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GENERALIZATIONS OF F. E. BROWDER'S SHARPENED FORM OF THE SCHAUDER FIXED POINT THEOREM

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Abstract

Let E be a Hausdorff topological vector space, let K be a nonempty compact convex subset of E and let f, g: $K \to 2^E$ be upper semicontinuous such that for each $x \in K$, $f(x)$ and $g(x)$ are nonempty compact convex. Let $\Omega \subset 2^E$ be convex and contain all sets of the form $x - f(x)$, $y - x + g(x) - f(x)$, for $x, y \in K$. Suppose p: $K \times \Omega \to \mathbf{R}$ satisfies: (i) for each $(x, A) \in K \times \Omega$ and for $\varepsilon > 0$, there exist a neighborhood U of x in K and an open subset set G in E with $A \subset G$ such that for all $(y, B) \in K \times \Omega$ with $y \in U$ and $B \subset G$, $|p(y, B) - p(x, A)| < \varepsilon$, and (ii) for each fixed $x \in K$, $p(x, \cdot)$ is a convex function on Ω. If $p(x, x - f(x)) \leqslant p(x, g(x) - f(x))$ for all $x \in K$, and if, for each $x \in K$ with $f(x) \cap g(x) = \varnothing$, there exists $y \in K$ with $p(x, y - x + g(x) - f(x)) < p(x, x - f(x))$, then there exists an $x_0 \in K$ such that $f(x_0) \cap g(x_0) \neq \varnothing$. Another coincidence theorem on a nonempty compact convex subset of a Hausdorff locally convex topological vector space is also given.

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1. Introduction and preliminaries

The classical Schauder fixed point theorem asserts that every continuous self-map of a nonempty compact convex subset of a Banach space has a fixed point. Obviously the Schauder fixed point theorem cannot be extended to non-self-maps without additional conditions. Many generalizations for single- or multi-valued

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maps have been obtained, for example see [2], [3], [5], [7] and [8]. Recently, F. E. Browder [4] gave a rather sharp improvement of these results for single-valued maps. Generalizations of those results in [4] to set-valued maps are obtained by S. Reich [12, 13], J. H. Jiang [9, 10] and others. In this paper, we shall extend some of Browder's results in [4] to set-valued maps in different directions, one of which extends a result of S. Reich in [12].

We shall denote by $\mathbf{R}$ the real line and, for any nonempty set X, by 2^X the collection of all nonempty subsets of X. Now let X and Y be topological spaces. Then a map $f\colon X \to 2^Y$ is said to be (i) *lower semicontinuous* (respectively, *upper semicontinuous*) [1] at $x_0 \in X$ if for each open set G in Y with $G \cap f(x_0) \neq \varnothing$ (respectively, with $f(x_0) \subset G$), there is a neighborhood U of x_0 in X such that $G \cap f(x) \neq \varnothing$ (respectively, $f(x) \subset G$) for all $x \in U$; (ii) *lower semicontinuous* (respectively, *upper semicontinuous*) *on* X if f is lower semicontinuous (respectively, upper semicontinuous) at each point of X; (iii) *continuous on* X if f is both lower semicontinuous on X and upper semicontinuous on X. Also if $\Omega \subset 2^Y$, then a map $p\colon X \times \Omega \to \mathbf{R}$ is said to be (iv) *ultimately continuous* at (x, A) if for each $\varepsilon > 0$, there exist a neighborhood U of x in X and an open set G in Y with $A \subset G$ such that $|p(y, B) - p(x, A)| < \varepsilon$ for all $(y, B) \in X \times \Omega$ with $y \in U$ and $B \subset G$; (v) *ultimately continuous on* $X \times \Omega$ if p is ultimately continuous at each point of $X \times \Omega$. We note that in the case $\Omega = \{\{y\}\colon y \in Y\}$, if we write $p(x, y) = p(x, \{y\})$, then the notions of ultimate continuity and continuity coincide. If $A \subset X$, $\mathrm{cl}(A)$ denotes the closure of A in X. Next let E be a vector space, let K be a nonempty subset of E and let $x \in K$; then the *inward set* and *outward set* [8] of K at x, denoted by $I_K(x)$ and $O_K(x)$, respectively, are defined by

$$I_K(x) = \{y \in E\colon \text{there exist } u \in K \text{ and } r > 0 \text{ such that } y = x + r(u - x)\}$$

and

$$O_K(x) = \{y \in E\colon \text{there exist } u \in K \text{ and } r > 0 \text{ such that } y = x - r(u - x)\}.$$

