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A FUNDAMENTAL SOLUTION FOR A NONELLIPTIC PARTIAL DIFFERENTIAL OPERATOR

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Let

(1) $$Z = \frac{\partial}{\partial z} + 2iz\bar{z}^2 \frac{\partial}{\partial t}$$

and set

(2) $$\mathscr{L} = \mathscr{L}_{z,t} = -\tfrac{1}{2}(Z\bar{Z} + \bar{Z}Z) = -\frac{\partial^2}{\partial z \partial \bar{z}} + 2i|z|^2 \frac{\partial}{\partial t}\left(Z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) - 4|z|^6 \frac{\partial^2}{\partial t^2}.$$

Here $z = x + iy$, $\partial/\partial z = \frac{1}{2}(\partial/\partial x - i\,\partial/\partial y)$. Z is the "unique" (modulo multiplication by nonzero functions) holomorphic vector-field which is tangent to the boundary of the "degenerate generalized upper half-plane"

(3) $$D = \{(z_1, z) \in \mathbf{C}^2; \rho = \operatorname{Im} z_1 - |z|^4 > 0\}.$$

In our terminology $t = \operatorname{Re} z_1$. We note that $\mathscr{L}$ is nowhere elliptic. To put it into context, $\mathscr{L}$ is of the type $\Box_b$, i.e. operators like $\mathscr{L}$ occur in the study of the boundary Cauchy-Riemann complex. For more information concerning this connection the reader should consult [**1**] and [**2**].

In this paper we give a fundamental solution, $F(z, t; w, s) = F_{(w,s)}(z, t)$ for $\mathscr{L}$, i.e.

(4) $$\langle F_{(w,s)}, \mathscr{L}(\phi)\rangle = \phi(w, s), \ \phi \in C_0^\infty(\mathbf{R}^3).$$

Here $z = x + iy$, $w = u + iv$ and with a mild abuse of notation (z, t) and (w, s) stand for $(x, y, t) \in \mathbf{R}^3$ and $(u, v, s) \in \mathbf{R}^3$, respectively. $\langle , \rangle$ denotes the action of distributions, as linear functionals, on $C_0^\infty(\mathbf{R}^3)$. We set

(5) $$A = \tfrac{1}{2}(|z|^4 + |w|^4 + i(t - s)),$$

and

(6) $$p = \begin{cases} \bar{z}w/A^{1/2} & \text{if } w \neq 0 \\ 0 & \text{if } w = 0. \end{cases}$$

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$A^{1/2}$ denotes the principal value of the square root, i.e. $A^{1/2} > 0$ if $A > 0$. We note that p is a C^∞-function of (z, t) whenever (w, s) is fixed.

(7) THEOREM. *If* $(z, t) \neq (w, s)$ *set*

$$(8) \quad F = \frac{i}{4\pi^2|A|} \cdot \frac{|1+p| + i|1-p|}{1+|p|^2} \times \frac{1}{|1-p|} \int_0^1 \left(\frac{|1+p|^2 + i|1-p^2|}{1+|p|^2} \xi - 1 \right)^{-1} d\xi.$$

Then F *is a fundamental solution of* $\mathcal{L}$.

Remark. In particular

$$(9) \quad F_{(0,s)}(z, t) = 4\pi^{-1}(|z|^8 + (t-s)^2)^{-1/2}.$$

The proof of Theorem 7 will be given in a series of steps. We note that

$$(10) \quad |p| = |\bar{z}w/A^{1/2}| \leqq 1,$$

and

$$(11) \quad \begin{cases} |p| = 1 \Leftrightarrow |z| = |w|, t = s, \\ p = \pm 1 \Leftrightarrow (z, t) = (\pm w, s). \end{cases}$$

An easy calculation yields

$$(12) \quad \left| 1 - \frac{|1+p|^2 + i|1-p^2|}{1+|p|^2} \right| = 1.$$

Therefore, if $0 \leqq \xi \leqq 1$,

$$1 - \frac{|1+p|^2 + i|1-p^2|}{1+|p|^2} \xi \neq 0$$

if $p \neq 1 \Leftrightarrow (z, t) \neq (w, s)$.

This justifies formula (8) if $(z, t) \neq (w, s)$. Next we derive two different representations for F. Set

$$(13) \quad \sigma^2 = |z^2 - w^2|^4 + (t + 2 \operatorname{Im} z^2\bar{w}^2)^2,$$

and

$$(14) \quad h(p, \bar{p}) = \frac{|1-p^2| - i(p+\bar{p})}{1+|p|^2}.$$

(15) PROPOSITION. *Assume* $(z, t) \neq (\pm w, s)$. *Then*

$$(16) \quad F_{(w,s)}(z, t) = \frac{1}{4\pi\sigma} + \frac{i}{2\pi^2\sigma} \log h.$$

