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NORMAL VARIATIONS OF INVARIANT HYPERSURFACES OF FRAMED MANIFOLDS

BY
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1. **Introduction.** A hypersurface of a globally framed f-manifold (briefly, a framed manifold), does not in general possess a framed structure as one may see by considering the 4-sphere S^4 in R^5 or S^5. For, a hypersurface so endowed carries an almost complex structure, or else, it admits a nonsingular differentiable vector field. Since an almost complex manifold may be considered as being globally framed, with no complementary frames, this situation is in marked contrast with the well known fact that a hypersurface (real codimension 1) of an almost complex manifold admits a framed structure, more specifically, an almost contact structure. For example, when considered as a unit sphere in E^6 or S^6, S^5 is framed [as an (almost) contact manifold]. In the examples cited, the immersions are not invariant, for, on the one hand, it is impossible to immerse a manifold as an invariant hypersurface of a contact space [2], and, on the other hand, an invariant hypersurface of an almost complex manifold is itself almost complex.

We are thereby led to consider hypersurfaces of framed manifolds immersed in such a way that they admit framed structures of the same rank. These hypersurfaces may be invariant or deviate from this property by a "normal variation" determined by some 1-form induced by the f-structure. Since S^7 is framed we might expect that S^6 can be immersed in it as an invariant hypersurface thereby showing that S^6 possesses a complex structure. However, S^6 is not parallelizable, hence it cannot be invariantly immersed (see §3). When the ambient space $\tilde{M}$ is affinely cosymplectic this situation was studied in some detail in [2]. The metric case was found to be particularly interesting. Indeed, if $\tilde{M}$ is cosymplectic, an invariant hypersurface or a normal variation of it, carries a Kaehlerian structure. The same situation prevails if the ambient space is a normal contact metric manifold.

2. **Framed manifolds.** An n-dimensional C^∞ manifold M carrying a linear transformation field f of class C^∞ satisfying the algebraic condition

$$f^3+f = 0$$

is called an *f-manifold* provided the *f-structure* f is of constant rank r at each point of M. The existence of such a structure is equivalent to a reduction of the structural

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group of the tangent bundle to $U(r/2)\times O(n-r)$. As examples, there are the almost complex structures for $n=2m$ and the almost contact structures for $n=2m-1$, the former having maximal rank and the latter having rank $2m-2$. Moreover, direct products of almost complex and almost contact spaces give rise to f-manifolds [1].

By putting

$$s=-f^2, \qquad t=f^2+I,$$

where I is the identity field,

$$\begin{gathered} s+t=I, \\ s^2=s, \qquad t^2=t, \\ st=ts=0, \\ f^2s=-s, \qquad ft=0. \end{gathered}$$

The operators s and t acting in the tangent space at each point of M are therefore complementary projection operators defining distributions S and T in M corresponding to s and t, respectively. The distribution S is r-dimensional and $\dim T=n-r$.

The union of the tangent spaces at each point of the distribution T has a bundle structure denoted by $\mathscr{T}$. It is clearly a subbundle of the tangent bundle of M and its dimension is $2n-r$. If the vector bundle $\mathscr{T}$ is trivial it may be naturally identified with $M\times R^{n-r}$. In this case, the latter admits an almost complex structure.

If there are $n-r$ vector fields E_a spanning $\mathscr{T}$ at each point of M, together with $n-r$ differential forms η^a satisfying

$$\eta^a(E_b)=\delta^a_b, \tag{2.1}$$

where δ^a_b, $a, b=1,\ldots,n-r$ is the 'Kronecker delta', and if

$$f^2=-I+\eta^a\otimes E_a, \tag{2.2}$$

where $\otimes$ denotes the tensor product, the summation convention being employed here and in the sequel, then M is called a *globally framed f-manifold* or, simply, a *framed manifold*. As examples, there are the almost complex manifolds for $n=2m$ and the almost contact spaces for $n=2m-1$. Moreover, direct products of almost complex and almost contact spaces are framed structures [1]. Parallelizable manifolds are globally framed. In this case, $f=0$ and $\eta^a\otimes E_a=I$.