Also, a subset Ω of 2^E is *convex* if for each A, $B \in \Omega$ and for each $t \in [0,1]$, $tA + (1 - t)B \in \Omega$. Moreover, if E is a topological vector space, we shall denote by $\mathscr{K}(E)$ the collection of all compact convex sets in 2^E and by $\mathscr{C}(E)$ the collection of all closed convex sets in 2^E. Finally we shall need the following fixed point theorem of K. Fan [6]:

THEOREM (K. Fan [6]). *Let K be a nonempty compact convex subset of a Hausdorff topological vector space E and let $S\colon K \to 2^K$. Suppose, for each $x \in K$, that $S(x)$ is convex, while for each $u \in K$, the set $S^{-1}(u) = \{y \in K\colon u \in S(y)\}$ is open in K. Then there exists $x_0 \in K$ such that $x_0 \in S(x_0)$.*

2. Main results

The following two propositions are easy consequences of the definitions.

PROPOSITION 2.1. *Let E be a topological vector space, let $K \subset E$ be nonempty, let f, g: $K \to 2^E$ be lower semicontinuous, let h: $K \to 2^E$ be upper semicontinuous and let $c \in \mathbb{R}$. Then $f + g$ and cg are lower semicontinuous, and ch is upper semicontinuous.*

PROPOSITION 2.2. *Let E be a topological vector space, let $K \subset E$ be nonempty and let f, g: $K \to 2^E$ be upper semicontinuous such that for each $x \in K$, $f(x)$ and $g(x)$ are both compact. Then $f + g$ is also upper semicontinuous.*

We note that Proposition 2.2 is false if the condition "f, g: $K \to 2^E$ be upper semicontinuous such that for each $x \in K$, $f(x)$ and $g(x)$ are both compact" is replaced by the condition "f, g: $K \to \mathscr{C}(E)$ be upper semicontinuous such that for each $x \in K$, at least one of $f(x)$ and $g(x)$ is compact." This can be seen from the following:

EXAMPLE 2.3. Let $E = \mathbb{R}^2$ and let $K = \{(x, y) \in \mathbb{R}^2\colon x^2 + y^2 \leqslant 1$ and $x, y > 0\}$. Define f: $K \to \mathscr{K}(E)$ by

$$f(r\cos\theta, r\sin\theta) = \{(t\cos\theta, t\sin\theta)\colon r \leqslant t \leqslant 2\}$$

for each $r \in (0,1]$ and $\theta \in (0, \pi/2)$. Define g: $K \to \mathscr{C}(E)$ by

$$g(x, y) = \{(z, 0)\colon z \geqslant x\}$$

for all $(x, y) \in K$. It can be easily checked that f and g are both upper semicontinuous (in fact, both continuous) but $f + g$ is not upper semicontinuous.

The following result generalizes Proposition 2 in [4] and also Theorem 1 in [7] to set-valued maps.

THEOREM 2.4. *Let E be a Hausdorff topological vector space, let $K \subset E$ be nonempty compact convex and let f, g: $K \to \mathscr{K}(E)$ be upper semicontinuous. Let $\Omega \subset 2^E$ be convex and contain all sets of the form $x - f(x)$, $y - x + g(x) - f(x)$, for x, $y \in K$. Suppose p: $K \times \Omega \to \mathbb{R}$ is ultimately continuous such that for each $x \in K$, $p(x, \cdot)$ is a convex function on Ω. Assume that*

(i) *$p(x, x - f(x)) \leqslant p(x, g(x) - f(x))$ for all $x \in K$, and*

(ii) *for each $x \in K$ with $f(x) \cap g(x) = \varnothing$, there exists $y \in K$ such that $p(x, y - x + g(x) - f(x)) < p(x, x - f(x))$.*

Then there exists an $x_0 \in K$ such that $f(x_0) \cap g(x_0) \neq \varnothing$.