Proof. First we note that the right hand side of (16) is well defined as long as $(z, t) \neq (\pm w, s)$. Next

$$\frac{p+\bar{p}}{1+|p|^2} = \frac{|1+p|^2}{1+|p|^2} - 1.$$

Hence

$$(17) \quad \frac{1}{4\pi\sigma} + \frac{i}{2\pi^2\sigma}\log\left(\frac{|1-p^2| - i(p+\bar{p})}{1+|p|^2}\right)$$
$$= \frac{1}{2\pi^2\sigma}\left\{\frac{\pi}{2} + i\log\left(i\left[1 - \frac{|1+p|^2 + i|1-p^2|}{1+|p|^2}\right]\right)\right\}$$
$$= \frac{i}{2\pi^2\sigma}\log\left(1 - \frac{|1+p|^2 + i|1-p^2|}{1+|p|^2}\right).$$

log denotes its principal value, i.e., $\log z > 0$ if $z > 1$. Since $(z, t) \neq (w, s)$

$$\log\left(1 - \frac{|1+p|^2 + i|1-p^2|}{1+|p|^2}\right)$$

is well defined. Furthermore

$$(18) \quad \sigma = 2|A|\,|1-p|\,|1+p|.$$

Now

$$(17) = \frac{i}{2\pi^2\sigma}\cdot\frac{|1+p|^2 + i|1-p^2|}{1+|p|^2} \times \int_0^1 \frac{d\xi}{\dfrac{|1+p|^2 + i|1-p^2|}{1+|p|^2}\xi - 1},$$

and simplifying the right hand side by $|1+p|$, see (18), we obtain $F_{(w,s)}(z, t)$. This proves Proposition (15).

(19) LEMMA. *Assume $w \neq 0$. Then p is near ± 1 if and only if (z, t) is near $(\pm w, s)$, respectively.*

Proof. Since $|p| \leqq 1$,

$$|1-p| \geqq \tfrac{1}{2}|1-p^2| \geqq \tfrac{1}{2}(1-|p|^2) = \frac{1}{2}\left(1 - \frac{2|z|^2|w|^2}{|z|^4+|w|^4}\cdot\frac{1}{\sqrt{1+\gamma^2}}\right),$$

where

$$\gamma = \frac{t-s}{|z|^4+|w|^4}.$$

We note that

$$\frac{2|z|^2|w|^2}{|z|^4+|w|^4} \leqq 1 \quad \text{and} \quad \frac{1}{\sqrt{1+\gamma^2}} \leqq 1.$$

It is easy to see that the first inequality in this proof implies

$$1 - \frac{2(|z|/|w|)^2}{1+(|z|/|w|)^4} \leqq 2|1-p|,$$
$$1 - (1+\gamma^2)^{-1/2} \leqq 2|1-p|.$$

If $2|1 - p| < \delta < 1$, then

$$||z|^2 - |w|^2| < 3|w|^2 \cdot \sqrt{\delta}(1 - \delta)^{-1}$$

$$|t - s| < 35|w|^2 \cdot \sqrt{\delta}(1 - \delta)^{-2}$$

i.e., $(|z|, t)$ must be arbitrarily near $(|w|, s)$ by choosing $|p|$ sufficiently near 1. The converse is clear, i.e., $(|z|, t)$ is near $(|w|, s) \Rightarrow |p|$ is near 1.

Finally set $z = |z|e^{i\theta}$, $w = |w|e^{i\omega}$. Then

$$1 - p = 1 - \frac{|z||w|}{|A|^{1/2}} e^{i(\omega - \theta)}.$$

Therefore p is near 1 if and only if $|z|\ |w|/|A|^{1/2}$ is near 1 and ω is near θ, i.e. p is near 1 if and only if (z, t) is near (w, s).

A similar argument shows that p is near -1 if and only if (z, t) is near $(-w, s)$. This proves Lemma 19.

(20) LEMMA. *Assume $w \neq 0$. Then when (z, t) is near $(-w, s)$ F can be written in the following form*

$$(21)\quad F = \frac{1}{4\pi^2|A|(p + \bar{p})} \int_0^1 \frac{d\xi}{1 + \dfrac{|1 - p^2|^2}{(p + \bar{p})^2}\xi^2}.$$

Proof. (17) yields

$$F = \frac{i}{2\pi^2\sigma} \log\left(\frac{-p - \bar{p} + i|1 - p^2|}{1 + |p|^2}\right).$$

If p is near -1 this gives

$$F = \frac{-1}{2\pi^2\sigma} \arctan \frac{|1 - p^2|}{-p - \bar{p}} = \frac{1}{2\pi^2\sigma} \cdot \frac{|1 - p^2|}{p + \bar{p}} \int_0^1 \frac{d\xi}{1 + \dfrac{|1 - p^2|^2}{(p + \bar{p})^2}\xi^2}$$

and now (18) implies (21).

(22) THEOREM. *$F_{(w,s)}(z, t)$ is a C^∞ function of $(z, t) = (x, y, t)$ as long as $(z, t) \neq (w, s)$.*

Proof. If $w = 0$ the result follows from (9). If $w \neq 0$, then $A^{-1/2}$, $\bar{A}^{-1/2}$, p and $\bar{p}$ are C^∞ functions of (z, t). Thus, if $w \neq 0$ and $(z, t) \neq (\pm w, s)$, then (16) is a C^∞ function of (z, t) and the result follows. Finally, if $w \neq 0$ and (z, t) is in a sufficiently small neighbourhood of $(-w, s)$, then (21) is a C^∞ function of p and $\bar{p}$, because $|1 - p^2|^2 = (1 - p^2)(1 - \bar{p}^2)$ is a C^∞ function of p, $\bar{p}$, hence the result follows in this case too. This proves Theorem 22.

