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SOLVING THE TRUNCATED MOMENT PROBLEM SOLVES THE FULL MOMENT PROBLEM

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Abstract. It is shown that the truncated multidimensional moment problem is more general than the full multidimensional moment problem.

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Introduction. Denote by $\mathbb{Z}_+^d$ (resp. $\mathbb{C}^d$) the set of all d-tuples of nonnegative integers (respectively complex numbers). If $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d$ and $z = (z_1, \dots, z_d) \in \mathbb{C}^d$, then we write $|\alpha| = \alpha_1 + \dots + \alpha_d$, $z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$ and $\bar{z} = (\bar{z}_1, \dots, \bar{z}_d)$. Let F be a nonempty closed subset of $\mathbb{C}^d$ and let $\boldsymbol{c}^{(n)} = \{c(\alpha, \beta) : \alpha, \beta \in \mathbb{Z}_+^d, \ |\alpha| + |\beta| \le n\}$ be a finite multisequence of complex numbers ($n \ge 0$). The *truncated (multidimensional and complex) F-moment problem* of order n consists in determining conditions under which there exists a positive Borel measure μ on $\mathbb{C}^d$ such that the closed support $\operatorname{supp} \mu$ of μ is contained in F and[1]

$$c(\alpha, \beta) = \int z^\alpha \bar{z}^\beta d\mu(z), \quad \alpha, \beta \in \mathbb{Z}_+^d, \ |\alpha| + |\beta| \le n. \tag{1}$$

A positive Borel measure μ on $\mathbb{C}^d$ satisfying (1) is called a *representing measure* of $\boldsymbol{c}^{(n)}$, while the numbers $\int z^\alpha \bar{z}^\beta d\mu(z)$ are customarily called *moments* of μ.

Let now $\boldsymbol{c} = \{c(\alpha, \beta) : \alpha, \beta \in \mathbb{Z}_+^d\}$ be a multisequence of complex numbers. The *full (multidimensional and complex) F-moment problem* entails determining whether there exists a positive Borel measure μ on $\mathbb{C}^d$ such that $\operatorname{supp} \mu \subseteq F$ and

$$c(\alpha, \beta) = \int z^\alpha \bar{z}^\beta d\mu(z), \quad \alpha, \beta \in \mathbb{Z}_+^d. \tag{2}$$

As above, a positive Borel measure μ on $\mathbb{C}^d$ satisfying (2) is called a *representing measure* of $\boldsymbol{c}$. We say that a multisequence of moments is *determinate* if it has precisely one representing measure.

The literature concerning the full F-moment problem (not necessarily complex) is extensive and it is still growing (see for instance [**6**, **7**, **18**, **19**, **1**, **23**, **3**, **22**, **4**, **25**, **16**, **30**] and [**8**, **26**, **27**, **15**, **28**, **5**, **29**] where semi-algebraic F's are considered). The truncated F-moment problem has been intensively studied since the early 90's mostly by Curto

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[1]Throughout the whole paper, we tacitly assume that all the functions under the integral sign are absolutely integrable. In particular, by (1), the measure μ is finite and consequently it is regular (e.g. see [**24**], Theorem 2.18]).

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and Fialkow (cf. [**2**, **20**, **21**, **11**, **12**, **10**, **13**, **14**]). In 1994 R. E. Curto asked a question[2] whether the truncated F-moment problem is more general than the full F-moment problem (see also [**21**, p. 5]). In the same year I answered this question in the affirmative (see [**11**] for the negative answer to the converse question). The present paper contains the proof of this statement (some ideas involved in it may appear in the literature under different circumstances; for the reader's convenience we include all the details).

An auxiliary result. Denote by $\mathcal{A}'$ the dual Banach space of a normed space $\mathcal{A}$ and by $\sigma(\mathcal{A}', \mathcal{A})$ the weak-star topology on $\mathcal{A}'$. Given a locally compact Hausdorff space X, we write $\mathcal{C}_0(X)$ for the Banach space (equipped with the supremum norm $\|\cdot\|_X$) of all continuous complex functions on X that vanish at infinity. $\mathcal{C}_c(X)$ stands for the set of all $f \in \mathcal{C}_0(X)$ such that the closed support of f is compact. The set $\mathcal{C}_c(X)$ is dense in $\mathcal{C}_0(X)$ (cf. [**24**, Theorem 3.17]). We attach to every complex Borel measure μ on X the functional $\widehat{\mu} \in \mathcal{C}_0(X)'$ defined by

$$\widehat{\mu}(f) = \int_X f d\mu, \quad f \in \mathcal{C}_0(X).$$

PROPOSITION 1. *Let F be a nonempty closed subset of $\mathbb{C}^d$ and let ρ be a non-negative continuous function on F. Assume that $\{\mu_\omega\}_{\omega\in\Omega}$ is a net of finite positive Borel measures on F and μ is a finite positive Borel measure on F such that*