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PROOF. Define $h\colon K \to \mathscr{K}(E)$ by $h(x) = x + f(x) - g(x)$ for all $x \in K$. Then h is upper semicontinuous by Propositions 2.1 and 2.2. Assume that for each $x \in K$, $f(x) \cap g(x) = \varnothing$, so that the set $S(x) = \{y \in K\colon p(x, y - h(x)) < p(x, x - f(x))\}$ is nonempty by hypothesis. Thus $S\colon K \to 2^K$. Let $x \in K$, y_1, $y_2 \in S(x)$ and $t \in [0,1]$; then $p(x, y_i - h(x)) < p(x, x - f(x))$ for $i = 1, 2$. Since $t(y_1 - h(x)) + (1 - t)(y_2 - h(x)) = ty_1 + (1 - t)y_2 - h(x)$, and since $p(x, \cdot)$ is convex, we see that

$$p(x, ty_1 + (1 - t)y_2 - h(x)) < p(x, x - f(x)),$$

so that $ty_1 + (1 - t)y_2 \in S(x)$. Hence $S(x)$ is convex for each $x \in K$.

Now let $u \in K$. We shall show that $S^{-1}(u)$ is open in K. Indeed, if $x \in S^{-1}(u)$, then $u \in S(x)$, so that $p(x, u - h(x)) < p(x, x - f(x))$. Let $\varepsilon = [p(x, x - f(x)) - p(x, u - h(x))]/2$. Since p is ultimately continuous at $(x, x - f(x))$, there exist an open neighborhood U_1 of x in K and an open set G in E with $x - f(x) \subset G$ such that $|p(y, A) - p(x, x - f(x))| < \varepsilon$ for all $(y, A) \in K \times \Omega$ with $y \in U_1$ and $A \subset G$. For each $a \in x - f(x)$, let N_a be an open neighborhood of 0 in E such that $a + N_a + N_a \subset G$. Since $x - f(x)$ is compact, there exist $a_1, \ldots, a_n \in x - f(x)$ such that $x - f(x) \subset \bigcup_{i=1}^n (a_i + N_{a_i})$. Since f is upper semicontinuous at x, and since $f(x) \subset x - \bigcup_{i=1}^n (a_i + N_{a_i})$, which is open, there exists an open neighborhood U_2 of x in K such that $f(y) \subset x - \bigcup_{i=1}^n (a_i + N_{a_i})$ for all $y \in U_2$. Let $V_1 = U_1 \cap U_2 \cap (x + \bigcap_{i=1}^n N_{a_i})$. Then V_1 is an open neighborhood of x in K. Let $y \in V_1$; as $y \in U_2$, we have $f(y) \subset x - \bigcup_{i=1}^n (a_i + N_{a_i})$, so that

$$(*) \qquad x - f(y) \subset \bigcup_{i=1}^{n} (a_i + N_{a_i});$$

as $y \in x + \bigcap_{i=1}^n N_{a_i}$, we have $y - x \in \bigcap_{i=1}^n N_{a_i}$, so that $y - f(y) = y - x + x - f(y) \subset \bigcap_{i=1}^n N_{a_i} + \bigcup_{i=1}^n (a_i + N_{a_i})$ by $(*)$. It follows that

$$(**) \qquad y - f(y) \subset \bigcup_{i=1}^{n} (a_i + N_{a_i} + N_{a_i}) \subset G;$$