Let

$$(23)\quad dv(z, t) = dxdydt \quad \text{and} \quad dv(z) = dxdy$$

denote Lebesgue measure on $\mathbf{R}^3$ and $\mathbf{R}^2$, respectively. We introduce a "regularization", F_ϵ, of F as follows. We set

$$(24) \quad A_\epsilon = \tfrac{1}{2}(|z|^4 + |w|^4 + \epsilon^4 + i(t - s)),$$

$$(25) \quad p_\epsilon = \bar{z}w/A_\epsilon^{1/2}.$$

This yields

$$(26) \quad \sigma_\epsilon^2 = (|z^2 - w^2|^2 + \epsilon^4)^2 + (t - s + 2 \operatorname{Im} z^2\bar{w}^2)^2,$$

and

$$(27) \quad F_\epsilon = F_{(w,s),\epsilon}(z, t) = \frac{1}{4\pi\sigma_\epsilon} + \frac{i \log h_\epsilon}{2\pi^2\sigma_\epsilon},$$

where, again

$$(28) \quad h_\epsilon(p, \bar{p}) = h(p_\epsilon, \bar{p}_\epsilon),$$

and h is given by (14). F_ϵ is C^∞ in all the variables if $\epsilon > 0$.

(29) PROPOSITION. $F_{(w,s)} = \lim_{\epsilon\to 0} F_{(w,s),\epsilon}$ *as a distribution in* $\mathbf{R}^3$.

Proof. Formulas (8) and (9) show that $F_{(w,s),\epsilon}(z, t) \to F_{(w,s)}(z, t)$, pointwise, as $\epsilon \to 0$, as long as $(z, t) \neq (w, s)$. Since $|h_\epsilon| = 1$,

$$(30) \quad |F_\epsilon| \leqslant C/\sigma,$$

with some $C > 0$, independent of $\epsilon > 0$. We shall show that $1/\sigma$ is locally integrable. Then the Lebesgue dominated convergence theorem implies that

$$F_{(w,s),\epsilon} \to F_{(w,s)} \text{ in } D'(\mathbf{R}^3), \text{ as } \epsilon \to 0.$$

The question of integrability occurs only at $(z, t) = (\pm w, s)$. We may as well set $s = 0$. To include the two points in question, or, possibly, one point, if $w = 0$, we shall estimate the integral of σ^{-1} on the domain $-1 \leqq t \leqq 1$, $|z| \leqq R$, where $R = 1 + 2|w|$. Then

$$(31) \quad \int_{-1}^{1} \frac{dt}{\sigma} = \int_{-1+2\operatorname{Im} z^2\bar{w}^2}^{1+2\operatorname{Im} z^2\bar{w}^2} \frac{ds}{(|z^2 - w^2|^4 + s^2)^{1/2}} < 2\int_0^{1+2R^4} \frac{ds}{(|z^2 - w^2|^4 + s^2)^{1/2}}$$
$$= 2 \log ([(1 + 2R^4)^2 + |z^2 - w^2|^4]^{1/2} + 1 + 2R^4) - 4 \log |z^2 - w^2|.$$

The first log term is clearly integrable on every compact domain in the z-plane. As for the second term

$$\int_{|z|<R} |\log |z - w| + \log |z + w||dv(z) \leqq 2\int_{|z|<2R} |\log |z||dv(z) < \infty.$$

This finishes the proof of Proposition 29.

Since the bounds in the proof of Proposition 29 can be chosen independently of (w, s) if (w, s) belongs to a compact set in $\mathbf{R}^3$, Fubini's Theorem implies

(32) COROLLARY. *$F_{(w,s)}(z, t)$ is locally integrable in $\mathbf{R}^3 \times \mathbf{R}^3$.*

A short heuristic explanation of the proof of Theorem 7 is in order. We shall show that $\mathscr{L} F_{(w,s)}(z, t) = 0$ if $(z, t) \neq (w, s)$, i.e.

$$\operatorname{supp} \mathscr{L}(F_{(w,s)}) \subset \{(w, s)\}.$$

According to Proposition 29

$$\mathscr{L}(F_{(w,s),\epsilon}) \to \mathscr{L}(F_{(w,s)})$$

in $D'(\mathbf{R}^3)$. Next we show that

$$\mathscr{L} F_{(w,s),\epsilon}(z, t) \in L^1(\mathbf{R}^3),$$

and

$$\|\mathscr{L}(F_{(w,s),\epsilon})\|_{L^1(\mathbf{R}^3)} < M,$$

M independent of $\epsilon > 0$. This yields

$$\mathscr{L}(F_{(w,s)}) = \lim_{\epsilon \to 0} \mathscr{L}(F_{(w,s),\epsilon}) = c\delta_{(w,s)},$$

where

$$c = \lim_{\epsilon \to 0} \int_{\mathbf{R}^3} \mathscr{L} F_{(w,s),\epsilon}(z, t) dv(z, t) = 1,$$

which proves Theorem 7.