(i) *the net $\{\widehat{\mu}_\omega\}_{\omega\in\Omega}$ is $\sigma(\mathcal{C}_c(F)', \mathcal{C}_c(F))$-convergent to $\widehat{\mu}$,*

(ii) $\sup_{\omega\in\Omega} \int_F \rho d\mu_\omega < \infty$.

Define the measures ν_ω and ν on F by $d\nu_\omega = \rho d\mu_\omega$ and $d\nu = \rho d\mu$. Then

(iii) *$\nu(F) < \infty$ and the net $\{\widehat{\nu}_\omega\}_{\omega\in\Omega}$ is $\sigma(\mathcal{C}_0(F)', \mathcal{C}_0(F))$-convergent to $\widehat{\nu}$.*

Moreover, if the set $\{z \in F : \rho(z) \le r\}$ is compact for some $r > 0$, then the net $\{\widehat{\mu}_\omega\}_{\omega\in\Omega}$ is $\sigma(\mathcal{C}_0(F)', \mathcal{C}_0(F))$-convergent to $\widehat{\mu}$ and $\int_F f d\mu = \lim_{\omega\in\Omega} \int_F f d\mu_\omega$ for every $f : F \to \mathbb{C}$ such that $\frac{f}{1+\rho} \in \mathcal{C}_0(F)$.

Proof. Assume that F is not compact. Let $\{F_n\}_{n=1}^\infty$ be an increasing sequence of compact subsets of F such that $F = \bigcup_{n=1}^\infty F_n$. By [**24**, Theorem 2.12], for every $n \ge 1$ there exists $\psi_n \in \mathcal{C}_c(F)$ such that $0 \le \psi_n \le 1$, and $\psi_n = 1$ on F_n. Applying the Lebesgue monotone convergence theorem, (i) and (ii) we obtain

$$\begin{aligned} \int_F \rho d\mu = \lim_{n\to\infty} \int_{Fn} \rho d\mu \le \limsup_{n\to\infty} \int_F \psi_n \rho d\mu \\ = \limsup_{n\to\infty} \lim_{\omega\in\Omega} \int_F \psi_n \rho d\mu_\omega \le \limsup_{\omega\in\Omega} \int_F \rho d\mu_\omega < \infty. \end{aligned} \tag{3}$$

According to (ii) and (i), the net $\{\widehat{\nu}_\omega\}_{\omega\in\Omega} \subseteq \mathcal{C}_0(F)'$ is uniformly bounded and pointwise convergent on a dense subspace $\mathcal{C}_c(F)$ of $\mathcal{C}_0(F)$ to $\widehat{\nu} \in \mathcal{C}_0(F)'$. Hence it is $\sigma(\mathcal{C}_0(F)', \mathcal{C}_0(F))$-convergent to $\widehat{\nu}$ as well[3].

[2]at the Semester on Linear Operators held in the Stefan Banach International Mathematical Center (the organizers: J. Janas, F. H. Szafraniec and J. Zemánek).

[3]Notice that the σ-compactness of F is not essential in the proof of part (iii) of Proposition 1; indeed, the continuity of ρ and the regularity of μ (cf. footnote [1]) imply the inner regularity of ν which, in turn, yields $\nu(F) \le \sup_{\omega\in\Omega} \int_F \rho d\mu_\omega < \infty$ (mimic (3)).