The framed structure on M will be denoted by $M(f, E_a, \eta^a)$. From (2.1) and (2.2) one easily obtains the relations

$$fE_a=0, \qquad \eta^a\circ f=0, \qquad a=1,\ldots,n-r. \tag{2.3}$$

Let $M(f, E_a, \eta^a)$ be an n-dimensional framed manifold of rank r. Since R^{n-r} admits a (trivial) framed structure, the direct product $M\times R^{n-r}$ admits a framed structure of rank r, and hence an underlying almost complex structure f' (see [1]). If f' is integrable, the framed structure on M is said to be *normal*, and in this case the tensor field

$$S_f\equiv[f,f]+d\eta^a\otimes E_a$$

vanishes [3], where $[f,f](X, Y)=[fX,fY]-f[fX, Y]-f[X,fY]+f^2[X, Y]$.

3. **Hypersurfaces of framed manifolds.** Let $\tilde{M}(\tilde{f}, \tilde{E}_a, \tilde{\eta}^a)$ be a framed manifold of dimension $n \geq 2$ and rank r, $a=1, \ldots, n-r$. We consider an $(n-1)$-dimensional hypersurface M immersed in $\tilde{M}$ with immersion $i: M \to \tilde{M}$ such that: *For each $m \in M$ the vectors $\tilde{E}_\alpha$, $\alpha=1, \ldots, n-r-1$ at $i(m)$ belong to the tangent hyperplane of $i(M)$ and*

$$(\tilde{E}_{n-r})_{i(m)} \notin i(M)_{i(m)}.$$

The vector field $\tilde{E}=\tilde{E}_{n-r}$ is then an affine normal to $i(M)$, so we may write

$$\tilde{f} i_* X = i_* f X + \theta(X)\tilde{E}, \tag{3.1}$$

$$\tilde{f}\tilde{E} = 0, \tag{3.2}$$

where f and θ are tensor fields on M of types (1, 1) and (0, 1), respectively, and i_* is the induced tangent map. If $\theta=0$, the submanifold is an invariant hypersurface of $\tilde{M}$. On the other hand, if $\theta \neq 0$, it provides a measure of the deviation of M from this property. Such a hypersurface will be called *noninvariant* or a *normal variation* of M. A hypersurface may, of course, be neither invariant nor non-invariant. However, in the sequel, unless otherwise specified, $i(M)$ will be a noninvariant hypersurface of the framed manifold $\tilde{M}$. We shall occasionally refer to M as the hypersurface.

Applying $\tilde{f}$ to both sides of (3.1), we get

$$-i_* X + \tilde{\eta}^a(i_* X)\tilde{E}_a = i_* f^2 X + \theta(fX)\tilde{E},$$

from which, since there are vector fields E_α on M such that

$$\tilde{E}_\alpha = i_* E_\alpha, \qquad \alpha = 1, \ldots, n-r-1,$$

we obtain

$$\begin{aligned} f^2 &= -I + \eta^\alpha \otimes E_\alpha, \\ C\theta &= \eta \end{aligned} \tag{3.3}$$

where

$$\eta^a = i^* \tilde{\eta}^a, \qquad a = 1, \ldots, n-r-1,$$

i^* is the dual map of i_*, $C\theta$ is the 1-form on M defined by

$$C\theta(X) = \theta(fX)$$

and

$$\eta = \eta^{n-r}.$$

From (2.1), $\eta^\alpha(E_\beta)=(i^*\tilde{\eta}^\alpha)(E_\beta)=\tilde{\eta}^\alpha(i_* E_\beta)=\tilde{\eta}^\alpha(\tilde{E}_\beta)=\delta^\alpha_\beta$. Moreover, by (2.3), $\tilde{f}\tilde{E}_\alpha=\tilde{f} i_* E_\alpha = i_* f E_\alpha + \theta(E_\alpha)\tilde{E}$, so $fE_\alpha=0$ and $\theta(E_\alpha)=0$, $\alpha=1, \ldots, n-r-1$.

THEOREM 1. *A noninvariant hypersurface of a framed manifold admits a framed structure of the same rank as the ambient space. Moreover, it admits a 1-form θ determining an $(n-r-1)$-dimensional distribution complementary to the distribution determined by the Pfaffian system*

$$\eta^\alpha = 0, \qquad \alpha = 1, \ldots, n-r-1.$$

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COROLLARY. *There are no noninvariant hypersurfaces of a parallelizable manifold, that is*, $\theta=0$.