as $y \in U_1$, by $(**)$, we have

$$(\dagger) \qquad |p(y, y - f(y)) - p(x, x - f(x))| < \varepsilon.$$

Next, since p is also ultimately continuous at $(x, u - h(x))$, there exist an open neighborhood U_3 of x in K and an open set G' in E with $u - h(x) \subset G'$ such that $|p(y, A) - p(x, u - h(x))| < \varepsilon$ for all $(y, A) \in K \times \Omega$ with $y \in U_3$ and $A \subset G'$. Since $h(x) \subset u - G'$, which is open, and since h is upper semicontinuous at x, there exists an open neighborhood U_4 of x in K such that $h(y) \subset u - G'$ for all $y \in U_4$. Let $V_2 = U_3 \cap U_4$. Then V_2 is an open neighborhood of x in K. Let $y \in V_2$; as $y \in U_4$, we have $h(y) \subset u - G'$, so that

$$(***) \qquad u - h(y) \subset G';$$

as $y \in U_3$, by (***), we have

$$(\dagger\dagger) \qquad |p(y, u - h(y)) - p(x, u - h(x))| < \varepsilon.$$

Let $V = V_1 \cap V_2$. Then V is an open neighborhood of x in K such that for each $y \in V$, (†) and (††) hold; it follows that

$$\begin{aligned} p(y, u - h(y)) &< p(x, u - h(x)) + \varepsilon \qquad (\text{by } (\dagger\dagger)) \\ &= p(x, x - f(x)) - \varepsilon \\ &< p(y, y - f(y)) \qquad (\text{by } (\dagger)) \end{aligned}$$

so that $u \in S(y)$ and hence $y \in S^{-1}(u)$ for all $y \in V$. Therefore $S^{-1}(u)$ is open for each $u \in K$.

By K. Fan's Theorem, there exists an $x_0 \in K$ such that $x_0 \in S(x_0)$); thus we have

$$p(x_0, g(x_0) - f(x_0)) = p(x_0, x_0 - h(x_0)) < p(x_0, x_0 - f(x_0)),$$

which contradicts (i). This shows that there must exist an $x_0 \in K$ such that $f(x_0) \cap g(x_0) \neq \varnothing$. This completes the proof.

By applying Theorem 2.4 and an argument similar to that used in proving Theorem 1 in [4], we obtain the following generalization of Theorem 1 in [4].

COROLLARY 2.5. *Let E be a Hausdorff topological vector space, let $K \subset E$ be nonempty compact convex and let f, g: $K \to \mathscr{K}(E)$ be upper semicontinuous. Let $\Omega \subset 2^E$ be convex and contain all sets of the form $x - f(x)$, $y - x + g(x) - f(x)$, for x, $y \in K$. Suppose p: $K \times \Omega \to \mathbb{R}$ is ultimately continuous on $K \times \Omega$. Assume that*

(i) *$p(x, x - f(x)) = p(x, g(x) - f(x))$ for all $x \in K$, and*

(ii) *for each $x \in K$ with $f(x) \cap g(x) = \varnothing$, there exists $y \in I_K(x)$ such that $p(x, y - x + g(x) - f(x)) < p(x, x - f(x))$.*

Then there exists an $x_0 \in K$ such that $f(x_0) \cap g(x_0) \neq \varnothing$.

By applying Corollary 2.5 and an argument similar to that used in proving Theorem 2 in [4], we obtain the following generalization of Theorem 2 in [4].

COROLLARY 2.6. *Let E be a Hausdorff topological vector space, let $K \subset E$ be nonempty comapct convex and let f, g: $K \to \mathscr{K}(E)$ be upper semi-continuous. Let $\Omega \subset 2^E$ be convex and contain all sets of the form $x - f(x)$, $y - x + g(x) - f(x)$, for x, $y \in K$. Suppose p: $K \times \Omega \to \mathbb{R}$ is ultimately continuous on $K \times \Omega$. Assume that*

(i) *$p(x, x - f(x)) = p(x, g(x) - f(x))$ for all $x \in K$, and*

(ii) *for each $x \in K$ with $f(x) \cap g(x) = \varnothing$, there exist $y \in O_K(x)$ such that $p(x, y - x + g(x) - f(x)) < p(x, x - f(x))$.*

Then there exists an $x_0 \in K$ such that $f(x_0) \cap g(x_0) \neq \varnothing$.