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Suppose that $K = \{z \in F : \rho(z) \le r\}$ is compact. Let $\psi \in \mathcal{C}_c(F)$ be such that $0 \le \psi \le 1$, and $\psi = 1$ on K. Since $\lim_\omega \int_F \psi d\mu_\omega = \int_F \psi d\mu$, there exists $\omega_0 \in \Omega$ such that $\int_F \psi d\mu_\omega \le M = \int_F \psi d\mu + 1$ for $\omega \ge \omega_0$. This implies that $\mu_\omega(K) \le \int_F \psi d\mu_\omega \le M$ for $\omega \ge \omega_0$. On the other hand, by (ii), we have $\mu_\omega(F \setminus K) \le \frac{1}{r}\int_{F\setminus K} \rho d\mu_\omega \le \frac{1}{r}\sup_{\tau\in\Omega} \int_F \rho d\mu_\tau$, for $\omega \in \Omega$, so that $\sup_{\omega\ge\omega_0} \mu_\omega(F) < \infty$. This means that the net $\{\widehat{\mu}_\omega\}_{\omega\ge\omega_0}$ is uniformly bounded and pointwise convergent on $\mathcal{C}_c(F)$ to $\widehat{\mu}$. Consequently, the net $\{\widehat{\mu}_\omega\}_{\omega\in\Omega}$ is $\sigma(\mathcal{C}_0(F)', \mathcal{C}_0(F))$-convergent to $\widehat{\mu}$. Since the net $\{\widehat{\mu_\omega + \nu_\omega}\}_{\omega\in\Omega}$ is $\sigma(\mathcal{C}_0(F)', \mathcal{C}_0(F))$-convergent to $\widehat{\mu + \nu}$, we get

$$\int_F f d\mu = \int_F \frac{f}{1+\rho} d(\mu+\nu) = \lim_{\omega\in\Omega} \int_F \frac{f}{1+\rho} d(\mu_\omega + \nu_\omega) = \lim_{\omega\in\Omega} \int_F f d\mu_\omega$$

for every $f : F \to \mathbb{C}$ such that $\frac{f}{1+\rho} \in \mathcal{C}_0(F)$. This completes the proof. □

Corollary 2. *Let $\{c(m,n) : m, n \ge 0,\ m+n \le 2N-1\}$ be a finite sequence of complex numbers ($N \ge 1$) and let F be a nonempty closed subset of $\mathbb{C}$. Assume that $\{\mu_\omega\}_{\omega\in\Omega}$ is a net of finite positive Borel measures on F and μ is a finite positive Borel measure on F such that*

(i) *the net $\{\widehat{\mu}_\omega\}_{\omega\in\Omega}$ is $\sigma(\mathcal{C}_c(F)', \mathcal{C}_c(F))$-convergent to $\widehat{\mu}$,*
(ii) *$c(m,n) = \int_F z^m \bar{z}^n d\mu_\omega(z)$ for $m, n \ge 0$ with $m+n \le 2N-1$ and $\omega \in \Omega$,*
(iii) $\sup_{\omega\in\Omega} \int_F z^N \bar{z}^N d\mu_\omega(z) < \infty$.

Then

(iv) *$c(m,n) = \int_F z^m \bar{z}^n d\mu(z)$ for $m, n \ge 0$ with $m+n \le 2N-1$.*

Proof. Apply Proposition 1 to the functions $\rho(z) = z^N \bar{z}^N$ and $f(z) = z^m \bar{z}^n$ ($z \in F$) with $m, n \ge 0$ such that $m+n \le 2N-1$ (notice that $\frac{f}{1+\rho} \in \mathcal{C}_0(F)$). □

We emphasize that Corollary 2 is optimum in a sense. Namely, it may happen that the equality in (ii) holds for all $\omega \in \Omega$ and for all integer lattice points (m,n) in the convex triangle Δ with vertices $(0,0)$, $(0,2N)$ and $(2N,0)$, though no integer lattice point (m,n) belonging to the edge of Δ joining $(0,2N)$ and $(2N,0)$ satisfies the equality in (iv) (cf. Figure 1). Moreover, the set of all representing measures of a truncated F-moment sequence of order $2N$ may not be $\sigma(\mathcal{C}_0(F)', \mathcal{C}_0(F))$-closed. Example 3 deals with the case $N=1$ and $F = \mathbb{C}$.

Example 3. Since $\sum_{j=1}^\infty \frac{1}{j} = \infty$, there exists a strictly increasing sequence of positive integers $\{\kappa_n\}_{n=1}^\infty$ such that

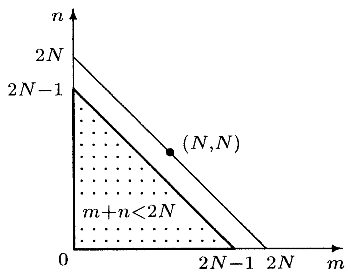

Figure 1. Integer lattice points involved in Corollary 2.