To carry out this procedure we need more precise information about $\mathscr{L} F_{(w,s),\epsilon}(z, t) = \mathscr{L}(F_{(w,s),\epsilon})$. We set

(33) $\sigma_\epsilon = \lambda_\epsilon^{1/2} \bar{\lambda}_\epsilon^{1/2}$,

where

(34) $\lambda_\epsilon = |z^2 - w^2|^2 + \epsilon^4 + i(t - s + 2 \operatorname{Im} z^2 \bar{w}^2) = 2(A - \bar{z}^2 w^2) + \epsilon^4 = \lambda + \epsilon^4$.
Then

(35) $Z(\lambda_\epsilon) = \partial \lambda_\epsilon / \partial z + 2i z\bar{z}^2 \, \partial \lambda_\epsilon / \partial t = 0,$

(36) $\bar{Z}(\lambda_\epsilon) = 4\bar{z}(z^2 - w^2),$

(37) $Z(\bar{\lambda}_\epsilon) = \overline{\bar{Z}(\lambda_\epsilon)} = 4z(\bar{z}^2 - \bar{w}^2),$

(38) $\bar{Z}(\bar{\lambda}_\epsilon) = \overline{Z(\lambda_\epsilon)} = 0.$

Next

$$\begin{aligned} Z\bar{Z}(\sigma_\epsilon^{-1}) &= Z\bar{Z}(\lambda_\epsilon^{-1/2} \bar{\lambda}_\epsilon^{-1/2}) = Z(-\tfrac{1}{2}\lambda_\epsilon^{-3/2}[4\bar{z}(z^2 - w^2)]\bar{\lambda}_\epsilon^{-1/2}) \\ &= -\tfrac{1}{2}\lambda_\epsilon^{-3/2}\{[4\bar{z}(z^2 - w^2)]Z(\bar{\lambda}_\epsilon^{-1/2}) + 8|z|^2 \bar{\lambda}_\epsilon^{-1/2}\} \\ &= -4|z|^2|\lambda_\epsilon|^{-3}(\bar{\lambda}_\epsilon - |z^2 - w^2|^2). \end{aligned}$$

Thus we have

$$(39)\quad -\tfrac{1}{2}(Z\bar{Z}+\bar{Z}Z)\left(\frac{1}{4\pi\sigma_\epsilon}\right) = \frac{1}{\pi}\frac{\epsilon^4|z|^2}{\sigma_\epsilon^3}.$$

In particular

$$(40)\quad \mathscr{L}(\sigma^{-1}) = 0 \quad \text{if} \quad (z, t) \neq (\pm w, s).$$

(41) PROPOSITION. *For all* $\epsilon > 0$

$$(42)\quad \int_{\mathbf{R}^3} \frac{1}{\pi}\frac{\epsilon^4|z|^2}{\sigma_\epsilon^3}\,dv(z, t) = 1.$$

Proof. First we evaluate the dt integral

$$\epsilon^4|z|^2\pi^{-1}\int_{-\infty}^{\infty} [(|z^2 - w^2|^2 + \epsilon^4)^2 + (t - s + 2\,\mathrm{Im}\, z^2\bar{w}^2)^2]^{3/2}dt$$

$$= \epsilon^4|z|^2\pi^{-1}\int_{-\infty}^{\infty} ((|z^2 - w^2|^2 + \epsilon^4)^2 + t^2)^{-3/2}dt$$

$$= 2\pi^{-1}\epsilon^4|z|^2(|z^2 - w^2|^2 + \epsilon^4)^{-2}.$$

Next we compute

$$I = 2\pi^{-1}\int_{\mathbf{R}^2} \epsilon^4|z|^2(|z^2 - w^2|^2 + \epsilon^4)^{-2}dv(z).$$

Let $r = |z|$ and set

$$r_1^2 = |z - w|^2 = r^2 + |w|^2 - 2r|w|\cos\theta,$$

$$r_2^2 = |z + w|^2 = r^2 + |w|^2 + 2r|w|\cos\theta.$$

Then, using $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, we have

$$|z^2 - w^2|^2 = r_1^2 r_2^2 = (r^2 + |w|^2)^2 - 4r^2|w|^2\cos^2\theta$$

$$= r^4 + |w|^4 - 2r^2|w|^2\cos 2\theta.$$

Therefore

$$I = 2\pi^{-1}\epsilon^4\int_0^\infty r^3dr\int_0^{2\pi} (r^4 + |w|^4 + \epsilon^4 - 2r^2|w|^2\cos 2\theta)^{-2}\,d\theta.$$

We use the formula

$$(43)\quad \int_0^{2\pi} (a - b\cos\theta)^{-2}d\theta = 2\pi a(a^2 - b^2)^{-3|2} \quad a > b \geqq 0,$$

which yields

$$I = 4\epsilon^4 \int_0^\infty (r^4 + |w|^4 + \epsilon^4) r^3 ((r^4 + |w|^4 + \epsilon^4)^2 - 4r^4|w|^4)^{-3/2} dr$$

$$= \epsilon^4 \int_0^\infty (t + |w|^4 + \epsilon^4)((t + |w|^4 + \epsilon^4)^2 - 4t|w|^4)^{-3/2} dt$$

$$= \epsilon^4 \int_0^\infty (t + |w|^4 + \epsilon^4)((t + \epsilon^4 - |w|^4)^2 + 4|w|^4\epsilon^4)^{-3/2} dt$$

$$= \epsilon^4 \int_{\epsilon^4 - |w|^4}^\infty (t + 2|w|^4)(t^2 + 4\epsilon^4|w|^4)^{-3/2} dt = 1,$$

which is the required result.

We would like to express our thanks to P. G. Rooney for computing I.

(44) *Remark.* Via contour integration

$$(45) \quad \int_0^{2\pi} \frac{d\theta}{a - b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}, \quad a > b \geqq 0.$$

Differentiating (45) with respect to a one obtains (43).