Let E be a locally convex topological vector space and let p be any continuous seminorm on E. If A, $B \subset E$ are nonempty, let $d_p(A, B) = \inf\{p(a - b)$: $a \in A$ and $b \in B\}$; if $A = \{a\}$, we shall write $d_p(a, B)$ instead of $d_p(\{a\}, B)$. The following result is motivated by the proof of Theorem 3.1 in [11].

LEMMA 2.7. *Let E be a Hausdorff locally convex topological vector space, let $K \subset E$ be nonempty comapct convex and let f, g: $K \to \mathscr{C}(E)$ be upper semicontinuous such that for each $x \in K$, either $f(x)$ or $g(x)$ is compact. Assume that for each continuous seminorm p on E, there exists an $x \in K$ such that $d_p(f(x), g(x)) = 0$. Then there exists an $x_0 \in K$ such that $f(x_0) \cap g(x_0) \neq \varnothing$.*

PROOF. Let $\mathscr{P}$ be the set of all continuous seminorms on E. For each $p \in \mathscr{P}$, let $K_p = \{x \in K$: $d_p(f(x), g(x)) = 0\}$. If $p \in \mathscr{P}$ is arbitrarily fixed, then K_p is nonempty by hypothesis; we shall show that K_p is also closed in K. Indeed, let $(x_\alpha)_{\alpha \in \Gamma}$ be a net in K_p such that $x_\alpha \to x$ for some $x \in K$. Suppose $r = d_p(f(x), g(x)) > 0$. Let $V_f = \{z \in E$: $d_p(z, f(x)) < r/3\}$ and $V_g = \{z \in E$: $d_p(z, g(x)) < r/3\}$. Then V_f and V_g are open in E, and $f(x) \subset V_f$ and $g(x) \subset V_g$. Since f and g are upper semicontinuous at x, there exists a neighborhood U of x in K such that for all $y \in U$, $f(y) \subset V_f$ and $g(y) \subset V_g$. Since $x_\alpha \to x$, there exists $\alpha_0 \in \Gamma$ such that $x_\alpha \in U$ for all $\alpha \geqslant \alpha_0$; it follows that, in particular, $f(x_{\alpha_0}) \subset V_f$ and $g(x_{\alpha_0}) \subset V_g$, so that $d_p(f(x_{\alpha_0}), g(x_{\alpha_0})) \geqslant r/3$, which contradicts our assumption that $d_p(f(x_{\alpha_0}), g(x_{\alpha_0})) = 0$. Thus $d_p(f(x), g(x)) = 0$, whence $x \in K_p$. Therefore K_p is closed in K for each $p \in \mathscr{P}$. Now let $\{p_1, \ldots, p_n\}$ be any finite subset of $\mathscr{P}$. Let $p = \sum_{i=1}^n p_i$. Then $p \in \mathscr{P}$, and $\bigcap_{i=1}^n K_{p_i} \supset K_p \neq \varnothing$. Thus the family $\{K_p : p \in \mathscr{P}\}$ has the finite intersection property, whence, by compactness of K, $\bigcap_{p \in \mathscr{P}} K_p \neq \varnothing$. It follows that there exists an $x_0 \in K$ such that $d_p(f(x_0), g(x_0)) = 0$ for all $p \in \mathscr{P}$. By the Hahn-Banach separation theorem, $f(x_0) \cap g(x_0) \neq \varnothing$. This completes the proof.

The following result generalizes part of Theorem 3 in [12]. We shall present a different proof than the one used in [12].

THEOREM 2.8. *Let E be a Hausdorff locally convex topological vector space, let $K \subset E$ be nonempty compact convex and let f, g: $K \to \mathscr{C}(E)$ be continuous such that for each $x \in K$, either $f(x)$ or $g(x)$ is compact. Suppose for each $x \in K$ and for each continuous seminorm p on E with $d_p(f(x), g(x)) > 0$, we have $d_p(K, x + f(x) - g(x)) < d_p(f(x), g(x))$. Then there exists an $x_0 \in K$ such that $f(x_0) \cap g(x_0) \neq \varnothing$.*