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$$u_n \overset{\text{df}}{=} \sum_{j=\kappa_n}^{\kappa_{n+1}} \frac{1}{j} \le \frac{1}{4} \text{ for } n \ge 1 \text{ and } \lim_{m\to\infty} u_m = \frac{1}{4}. \tag{4}$$

Therefore we have

$$a_n \overset{\text{df}}{=} \frac{1}{2} + u_n - \sum_{j=\kappa_n}^{\kappa_{n+1}} \frac{1}{j^2} > 0,$$

$$2b_n \overset{\text{df}}{=} \frac{1}{2} - u_n - \sum_{j=\kappa_n}^{\kappa_{n+1}} \frac{1}{j^{3/2}} > 0$$

and

$$2c_n \overset{\text{df}}{=} \frac{1}{2} - u_n + \sum_{j=\kappa_n}^{\kappa_{n+1}} \frac{1}{j^{3/2}} > 0 \text{ for } n \ge 1.$$

Because $\sum_{j=1}^{\infty} \frac{1}{j^{3/2}} < \infty$, (4) yields $\lim_{n\to\infty} a_n = \frac{3}{4}$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \frac{1}{8}$. Denote by δ_z the probability Borel measure on $\mathbb{C}$ concentrated at the point z. Set

$$\mu \overset{\text{df}}{=} \frac{3}{4}\delta_0 + \frac{1}{8}\delta_1 + \frac{1}{8}\delta_{-1}$$

and

$$\mu_n \overset{\text{df}}{=} a_n\delta_0 + b_n\delta_1 + c_n\delta_{-1} + \sum_{j=\kappa_n}^{\kappa_{n+1}} \frac{1}{j^2}\,\delta_{\sqrt{j}} \text{ for } n \ge 1.$$

It is a matter of direct verification that $\int f d\mu = \lim_{n\to\infty} \int f d\mu_n$, for every bounded continuous complex function f on $\mathbb{C}$ (in particular $\{\widehat{\mu}_n\}_{n=1}^{\infty}$ is $\sigma(\mathcal{C}_0(\mathbb{C})', \mathcal{C}_0(\mathbb{C}))$-convergent to $\widehat{\mu}$). Moreover, for every $n \ge 1$, the following conditions hold true

$$\int z^0\bar{z}^0 d\mu_n(z) = \int z^0\bar{z}^0 d\mu(z) = 1,$$
$$\int z^1\bar{z}^0 d\mu_n(z) = \int z^0\bar{z}^1 d\mu_n(z) = \int z^1\bar{z}^0 d\mu(z) = \int z^0\bar{z}^1 d\mu(z) = 0,$$
$$\int z^2\bar{z}^0 d\mu_n(z) = \int z^1\bar{z}^1 d\mu_n(z) = \int z^0\bar{z}^2 d\mu_n(z) = \frac{1}{2},$$
$$\int z^2\bar{z}^0 d\mu(z) = \int z^1\bar{z}^1 d\mu(z) = \int z^0\bar{z}^2 d\mu(z) = \frac{1}{4}.$$

The main result.

THEOREM 4. *Let F be a nonempty closed subset of $\mathbb{C}^d$ and let $\{c(\alpha,\beta)\}_{\alpha,\beta\in\mathbb{Z}_+^d}$ be a multisequence of complex numbers. If for every $n \ge 0$ there exists a positive Borel measure μ_n on F such that*

(i) *$c(\alpha,\beta) = \int_F z^\alpha\bar{z}^\beta d\mu_n(z)$, for all $\alpha,\beta \in \mathbb{Z}_+^d$ with $|\alpha| + |\beta| \le n$,*

then there exists a positive Borel measure μ on F such that $c(\alpha,\beta) = \int_F z^\alpha\bar{z}^\beta d\mu(z)$ for all $\alpha,\beta \in \mathbb{Z}_+^d$.

