \right\}.$$

Combining these two results, we find that, when $\sin \theta \cos \phi \neq 0$,

$$Q_n (\cos \theta) = \frac{i^{n+1}}{4} \int_{-\pi}^{\pi} \frac{\sin^n u\, f(u)\, du}{(\sin \theta \cos \phi + i \sin \theta \sin \phi \cos u + i \cos \theta \sin u)^{n+1}}.$$

Putting $\phi = 0$, we obtain

Theorem 2. *If* $0 < \theta < \pi$,

$$Q_n (\cos \theta) = \frac{i^{n+1}}{4} \int_{-\pi}^{\pi} \frac{\sin^n u\, f(u)\, du}{(\sin \theta + i \cos \theta \sin u)^{n+1}}$$

where $f(u) = +1$ *when* $u > 0$, $= -1$ *when* $u < 0$.

This formula for Q_n (cos θ) can be deduced from the integral representation

$$Q_n (z) = \int_0^{\infty} \{z + (z^2 - 1)^{\frac{1}{2}} \cosh u\}^{-n-1}\, du$$

which is valid in the plane of the complex variable z, cut along the real axis from -1 to $+1$, and the definition

$$Q_n (\cos \theta) = \tfrac{1}{2} \{Q_n (\cos \theta + 0i) + Q_n (\cos \theta - 0i)\}.$$

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* Hobson, *Spherical and Ellipsoidal Harmonics* (Cambridge, 1931), 70.

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