If $\tilde{M}$ is integrable, we obtain

THEOREM 2. *A noninvariant hypersurface of a normal framed manifold* $\tilde{M}(\tilde{f}, \tilde{E}_a, \tilde{\eta}^a)$ *is a normal framed manifold of the same rank r carrying a* 1*-form whose differential has bidegree* (1, 1) *with respect to the induced f-structure.*

Proof. Given a symmetric affine connection $\tilde{D}$ on $\tilde{M}$, an affine connection D is defined on M with respect to the affine normal $\tilde{E}$ by the Gauss equation

$$\tilde{D}_{i_*X} i_* Y = i_* D_X Y + h(X, Y)\tilde{E}, \tag{3.4}$$

where h is a symmetric tensor field of type (0, 2) on M, namely, the *second fundamental tensor* of M with respect to $\tilde{E}$.

Since $(\tilde{f}, \tilde{E}_a, \tilde{\eta}^a)$ is normal, $S_{\tilde{f}}$ vanishes, so by [3]

$$L_{\tilde{E}}\tilde{f} = 0, \qquad L_{\tilde{E}}\tilde{\eta} = 0, \tag{3.5}$$

where $\tilde{\eta}=\tilde{\eta}^{n-r}$ and L_X denotes the Lie derivative with respect to the vector field X. Expressing $S_{\tilde{f}}(x, y)$, where x and y are any two vector fields on $\tilde{M}$, in the form

$$\begin{aligned} S_{\tilde{f}}(x, y) = (\tilde{D}_{\tilde{f}x}\tilde{f})y-(\tilde{D}_{\tilde{f}y}\tilde{f})x+\tilde{f}(\tilde{D}_y\tilde{f})x \\ -\tilde{f}(\tilde{D}_x\tilde{f})y+[(\tilde{D}_x\tilde{\eta}^a)(y)-(\tilde{D}_y\tilde{\eta}^a)(x)]\tilde{E}_a, \end{aligned} \tag{3.6}$$

then by judiciously applying (2.2), (3.1) and (3.4),

$$\begin{aligned}
\tilde{S}_f(i_*X, i_*Y) &= (\tilde{D}_{i_*fX+\theta(X)\tilde{E}}\tilde{f})i_*Y-(\tilde{D}_{i_*fY+\theta(Y)\tilde{E}}\tilde{f})i_*X \\
&\quad +\tilde{f}\{(\tilde{D}_{i_*Y}\tilde{f})i_*X-(\tilde{D}_{i_*X}\tilde{f})i_*Y\} \\
&\quad +\{(\tilde{D}_{i_*X}\tilde{\eta}^a)(i_*Y)-(\tilde{D}_{i_*Y}\tilde{\eta}^a)(i_*X)\}\tilde{E}_a \\
&= \tilde{D}_{i_*fX}(\tilde{f}i_*Y)-\tilde{f}\tilde{D}_{i_*fX}i_*Y+\theta(X)(\tilde{D}_{\tilde{E}}\tilde{f})i_*Y \\
&\quad -\tilde{D}_{i_*fY}(\tilde{f}i_*X)+\tilde{f}\tilde{D}_{i_*fY}i_*X-\theta(Y)(\tilde{D}_{\tilde{E}}\tilde{f})i_*X \\
&\quad +\tilde{f}\{\tilde{D}_{i_*Y}(\tilde{f}i_*X)-\tilde{f}\tilde{D}_{i_*Y}(i_*X)-\tilde{D}_{i_*X}(\tilde{f}i_*Y)+\tilde{f}\tilde{D}_{i_*X}(i_*Y)\} \\
&\quad +\{\tilde{D}_{i_*X}(\tilde{\eta}^a(i_*Y))-\tilde{\eta}^a(\tilde{D}_{i_*X}i_*Y) \\
&\quad -\tilde{D}_{i_*Y}(\tilde{\eta}^a(i_*X))+\tilde{\eta}^a(\tilde{D}_{i_*Y}i_*X)\}\tilde{E}_a \\
&= i_*D_{fX}(fY)+h(fX, fY)\tilde{E}+(fX\cdot\theta(Y))\tilde{E} \\
&\quad +\theta(Y)\tilde{D}_{i_*fX}\tilde{E}-i_*fD_{fX}Y-\theta(D_{fX}Y)\tilde{E}+\theta(X)(\tilde{D}_{\tilde{E}}\tilde{f})i_*Y \\
&\quad -i_*D_{fY}(fX)-h(fX, fY)\tilde{E}-(fY\cdot\theta(X))\tilde{E} \\
&\quad -\theta(X)\tilde{D}_{i_*fY}\tilde{E}+i_*fD_{fY}X+\theta(D_{fY}X)\tilde{E}-\theta(Y)(\tilde{D}_{\tilde{E}}\tilde{f})i_*X \\
&\quad +\tilde{f}\{i_*D_Y(fX)+h(Y, fX)\tilde{E}\}+\theta(X)\tilde{f}\tilde{D}_{i_*Y}\tilde{E} \\
&\quad +i_*D_YX-\tilde{\eta}(i_*D_YX)\tilde{E}-\tilde{f}\{i_*D_X(fY)+h(X, fY)\tilde{E}\} \\
&\quad -\theta(Y)\tilde{f}\tilde{D}_{i_*X}\tilde{E}-i_*D_XY+\tilde{\eta}(i_*D_XY)\tilde{E} \\
&\quad +\{D_X((\theta\circ f)Y)-\tilde{\eta}^a(i_*D_XY)-D_Y((\theta\circ f)X)+\tilde{\eta}^a(i_*D_YX)\}\tilde{E}_a \\
&= i_*\{[f, f](X, Y)+d\eta^a(X, Y)E_a\} \\
&\quad +L_{\tilde{E}}\tilde{f}\{\theta(X)i_*Y-\theta(Y)i_*X\}+\{d\theta(fX, Y)+d\theta(X, fY)\}\tilde{E}.
\end{aligned}$$