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Proof. Assume that F is not compact (the other case is simpler). Since F is locally compact metrizable and separable, one can see — applying [**9**, Theorem V.6.6] to the one-point compactification[4] of F — that[5]

$$\mathcal{C}_0(F) \text{ is a separable Banach space.} \tag{5}$$

Given $\alpha, \beta \in \mathbb{Z}_+^d$, we define the function $\varphi_{\alpha,\beta} : \mathbb{C}^d \to \mathbb{C}$ by

$$\varphi_{\alpha,\beta}(z) = \frac{z^\alpha \bar{z}^\beta}{\prod_{j=1}^d (1+|z_j|^2)^{\alpha_j+\beta_j+1}}, \quad (z \in \mathbb{C}^d).$$

Since the functions $z \mapsto \frac{z^m \bar{z}^n}{(1+|z|^2)^{m+n+1}}$ $(m, n \geq 0)$ are in $\mathcal{C}_0(\mathbb{C})$ and the d-fold tensor product of $\mathcal{C}_0$ functions is again a $\mathcal{C}_0$ function, we conclude that

$$\varphi_{\alpha,\beta} \in \mathcal{C}_0(F), \quad \alpha, \beta \in \mathbb{Z}_+^d. \tag{6}$$

It follows from (i) that $|\widehat{\mu}_n(f)| \leq \int_F z^0 \bar{z}^0 d\mu_n(z) \|f\|_F = c(0,0)\|f\|_F$, for every $f \in \mathcal{C}_0(F)$ and so $\widehat{\mu}_n$ belongs to $c(0,0)\boldsymbol{B}$, where $\boldsymbol{B}$ is the closed unit ball in $\mathcal{C}_0(F)'$. By (5), the set $c(0,0)\boldsymbol{B}$ is weak-star metrizable and weak-star compact (cf. [**9**, Theorems V.3.1 and V.5.1]). Hence there exists a subsequence $\{\widehat{\mu}_{k_n}\}_{n=0}^\infty$ of $\{\widehat{\mu}_n\}_{n=0}^\infty$ that is weak-star convergent to a functional $\Lambda \in c(0,0)\boldsymbol{B}$. Notice that if $f \in \mathcal{C}_0(F)$ and $f \geq 0$, then $\Lambda(f) = \lim_{n\to\infty} \widehat{\mu}_{k_n}(f) \geq 0$ and so by the Riesz representation theorem (cf. [**24**, Theorems 2.14 and 6.19] or [**17**, § 56]) there exists a finite positive Borel measure μ on F such that $\Lambda = \widehat{\mu}$. If $n(\alpha) \in \mathbb{Z}_+$ is chosen so that $k_{n(\alpha)} \geq 2|\alpha|$, then, by (i), we have

$$\int_F z^\alpha \bar{z}^\alpha d\mu_{k_n}(z) = c(\alpha,\alpha) \text{ for } n \geq n(\alpha) \ (\alpha \in \mathbb{Z}_+^d).$$

Applying Proposition 1 to $\rho(z) = z^\alpha \bar{z}^\alpha$ gives us $\int_F z^\alpha \bar{z}^\alpha d\mu(z) < \infty$ for $\alpha \in \mathbb{Z}_+^d$ and

$$\lim_{n\to\infty} \int_F f(z) z^\alpha \bar{z}^\alpha d\mu_{k_n}(z) = \int_F f(z) z^\alpha \bar{z}^\alpha d\mu(z), \quad f \in \mathcal{C}_0(F), \ \alpha \in \mathbb{Z}_+^d. \tag{7}$$

It follows from (i), (6) and (7) that

$$\begin{aligned} c(\alpha,\beta) &= \lim_{n\to\infty} \int_F z^\alpha \bar{z}^\beta d\mu_{k_n}(z) \\ &= \lim_{n\to\infty} \int_F \varphi_{\alpha,\beta}(z) \prod_{j=1}^d (1+|z_j|^2)^{\alpha_j+\beta_j+1} d\mu_{k_n}(z) \\ &= \int_F \varphi_{\alpha,\beta}(z) \prod_{j=1}^d (1+|z_j|^2)^{\alpha_j+\beta_j+1} d\mu(z) \\ &= \int_F z^\alpha \bar{z}^\beta d\mu(z), \quad \alpha, \beta \in \mathbb{Z}_+^d, \end{aligned}$$

which completes the proof. □

[4]One can show that if X is a locally compact Hausdorff space that is not compact, then the one-point compactification of X is metrizable if and only if X is metrizable and separable.
[5]The separability of $\mathcal{C}_0(F)$ can also be deduced from the Stone-Weierstrass theorem.