Next we set

$$(46) \quad g_\epsilon = i \log h_\epsilon,$$

and compute $\mathscr{L}(g_\epsilon/\sigma_\epsilon)$. We note that g_ϵ is real. Then

$$\bar{Z}Z(g_\epsilon/\sigma_\epsilon) = \bar{Z}Z(\sigma_\epsilon^{-1})g_\epsilon + Z(\sigma_\epsilon^{-1})\bar{Z}(g_\epsilon) + \bar{Z}(\sigma_\epsilon^{-1})Z(g_\epsilon) + \sigma_\epsilon^{-1}\bar{Z}Z(g_\epsilon).$$

Therefore

$$(47) \quad (\bar{Z}Z + Z\bar{Z})(g_\epsilon/\sigma_\epsilon) = (\bar{Z}Z + Z\bar{Z})(\sigma_\epsilon^{-1})g_\epsilon + 2Z(\sigma_\epsilon^{-1})\bar{Z}(g_\epsilon) + 2\bar{Z}(\sigma_\epsilon^{-1})Z(g_\epsilon) + \sigma_\epsilon^{-1}(\bar{Z}Z + Z\bar{Z})(g_\epsilon).$$

Using (33)–(38) we obtain

$$(48) \quad Z(\sigma_\epsilon^{-1}) = -2z(\bar{z}^2 - \bar{w}^2)\lambda_\epsilon/\sigma_\epsilon^3 = \overline{\bar{Z}(\sigma_\epsilon^{-1})}.$$

Similarly

$$(49) \quad Z(A_\epsilon) = \partial A_\epsilon/\partial z + 2iz\bar{z}^2\, \partial A_\epsilon/\partial t = 0 = \bar{Z}(\bar{A}_\epsilon),$$

$$(50) \quad Z(\bar{A}_\epsilon) = 2z\bar{z}^2 = \bar{Z}(A_\epsilon),$$

$$(51) \quad Z(p_\epsilon) = 0 = \bar{Z}(\bar{p}_\epsilon),$$

$$(52) \quad Z(\bar{p}_\epsilon) = \frac{\bar{w}}{\bar{A}_\epsilon^{1/2}}\left(1 - \frac{|z|^4}{\bar{A}_\epsilon}\right) = \overline{\bar{Z}(p_\epsilon)}.$$

We recall that $g_\epsilon = g(p_\epsilon, \bar{p}_\epsilon)$. Hence

$$(53) \quad Z(g_\epsilon) = \frac{\bar{w}}{\bar{A}_\epsilon^{1/2}}\left(1 - \frac{|z|^4}{\bar{A}_\epsilon}\right)\frac{\partial g_\epsilon}{\partial \bar{p}_\epsilon} = \overline{\bar{Z}(g_\epsilon)}.$$

Therefore

$$(54)\quad (Z\bar{Z}+\bar{Z}Z)(g_\epsilon) = -2|z|^2\left(\frac{\bar{p}_\epsilon}{\bar{A}_\epsilon}\frac{\partial g_\epsilon}{\partial p_\epsilon}+\frac{p_\epsilon}{A_\epsilon}\frac{\partial g_\epsilon}{\partial p_\epsilon}\right) + 2\frac{|w|^2}{|A_\epsilon|}\left|1-\frac{|z|^4}{A_\epsilon}\right|^2\frac{\partial^2 g_\epsilon}{\partial p_\epsilon\partial\bar{p}_\epsilon}.$$

From (26) we also have

$$(55)\quad \sigma_\epsilon^2 = 4|A_\epsilon - \bar{z}^2w^2|^2 = 4|A_\epsilon|^2\,|1-p_\epsilon^2|^2.$$

Therefore (47), (48), (53)—(55) and (39) yield

$$\begin{aligned}(Z\bar{Z}+\bar{Z}Z)(g_\epsilon/\sigma_\epsilon) &= -8|z|^2\epsilon^4 g_\epsilon/\sigma_\epsilon^3 \\ &\quad + 8\,\mathrm{Re}\,\{2z(w^2-\bar{z}^2)(A_\epsilon - \bar{z}^2w^2)\bar{Z}(g_\epsilon)\}/\sigma_\epsilon^3 + 4|A_\epsilon - \bar{z}^2w^2|^2 \\ &\quad \times\left\{-4|z|^2\,\mathrm{Re}\left(\frac{p_\epsilon}{A_\epsilon}\frac{\partial g_\epsilon}{\partial p_\epsilon}\right)+\frac{2|w|^2}{|A_\epsilon|}\left|1-\frac{|z|^4}{A_\epsilon}\right|^2\frac{\partial^2 g_\epsilon}{\partial p_\epsilon\partial\bar{p}_\epsilon}\right\}\Big/\sigma_\epsilon^3.\end{aligned}$$

We calculate the necessary derivatives.

$$\frac{\partial g_\epsilon}{\partial p_\epsilon} = \frac{1-\bar{p}_\epsilon^2}{|1-p_\epsilon^2|}\frac{1}{1+|p_\epsilon|^2},$$

$$\frac{\partial^2 g_\epsilon}{\partial p_\epsilon\partial\bar{p}_\epsilon} = \frac{-2\,\mathrm{Re}\,p_\epsilon}{|1-p_\epsilon^2|(1+|p_\epsilon|^2)^2}.$$