Applying (3.5), we obtain

$$S_{\tilde{f}}(i_*X, i_*Y) = i_*S_f(X, Y)+\{d\theta(fX, Y)+d\theta(X, fY)\}\tilde{E}.$$

The left-hand side being zero, $S_{\tilde{f}}$ must vanish and $d\theta$ must be of bidegree (1, 1) with respect to f.

Corollary 1. *An invariant hypersurface of a normal framed manifold is a normal framed manifold of the same rank.*

Since S^7 is parallelizable we might expect that S^6 can be immersed in it as an invariant hypersurface or a normal variation of it thereby showing that S^6 possesses an integrable almost complex structure. But S^6 is not parallelizable, hence, it cannot be so imbedded. For the same reason S^2 cannot be invariantly or noninvariantly immersed in S^3, but it does possess a complex structure.

If $\dim \tilde{M}=2m+1$, it has an almost contact structure $(\tilde{f}', \tilde{E}_{2n-r+1}, \tilde{\eta}^{2n-r+1})$, where

$$\tilde{f}' = \tilde{f}+\tilde{\eta}^{2i}\otimes \tilde{E}_{2i-1}-\tilde{\eta}^{2i-1}\otimes \tilde{E}_{2i}, \qquad i = 1, \ldots, m-\frac{r}{2}.$$

Thus,

$$\tilde{f}'i_*X = i_*f'X+\theta(X)\tilde{E},$$

where

$$f' = f+\eta^{2i}\otimes E_{2i-1}-\eta^{2i-1}\otimes E_{2i}$$

is the induced almost complex structure on M. Applying [3, Theorem 1], we obtain the following generalization of [2, Theorem 1].

Corollary 2. *A noninvariant hypersurface of an odd dimensional normal framed manifold is a complex manifold carrying a* 1*-form* θ *whose differential has bidegree* (1, 1) *with respect to the almost complex structure.*

Corollary 3. *An invariant hypersurface of an odd dimensional normal framed manifold is a complex manifold.*

The direct product of an odd number of normal almost contact manifolds being a normal framed manifold [1, Theorem 3], we obtain the following generalization of [2, Theorem 1].