REMARK 5. In fact, we have proved that if for every $n \geq 0$ there exists a positive Borel measure μ_n on F satisfying condition (i) of Theorem 4, then there exists a positive Borel measure μ on F with all its moments finite and a subsequence $\{\mu_{k_n}\}_{n=0}^{\infty}$ of $\{\mu_n\}_{n=0}^{\infty}$ such that $\{\widehat{\mu_{k_n}^{\alpha}}\}_{n=0}^{\infty}$ is $\sigma(\mathcal{C}_0(F)', \mathcal{C}_0(F))$-convergent to $\widehat{\mu^{\alpha}}$ for $\alpha \in \mathbb{Z}_+^d$; here $d\nu^{\alpha}(z) = z^{\alpha}\bar{z}^{\alpha} d\nu(z)$ for $\nu = \mu, \mu_n$. This, in turn, has enabled us to show that μ is a representing measure of $\{c(\alpha, \beta)\}_{\alpha,\beta \in \mathbb{Z}_+^d}$. It is clear that all representing measures of $\{c(\alpha, \beta)\}_{\alpha,\beta \in \mathbb{Z}_+^d}$ can be obtained by way of this limit procedure. In case μ is unique we can prove more.

THEOREM 6. *Let* $\{c(\alpha, \beta)\}_{\alpha,\beta \in \mathbb{Z}_+^d}$, F, μ_n *and* μ *be as in Theorem 4. If, moreover, the multisequence* $\{c(\alpha, \beta)\}_{\alpha,\beta \in \mathbb{Z}_+^d}$ *is determinate, then the sequence* $\{\widehat{\mu_n^{\alpha}}\}_{n=0}^{\infty}$ *is* $\sigma(\mathcal{C}_0(F)', \mathcal{C}_0(F))$*-convergent to* $\widehat{\mu^{\alpha}}$*, for every* $\alpha \in \mathbb{Z}_+^d$.

Proof. Analysis similar to that in the proof of Theorem 4 (cf. Remark 5) shows that for every subsequence $\{\mu_{k_n}\}_{n=0}^{\infty}$ of $\{\mu_n\}_{n=0}^{\infty}$ there exists a subsequence $\{\mu_{k_{l_n}}\}_{n=0}^{\infty}$ of $\{\mu_{k_n}\}_{n=0}^{\infty}$ such that $\{\widehat{\mu_{k_{l_n}}^{\alpha}}\}_{n=0}^{\infty}$ is $\sigma(\mathcal{C}_0(F)', \mathcal{C}_0(F))$-convergent to $\widehat{\mu^{\alpha}}$ for $\alpha \in \mathbb{Z}_+^d$ (use the fact that the representing measure μ is unique). Hence the general topological characterization of convergent sequences yields the conclusion. □

It is worth while to notice that if the multisequence of moments $\{c(\alpha, \beta)\}_{\alpha,\beta \in \mathbb{Z}_+^d}$ is not determinate, then $\{\widehat{\mu}_n\}_{n=0}^{\infty}$ may not be $\sigma(\mathcal{C}_0(F)', \mathcal{C}_0(F))$-convergent. Indeed, if $\mu \neq \nu$ are two representing measures of $\{c(\alpha, \beta)\}_{\alpha,\beta \in \mathbb{Z}_+^d}$ and the sequence $\{\mu_n\}_{n=0}^{\infty}$ is defined by $\mu_{2k} = \mu$ and $\mu_{2k+1} = \nu$ for $k \geq 0$, then $\{\mu_n\}_{n=0}^{\infty}$ satisfies condition (i) of Theorem 4 but $\{\widehat{\mu}_n\}_{n=0}^{\infty}$ is not $\sigma(\mathcal{C}_0(F)', \mathcal{C}_0(F))$-convergent (indeed, otherwise it must be $\widehat{\mu} = \widehat{\nu}$ which, by the Riesz representation theorem (see also footnote [1]), gives us $\mu = \nu$, a contradiction).

Theorems 4 and 6 can easily be adapted to the context of other classical moment problems and in particular to the multidimensional Hamburger moment problem.

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