We recall

$$g_\epsilon = i\log h_\epsilon,$$

and

$$h_\epsilon = \frac{|1-p_\epsilon^2| + 2i\,\mathrm{Re}\,p_\epsilon}{1+|p_\epsilon|^2}.$$

Substituting for these derivatives we obtain

$$\begin{aligned}(Z\bar{Z}+\bar{Z}Z)(g_\epsilon/\sigma_\epsilon) &= -8|z|^2\epsilon^4 g_\epsilon/\sigma_\epsilon^3 + \frac{16|1-p_\epsilon^2|}{\sigma_\epsilon^3(1+|p_\epsilon|^2)} \\ &\quad\times\mathrm{Re}\left\{\frac{zw}{A_\epsilon^{1/2}}(\bar{w}^2-\bar{z}^2)(A_\epsilon-|z|^4) - |z|^2\frac{\bar{z}w}{A_\epsilon^{1/2}}(A_\epsilon - z^2\bar{w}^2)\right. \\ &\quad\left. - |\bar{w}|^2|A_\epsilon - |z|^4|^2\frac{\bar{z}w}{A_\epsilon^{1/2}}\frac{1}{|A_\epsilon|+|zw|^2}\right\}.\end{aligned}$$

We multiply through by $|A_\epsilon| + |zw|^2$ and in $\{\cdots\}$ collect the terms as coefficient of $A_\epsilon^{-1/2}$, e.g.,

$$\frac{1}{A_\epsilon^{1/2}}|A| = \frac{1}{\bar{A}_\epsilon^{1/2}}\bar{A}_\epsilon.$$

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This yields

$$(Z\bar{Z} + \bar{Z}Z)(g_\epsilon/\sigma_\epsilon) = \frac{-8|z|^2\epsilon^4}{\sigma_\epsilon^3} g_\epsilon + \frac{16|1 - p_\epsilon^2|}{(1 + |p_\epsilon|^2)^2 |A_\epsilon| \sigma_\epsilon^3} \operatorname{Re} \left\{ \frac{K_1(\epsilon)}{A_\epsilon^{1/2}} \right\},$$

where

(56) $$K_1(\epsilon) = zw|zw|^2(\bar{w}^2 - \bar{z}^2)(A_\epsilon - |z|^4) + \bar{z}\bar{w}(w^2 - z^2)A_\epsilon(\bar{A}_\epsilon - |z|^4) - |z|^2 z\bar{w}A_\epsilon(A_\epsilon - \bar{z}^2w^2) - |z|^2|zw|^2\bar{z}w(\bar{A}_\epsilon - z^2\bar{w}^2) - |w|^2\bar{z}w|A_\epsilon - |z|^4|^2.$$

(57) LEMMA. $K_1(0) = 0$. *In particular*

$$\mathscr{L}\left(\frac{i \log h}{\sigma}\right) = 0$$

if $(z, t) \neq (\pm w, s)$ *and* $w \neq 0$.

Proof. $K_1(0) = a(t - s)^2 + b(t - s) + c$ and a simple but tedious calculation yields $a = b = c = 0$. The rest follows if we note that the above calculation yields

$$(Z\bar{Z} + \bar{Z}Z)\left(\frac{i \log h}{\sigma}\right) = \frac{16|1 - p^2|}{(1 + |p|^2)^2\sigma^3|A|} \operatorname{Re}\left(\frac{K_1(0)}{A^{1/2}}\right) = 0.$$

Finally, Lemma 57 yields

(58) $$(Z\bar{Z} + \bar{Z}Z)\left(\frac{i \log h_\epsilon}{\sigma_\epsilon}\right) = -\frac{8|z|^2\epsilon^4}{\sigma_\epsilon^3} i \log h_\epsilon + \frac{8|1 - p_\epsilon^2|\epsilon^4}{|A_\epsilon|(1 + |p_\epsilon|^2)^2\sigma_\epsilon^3} \times \operatorname{Re}(A_\epsilon^{-1/2}K(\epsilon)),$$

where

(59) $$K(\epsilon) = zw|zw|^2(\bar{w}^2 - \bar{z}^2) + \bar{z}\bar{w}(w^2 - z^2)(A_\epsilon + \bar{A} - |z|^4) - |z|^2 z\bar{w}(A_\epsilon + A - \bar{z}^2w^2) - |z|^2|zw|^2\bar{z}w - |w|^2\bar{z}w(A_\epsilon + \bar{A} - 2|z|^4).$$

(60) PROPOSITION. $\mathscr{L}(F) = 0$ *as long as* $(z, t) \neq (w, s)$.

Proof. (i) $w = 0$. Then, according to Proposition 15, $F_{(w,s)}(z, t) = 1/4\pi\sigma$, hence (40) is the required result.

(ii) $w \neq 0$. In this case Proposition 15, (40) and Lemma 57 imply that

$$\mathscr{L}F_{w,s}(z, t) = 0$$

as long as $(z, t) \neq (\pm w, s)$. On the other hand, according to Theorem 22, $F_{(w,s)}(z, t)$ is C^∞ in a neighbourhood of $(-w, s)$. Therefore $\mathscr{L}F_{(w,s)}(z, t) = 0$ in a neighbourhood of $(-w, s)$, which yields Proposition 60.