Corollary 4. *An invariant or noninvariant hypersurface of the direct product of an odd number of normal almost contact manifolds is a complex manifold.*

The above computation also yields

Theorem 3. *If* $\tilde{E}$ *is an infinitesimal automorphism of the framed structure* $\tilde{M}(\tilde{f}, \tilde{E}_a, \tilde{\eta}^a)$, *and if for every noninvariant hypersurface the induced* f*-structure*

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$M(f, E_\alpha, \eta^\alpha)$ is normal, and the differential of the 1*-form θ is of bidegree* (1, 1) *with respect to f, then $(\tilde{f}, \tilde{E}_a, \tilde{\eta}^a)$ is integrable.*

COROLLARY. *The framed structure $(\tilde{f}, \tilde{E}_a, \tilde{\eta}^a)$ is normal if for every invarian immersion, the induced structure $(f, E_\alpha, \eta^\alpha)$ is normal.*

4. **Framed metric manifolds.** The framed manifold $\tilde{M}(\tilde{f}, \tilde{E}_a, \eta^a)$ is called a framed metric manifold if it carries a Riemannian metric $\tilde{g}$ such that

(i) $\tilde{\eta}^a=\tilde{g}(\tilde{E}_a, \ )$, $a=1,\ldots, n-r$

and

(ii) $\tilde{f}$ is skew symmetric with respect to $\tilde{g}$.

It can be shown that a framed manifold carries a metric with these properties. We put

$$\tilde{F}(X, Y) = \tilde{g}(\tilde{f}\tilde{X}, \tilde{Y})$$

and call it the *fundamental* 2*-form* of the structure. Let $g=i^*\tilde{g}$ and let F be the fundamental 2-form of the induced f-structure on M. Relating $\tilde{F}$ with F, we get

$$\begin{aligned}\tilde{F}(i_*X, i_*Y) &= \tilde{g}(\tilde{f}i_*X, i_*Y)\\ &= g(fX, Y)+\theta(X)\tilde{\eta}(i_*Y)\\ &= F(X, Y)+(\theta \wedge C\theta)(X, Y),\end{aligned}$$

that is,

$$i^*\tilde{F} = F+\theta \wedge C\theta.$$

Since $\tilde{f}$ is not of maximal rank, the tensor field

$$\gamma = g-C\theta \otimes C\theta$$

is not a Riemannian metric. However, if $n=2m+1$ and $r=2m$, f is of maximal rank, so i being a regular map, γ defines a positive definite metric. In this case, it is easily checked that γ is hermitian with respect to f. In fact, if $\tilde{F}$ is closed, γ is an almost Kaehler metric and $F+\theta\wedge C\theta$ is the fundamental 2-form of the almost Kaehler manifold $M(f, \gamma)$. If the structure on $\tilde{M}$ is integrable, then M is Kaehlerian.

THEOREM 4. *In addition to the canonical framed Aetric structure (f, η^α, g) the noninvariant hypersurface $M(f, E_\alpha, \eta^\alpha)$ admits the framed Aetric structure (f, η^α, g^*), where*

$$g^* = g+\theta \otimes \theta.$$

Proof. By (ii) and (3.1),

$$\tilde{g}(i_*fX, i_*Y)+\theta(X)\tilde{\eta}(i_*Y) = -\tilde{g}(i_*X, i_*fY)-\theta(Y)\tilde{\eta}(i_*X).$$

Hence, by (3.3), $g(fX, Y)+\theta(X)C\theta(Y)=-g(X,fY)-\theta(Y)C\theta(X)$, that is

$$(g+\theta \otimes \theta)(fX, Y) = -(g+\theta \otimes \theta)(X, fY).$$

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Moreover,

$$\eta^\alpha(X) = \tilde{\eta}^\alpha(i_*X) = \tilde{g}(i_*X, \tilde{E}_\alpha) = \tilde{g}(i_*X, i_*E_\alpha) = g(X, E_\alpha)$$
$$= g^*(X, E_\alpha) - \theta(X)\theta(E_\alpha) = g^*(X, E_\alpha).$$

COROLLARY. *A noninvariant hypersurface $M(f, g, \theta)$ of an almost contact manifold admits the hermitian metric $g+\theta\otimes\theta$.*

This fact is easily generalized. In fact, since $\tilde{f}$ is skew symmetric with respect to $\tilde{g}$, so is

$$\tilde{f}' = \tilde{f} + \tilde{\eta}^{2i} \otimes \tilde{E}_{2i-1} - \tilde{\eta}^{2i-1} \otimes E_{2i}.$$

Thus, since $\theta \circ f' = \theta \circ f, f'$ is skew symmetric with respect to g^*.