PROOF. Define $h: K \to \mathscr{C}(E)$ by $h(x) = x + f(x) - g(x)$ for all $x \in K$. Then h is lower semicontinuous on K by Proposition 2.1. Let $\mathscr{P}$ be the set of all continuous seminorms on E. By Lemma 2.7, it is sufficient to show that for each $p \in \mathscr{P}$, there exists an $x \in K$ such that $d_p(f(x), g(x)) = 0$. If not, then there exists a $p \in \mathscr{P}$ such that $d_p(f(x), g(x)) > 0$ for all $x \in K$, so that the set $S(x) = \{y \in K: d_p(y, h(x)) < d_p(f(x), g(x))\}$ is nonempty for all $x \in K$, by hypothesis. Thus $S: K \to 2^K$. Let $x \in K$. As $h(x)$ is convex, $d_p(\cdot, h(x))$ is a convex function on K, and hence $S(x)$ is convex. Let $u \in K$. We shall show that $S^{-1}(u)$ is open in K. Indeed, if $x \in S^{-1}(u)$, then $u \in S(x)$, so that $d_p(u, h(x)) < d_p(f(x), g(x))$. Let $\varepsilon = [d_p(f(x), g(x)) - d_p(u, h(x))]/4$. Choose $w_0 \in h(x)$ such that $p(u - w_0) < d_p(u, h(x)) + \varepsilon$. Let $G = \{z \in K: p(z - w_0) < \varepsilon\}$. Then G is open in K, and $G \cap h(x) \neq \varnothing$. Since h is lowe semicontinuous at x, there exists an open neighborhood V_1 of x in K such that $h(y) \cap G \neq \varnothing$ for all $y \in V_1$. Let $V_2 = V_1 \cap \{z \in K: p(z - x) < \varepsilon\}$. Then V_2 is an open neighborhood of x in K. Let $y \in V_2$. Then $h(y) \cap G \neq \varnothing$, and if we choose any $w \in h(y) \cap G$, we have

$$(*) \qquad \begin{aligned} d_p(u, h(y)) &\leqslant p(u - w) \leqslant p(u - w_0) + p(w_0 - w) \\ &< d_p(u, h(x)) + \varepsilon + \varepsilon = d_p(u, h(x)) + 2\varepsilon. \end{aligned}$$

Next, note that for $V_f = \{z \in K: d_p(z, f(x)) < \varepsilon/2\}$ and $V_g = \{z \in K; d_p(z, g(x)) < \varepsilon/2\}$, V_f and V_g are open in K, and they contain $f(x)$ and $g(x)$, respectively. Since f and g are upper semicontinuous at x, there exists an open neighborhood V_3 of x in K such that $f(y) \subset V_f$ and $g(y) \subset V_g$ for all $y \in V_3$. Let $y \in V_3$, and then choose $a \in f(y)$ and $b \in g(y)$ such that $p(a - b) < d_p(f(y), g(y)) + \varepsilon$. Since $a \in f(y) \subset V_f$ and $b \in g(y) \subset V_g$, there are $a_0 \in f(x)$ and $b_0 \in g(x)$ with $p(a - a_0) < \varepsilon/2$ and $p(b - b_0) < \varepsilon/2$. It follows that

$$(**) \qquad \begin{aligned} d_p(f(x), g(x)) &\leqslant p(a_0 - b_0) \\ &\leqslant p(a_0 - a) + p(a - b) + p(b - b_0) \\ &< \frac{\varepsilon}{2} + d_p(f(y), g(y)) + \varepsilon + \frac{\varepsilon}{2} \\ &= d_p(f(y), g(y)) + 2\varepsilon. \end{aligned}$$

If now $V = V_2 \cap V_3$, then V is an open neighborhood of x in K, and for each $y \in V$, we have

$$\begin{aligned} d_p(u, h(y)) &< d_p(u, h(x)) + 2\varepsilon, \quad \text{by } (*) \\ &= d_p(f(x), g(x)) - 2\varepsilon \\ &< d_p(f(y), g(y)), \quad \text{by } (**) \end{aligned}$$

so that $u \in S(y)$, and hence $y \in S^{-1}(u)$ for all $y \in V$. Therefore $S^{-1}(u)$ is open in K for each $u \in K$. By K. Fan's Theorem, there exists an $x_0 \in K$ such that $x_0 \in S(x_0)$, so that $d_p(x_0, h(x_0)) < d_p(f(x_0), g(x_0))$, and this is impossible because $d_p(x_0, h(x_0)) = d_p(f(x_0), g(x_0))$. This completes the proof.