(61) LEMMA. $\mathscr{L}F_{(w,s),\epsilon}(z, t) \to 0$ *uniformly on compact subsets of* $\mathbf{R}^3$ *which do not contain the point* (w, s) *as* $\epsilon \to 0$.

Proof. (i) $w = 0$. From (39)

$$\mathscr{L}_{z,t}((4\pi||z|^4 + \epsilon^4 + i(t - s)|)^{-1}) = \epsilon^4|z|^2\pi^{-1}||z|^4 + \epsilon^4 + i(t - s)|^{-3} \to 0,$$

uniformly on compact sets which exclude the point $(0, s)$ as $\epsilon \to 0$.

(ii) $w \neq 0$ and let N be a compact subset of $\mathbf{R}^3$ which excludes the points $(\pm w, s)$. Since $A_\epsilon \to A$, uniformly on N and since $|A_\epsilon|$ is bounded away from zero, independently of $\epsilon > 0$, $p_\epsilon \to p$, uniformly on N. Furthermore, since N misses a neighbourhood of $(\pm w, s)$, p misses a neighbourhood of ± 1 (see Lemma 19). Therefore, for sufficiently small $\epsilon > 0$, there exists $\delta > 0$, such that $|1 + p_\epsilon| > \delta$ and $|1 - p_\epsilon| > \delta$ on N. Recall that

$$\sigma_\epsilon^2 = 4|A_\epsilon|^2|1 - p_\epsilon^2|^2.$$

Therefore, Proposition 15, (39) and (58) imply that

$$\mathscr{L}_{z,t}(F_{(w,s),\epsilon}(z, t)) \to 0,$$

uniformly on N as $\epsilon \to 0$.

(iii) Finally assume $w \neq 0$ and (z, t) is in a sufficiently small neighbourhood, U, of $(-w, s)$. By Lemma 20

$$F_{(w,s),\epsilon}(z, t) = \frac{1}{4\pi^2|A_\epsilon|(p_\epsilon + \bar{p}_\epsilon)} \int_0^1 \frac{d\xi}{1 + \dfrac{|1 - p_\epsilon^2|^2}{(p_\epsilon + \bar{p}_\epsilon)^2}\xi^2},$$

where p_ϵ is in a sufficiently small neighbourhood of -1. Clearly, all derivatives $D^\alpha_{z,t} F_{(w,s),\epsilon}(z, t)$ converge, uniformly in U to $D^\alpha_{z,t} F_{(w,s)}(z, t)$. In particular,

$$\mathscr{L} F_{(w,s),\epsilon}(z, t) \to \mathscr{L} F_{(w,s)}(z, t) = 0,$$

as $\epsilon \to 0$, uniformly for $(z, t) \in U$ (see Proposition 60). This proves Lemma 61.

(62) LEMMA. *For every fixed* (w, s)

(63) $$\int_{\mathbf{R}^3} |\mathscr{L} F_{(w,s),\epsilon}(z, t)| dv(z, t) < C,$$

for some $C > 0$, C *independent of* $\epsilon > 0$.

Proof. First of all we have

(64) $$|\mathscr{L} F_{(w,s),\epsilon}(z, t)| \leqq C \frac{1 + |z|^2}{\sigma_\epsilon^3} \epsilon^4,$$

which follows immediately from (40), (58) and (59). We note that

$$-\pi < i \log h_\epsilon < \pi,$$

since $|h_\epsilon| = 1$ and $|p_\epsilon| < 1$ if $\epsilon > 0$. Thus to show that (64) implies (63) all we have to show is that

(65) $$\int_{\mathbf{R}^3} (\epsilon^4/\sigma_\epsilon^3) dv(z, t) < \infty,$$

uniformly in $\epsilon > 0$, if $w \neq 0$. The case $w = 0$ follows from (16) and (42).

Now

$$\int_{\mathbf{R}^3} \epsilon^4 \sigma_\epsilon^{-3} dv(z, t) < C_1 \int_{|z|<|w|/2} dv(z) \int_{-\infty}^{\infty} \sigma^{-3} dt + \int_{|z|\geqq|w|/2} dv(z) \int_{-\infty}^{\infty} \epsilon^4 \sigma_\epsilon^{-3} dt = I_1 + I_2.$$

First

$$I_1 = C_1 \int_{|z|<|w|/2} dv(z) \int_{-\infty}^{\infty} (|z^2 - w^2|^4 + t^2)^{-3/2} dt = C_1 \int_{|z|<|w|/2} |z^2 - w^2|^{-4} dv(z) \int_{-\infty}^{\infty} (1 + t^2)^{-3/2} dt < \infty,$$

since $|z| < |w|/2 \Rightarrow |z - w| > |w|/2$ and $|z + w| > |w|/2$. C_1 may be chosen to be one if $0 < \epsilon < 1$.

Next we note that

$$|z| \geqq |w|/2 \Rightarrow 1 < C_2|z|^2.$$

Therefore

$$I_2 < C_2 \int_{|z|\geqq|w|/2} dv(z) \int_{-\infty}^{\infty} \epsilon^4 |z|^2 \sigma_\epsilon^{-3} dv(z, t) < C_2 \int_{\mathbf{R}^3} \epsilon^4 |z|^2 \sigma_\epsilon^{-3} dv(z, t) = \pi C_2,$$

where C_2 is independent of $\epsilon > 0$. This proves Lemma 62.