THEOREM 5. *A noninvariant hypersurface of an odd dimensional framed metric manifold admits the hermitian structure $(f', g+\theta\otimes\theta)$.*

5. **Hypersurfaces of special manifolds.** Weingarten's equation for the immersion i and the connection $\tilde{D}$ is

$$\tilde{D}_{i_*X}\tilde{E} = -i_*HX + \omega(X)\tilde{E}, \tag{5.1}$$

where H is the *second fundamental tensor* of type (1, 1) of M with respect to $\tilde{E}$, and ω is a 1-form on M defining the connection of the affine normal bundle.

Covariant differentiation of (3.1) yields after applying (3.4) and (5.1), then (3.1)

$$\begin{aligned}(\tilde{D}_{i_*X}\tilde{f})i_*Y = i_*\{(D_Xf)Y - \theta(Y)HX\} \\ +\{h(X, fY) + (D_X\theta)(Y) + \omega(X)\theta(Y)\}\tilde{E}.\end{aligned} \tag{5.2}$$

A framed manifold $\tilde{M}(\tilde{f}, \tilde{E}_a, \tilde{\eta}^a)$ with a symmetric affine connection $\tilde{D}$ is called a *K-manifold*(2) if

$$\tilde{D}\tilde{f} = 0, \qquad \tilde{D}\tilde{\eta}^a = 0, \tag{5.3}$$

$a=1, \ldots, n-r$. Clearly, then, by (3.6), the framed structure is normal. Moreover,

$$\tilde{D}\tilde{E}_a = 0, \qquad a = 1, \ldots, n-r,$$

so all the structure tensors are parallel fields with respect to $\tilde{D}$. Hence, by (5.1), $HX=0$ and $\omega(X)=0$. Furthermore, by (3.2)

$$Df = 0,$$
$$(D_X\theta)(Y) = -h(X, fY).$$

(2) In a previous paper [4] this name was used exclusively for even dimensional framed metric manifolds. In this case it was shown, if the structure tensors are closed, that there is an underlying Kaehlerian structure.

The direct product of any number of cosymplectic manifolds is a K-manifold. Moreover, R^n and the n-torus have K-structures.

If for every vector field X on the hypersurface M, $HX=0$, then, by (5.1), the fields $\tilde{D}_{i_*X}\tilde{E}$ and $\tilde{E}$ are proportional. Hence, the affine normals are parallel at each point of M and M is said to be *totally flat*.

THEOREM 6. *A noninvariant hypersurface of a K-manifold is a totally flat K-manifold and the connection in the affine normal bundle is trivial. If the hypersurface is invariant it is also totally geodesic.*

COROLLARY. *A noninvariant hypersurface of a product of an odd number of cosymplectic manifolds is a totally flat Kaehler manifold and the connection in the normal bundle is trivial. If the hypersurface is invariant it is also totally geodesic.*

That h vanishes for invariant hypersurfaces may be seen as follows. Since

$$h(X, f^2 Y) = 0, \qquad h(X, Y) = \eta^\alpha(Y)h(X, E_\alpha).$$

But, $\tilde{D}_{i_*X}\tilde{E}_\alpha = \tilde{D}_{i_*X} i_* E_\alpha = i_* D_X E_\alpha + h(X, E_\alpha)\tilde{E}$.

A framed manifold with a symmetric affine connection $\tilde{D}$ is called *affinely exact* if it is integrable and

$$= \tilde{D}\tilde{E}.$$

THEOREM 7. *On a nonvariant hypersurface of an affinely exact manifold*

$$f = -H \tag{5.4}$$

and

$$\theta = \omega.$$

This is a consequence of the relations (3.1) and (5.1).

A contact manifold is said to be *normal* if its underlying framed structure is normal.

A normal contact metric manifold being affinely exact, we obtain Proposition 3 and Theorem 5 of [2].

COROLLARY. *There are no invariant or noninvariant hypersurfaces of a normal contact manifold.*

This is particularly true of the unit sphere S^7 in E^8.

Proof. A normal contact manifold with a compatible metric $\tilde{g}$ is affinely exact with respect to the Riemannian connection. Since H is symmetric and f is skew symmetric with respect to the induced metric $i^*\tilde{g}$, the relation (5.4) cannot hold.

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