Analogous to Corollary 2.5 and Corollary 2.6, we have the following results, which form generalizations of Corollary 1 (respectively, Corollary 1′) in [4].

COROLLARY 2.9. *Let E be a Hausdorff locally convex topological vector space, let* $K \subset E$ *be nonempty compact convex and let* $f, g\colon K \to \mathscr{C}(E)$ *be continuous such that for each* $x \in K$, *either* $f(x)$ *or* $g(x)$ *is compact. Suppose for each* $x \in K$ *and for each continuous seminorm p on E with* $d_p(f(x), g(x)) > 0$, *there exist* $y \in I_K(x)$ *(respectively,* $y \in O_K(x)$*) such that* $d_p(y, x + f(x) - g(x)) < d_p(f(x), g(x))$. *Then there exists an* $x_0 \in K$ *such that* $f(x_0) \cap g(x_0) \neq \varnothing$.

The following is an immediate consequence of Corollary 2.9.

COROLLARY 2.10. *Let E be a Hausdorff locally convex topoological vector space, let* $K \subset E$ *be nonempty compact convex and let* $f, g\colon K \to \mathscr{C}(E)$ *be continuous such that for each* $x \in K$, *either* $f(x)$ *or* $g(x)$ *is compact. Suppose for each* $x \in K$ *and for each continuous seminorm p on E with* $d_p(f(x), g(x)) > 0$, *there exists* $y \in \mathrm{cl}(I_K(x))$ *(respectively,* $y \in \mathrm{cl}(O_K(x))$*) such that* $d_p(y, x + f(x) - g(x)) < d_p(f(x), g(x))$. *Then there exist an* $x_0 \in K$ *such that* $f(x_0) \cap g(x_0) \neq \varnothing$.

References

[1] C. Berge, *Topological Spaces* (Oliver & Boyd, Edinburgh and London, 1963).
[2] F. E. Browder, A new generalization of the Schauder fixed point theorem, *Math. Ann.* **174** (1967), 285–290.
[3] F. E. Browder, The fixed point theory of multi-valued mappings in topological vector spaces, *Math. Ann.* **177** (1968), 283–301.
[4] F. E. Browder, On a sharpened form of the Schauder fixed-point theorem, *Proc. Nat. Acad. Sci. U. S. A.* **74** (1977), 4749–4751.
[5] J. Caristi, Fixed point theorem for mapping satisfying inwardness conditions, *Trans. Amer. Math. Soc.* **215** (1976), 241–251.
[6] K. Fan, A generalization of Tychonoff's fixed point theorem, *Math. Ann.* **142** (1961), 305–310.
[7] K. Fan, Extensions of two fixed point theorems of F. E. Browder, *Math. Z.* **112** (1969), 234–240.
[8] B. Halpern and G. Bergman, A fixed point theorem for inward and outward maps, *Trans. Amer. Math. Soc.* **130** (1968), 353–358.
[9] J. H. Jiang, Fixed point theorems for multi-valued mappings in locally convex spaces, *Acta Math. Sinica* **25** (1982), 365–373.

[10] J. H. Jiang, Generalization of two fixed point theorems of S. Reich, *Acta Math. Sinica* **24** (1981), 359–364.
[11] S. Reich, Approximate selections, best approximations, fixed points, and invariant sets, *J. Math. Anal. Appl.* **62** (1978), 104–113.
[12] S. Reich, Fixed point theorems for set-valued mappings, *J. Math. Anal. Appl.* **69** (1979), 353–358.
[13] S. Reich, Some problems and results in fixed point theory, *Contemporary Math.* **21** (1983), 179–187.

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