(66) LEMMA. *For all* $\epsilon > 0$

$$\int_{\mathbf{R}^3} \mathscr{L}_{z,t}\left(\frac{i \log h_\epsilon}{\sigma_\epsilon}\right) dv(z, t) = 0.$$

Proof. This follows immediately from

(i) $\mathscr{L}_{z,t}\left(\frac{i \log h_\epsilon}{\sigma_\epsilon}\right) \in L^1(\mathbf{R}^3)$, according to Lemma 62, and

(ii) from (58) one sees that

$$\left(\mathscr{L}\left(\frac{i \log h_\epsilon}{\sigma_\epsilon}\right)\right)(z, t) = -\left(\mathscr{L}\left(\frac{i \log h_\epsilon}{\sigma_\epsilon}\right)\right)(-z, t).$$

Proof of Theorem 7. Let $\phi \in C_0^\infty(\mathbf{R}^3)$. Recall that we want to show that $\phi(w, s) = \langle F_{(w,s)}, \mathscr{L}(\phi)\rangle$. According to Proposition 29

$$\langle F_{(w,s)}, \mathscr{L}(\phi)\rangle = \lim_{\epsilon\to 0} \langle F_{(w,s),\epsilon}, \mathscr{L}(\phi)\rangle = \lim_{\epsilon\to 0} \langle \mathscr{L}(F_{(w,s),\epsilon}), \phi\rangle.$$

Furthermore, by Proposition 41 and Lemmas 62 and 66 we can write

$$\langle \mathscr{L}(F_{(w,s),\epsilon}), \phi\rangle = \int_{\mathbf{R}^3} \mathscr{L} F_{(w,s),\epsilon}(z, t)\phi(z, t) dv(z, t) = \phi(w, s) + \int_{\mathbf{R}^3} \mathscr{L} F_{(w,s),\epsilon}(z, t)(\phi(z, t) - \phi(w, s)) dv(z, t).$$

Next let U be a neighbourhood of (w, s). Then

$$\lim_{\epsilon\to 0} \int_{\mathbf{R}^3-U} \mathscr{L} F_{(w,s),\epsilon}(z, t)(\phi(z, t) - \phi(w, s))dv(z, t) = 0,$$

because $\mathscr{L} F_{(w,s),\epsilon}(z, t) \to 0$ as $\epsilon \to 0$, uniformly on $\{\text{supp } \phi\} \cap \{\mathbf{R}^3 - U\}$ (see Lemma 61). Furthermore, according to Lemma 62

$$\left| \int_U \mathscr{L} F_{(w,s),\epsilon}(z, t)(\phi(z, t) - \phi(w, s))dv(z, t) \right|$$
$$\leqq C \sup_{(z,t)\in U} |\phi(z, t) - \phi(w, s)|.$$

Since U is arbitrary, we see that

$$\phi(w, s) = \lim_{\epsilon\to 0} \langle \mathscr{L}(F_{(w,s),\epsilon}), \phi\rangle = \langle F_{(w,s)}, \mathscr{L}(\phi)\rangle,$$

which proves Theorem 7.

(67) COROLLARY. *Let* $\phi \in C_0^\infty(\mathbf{R}^3)$. *Then the distribution*

$$u(z, t) = \int_{\mathbf{R}^3} F_{(w,s)}(z, t)\phi(w, s)dv(w, s)$$

solves

$$\mathscr{L}(u) = \phi.$$

Furthermore, $u \in C^\infty(\mathbf{R}^3)$.

Proof. Corollary 32 shows that u is a locally integrable distribution. Let $\psi \in C_0^\infty(\mathbf{R}^3)$. Then

$$\left\langle \psi, \mathscr{L} \int_{\mathbf{R}^3} F_{(w,s)}(z, t)\phi(w, s)dv(w, s)\right\rangle$$
$$= \left\langle \mathscr{L}(\psi), \int_{\mathbf{R}^3} F_{(w,s)}(z, t)\phi(w, s)dv(w, s)\right\rangle$$
$$= \left\langle \int_{\mathbf{R}^3} F_{(w,s)}(z, t)\mathscr{L}(\psi)(z, t)dv(z, t), \phi\right\rangle$$
$$= \int_{\mathbf{R}^3} \psi(w, s)\phi(w, s)dv(w, s)$$

by Theorem 7. This implies $\mathscr{L}(u) = \phi$. Finally $u \in C^\infty(\mathbf{R}^3)$ because $\mathscr{L} = -(\text{Re } Z)^2 - (\text{Im } Z)^2$ is hypoelliptic (see [**3**], [**4**] and [**5**]).

(68) *Remark.* It is interesting to compute the singularity of $F_{(w,s)}(z, t)$ when (z, t) is near (w, s). First assume $w \neq 0$. Then $p = 1$ at $(z, t) = (w, s)$ and

$$z = \frac{|1 + p|^2 + i|1 - p^2|}{1 + |p|^2} = 2 \quad \text{at } p = 1.$$

We note that Im $z > 0$ if $p \neq \pm 1$. Next,

$$\int_0^1 \frac{dt}{zt - 1}\bigg|_{z=2} = \frac{1}{z} \log (1 - z)\bigg|_{z=2} = \frac{-i\pi}{2}.$$

Therefore, according to (8), F has the following singularity when (z, t) is near (w, s):

$$\frac{1}{8\pi|A|^{1/2}|A^{1/2} - \bar{z}w|}.$$

This holds even when $w = 0$.

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