Canad. J. Math. Vol. 45 (3) 1993, pp. 650–661.

# GROUP ACTIONS ON FLAG MANIFOLDS AND COBORDISM

P. SANKARAN AND K. VARADARAJAN

ABSTRACT In this paper we show that any $\alpha \in \mathcal{N}_m$ (respectively $\alpha \in \Omega_m$) can be represented by a closed smooth (respectively closed, oriented smooth) manifold $M^m$ admitting a smooth $(Z/2)^m$ (respectively $S^1$)-action with a finite stationary set We also completely determine the Grassman manifolds $\mathcal{C}G_{n,k}$, $\mathcal{H}G_{n,k}$ and $\tilde{G}_{n,k}$ which are oriented boundaries as well as those which represent non-torsion elements in $\Omega_*$

1. **Introduction.** Let $\mathcal{F}$ denote one of the division rings $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{H}$. $\mathcal{F}^n$ is given the usual inner product on it. Let $\mathcal{F}G(n_1, \ldots, n_s)$ denote the $\mathcal{F}$-flag manifold of all $\mathcal{F}$-flags $(A_1, \ldots, A_s)$ where $A_i$ is a (left) vector subspace of $\mathcal{F}^n$, $A_i \perp A_j$, for $i \neq j$, $\dim_{\mathcal{F}} A_i = n_i$, $1 \leq i, j \leq s$, $n = \sum n_i$. $\mathcal{F}G(n_1, \ldots, n_s)$ is a smooth manifold of real dimension $d(\sum_{1 \leq i < j \leq s} n_i n_j)$, where $d = \dim_{\mathcal{R}} F$. Let $\xi_j^{\mathcal{F}}$ (or simply $\xi_j$) denote the $\mathcal{F}$-vector bundle of rank $n_j$ over $\mathcal{F}G(n_1, \ldots, n_s)$ whose fibre over $(A_1, \ldots, A_s) \in \mathcal{F}G(n_1, \ldots, n_s)$ is the $\mathcal{F}$-vector space $A_j$. One has the bundle isomorphism $\xi_1 \oplus \cdots \oplus \xi_s \approx n\varepsilon$, where $\varepsilon$ denotes a trivial $\mathcal{F}$-line bundle. The tangent bundle $\tau\mathcal{F}G(n_1, \ldots, n_s)$ of the flag manifold $\mathcal{F}G(n_1, \ldots, n_s)$ has the following description as a $Z(\mathcal{F})$-vector bundle, due to Lam [13].

$$\tau\mathcal{F}G(n_1, \ldots, n_s) \approx \bigoplus_{1 \leq i < j \leq s} (\bar{\xi}_i \otimes_{\mathcal{F}} \xi_j) \tag{1}$$

where $\bar{\xi}$ denotes the 'conjugate' $\mathrm{Hom}_{\mathcal{F}}(\xi, \mathcal{F})$ of $\xi$ and where $Z(\mathcal{F})$ denotes the centre of $\mathcal{F}$. The flag manifold $\mathcal{F}G(k, n-k)$ is the same as the Grassmann manifold $\mathcal{F}G_{n,k}$ of $k$-planes in $\mathcal{F}^n$. In particular $\mathcal{F}G(1, n-1)$ is the $\mathcal{F}$-projective space $\mathcal{F}P^{n-1}$. In the case of Grassmann manifolds the bundles $\xi_1^{\mathcal{F}}$ and $\xi_2^{\mathcal{F}}$ are usually denoted as $\gamma_{n,k}^{\mathcal{F}}$ and $\beta_{n,k}^{\mathcal{F}}$ respectively.

In [18] and [19] one of the authors has completely determined the $\mathcal{F}$-Grassmann manifolds which are unoriented cobordant to zero. See also [25]. In Section 2 of this paper we address the question as to which of the flag manifolds are unoriented boundaries. One has the following result which reduces this problem to the consideration of only the real case:

THEOREM 1.1. *Let $\mathcal{F} = \mathcal{R}$, $\mathcal{C}$, or $\mathcal{H}$ and let $d = \dim_{\mathcal{R}} \mathcal{F}$. Then following relation holds among the unoriented cobordism classes of $\mathcal{F}$-flag manifolds:* $[\mathcal{F}G(n_1, \ldots, n_s)]_2 = [\mathcal{R}G(n_1, \ldots, n_s)]_2^d$. ∎

Both authors were supported in part by NSERC grant A-8225
Received by the editors January 13, 1992
AMS subject classification 57R75, 57R85
Key words and phrases (Oriented) cobordism, stationary set, Pontrjagin numbers

The above theorem is proved by comparing the mod 2 cohomology rings of the real, complex, and quaternionic flag manifolds and the formulae for their Stiefel-Whitney classes. (*cf.* [17] and [18]).

In case $\mathcal{F} = \mathcal{C}$, or $\mathcal{H}$ the manifold $\mathcal{F}G(n_1, \ldots, n_s)$ is simply connected and is therefore orientable. In the real case $\mathcal{R}G(n_1, \ldots, n_s)$ is orientable if and only if $n_1 \equiv \cdots \equiv n_s \bmod 2$ [12]. In Section 3, we determine completely the manifolds $\widetilde{G}_{n,k}$, $\mathcal{C}G_{n,k}$ and $\mathcal{H}G_{n,k}$ which are oriented boundaries. We also establish that for suitable choice of orientations, $[\mathcal{R}G(2n_1, \ldots, 2n_s)] = [\mathcal{H}G(n_1, \ldots, n_s)]$ and obtain partial results concerning $\mathcal{F}$-flag manifolds which are oriented boundaries ($\mathcal{F} = \mathcal{R}, \mathcal{C}$ or $\mathcal{H}$).

In their monumental work [7], Conner and Floyd consider cobordism with group actions. It is shown in [7] that the unoriented cobordism class of a manifold $M$ on which there exists an action $\phi$ of $(\mathcal{Z}/2)^k$ with finite stationary point set $S$ is determined by the tangential $(\mathcal{Z}/2)^k$-modules $T_xM$, $x \in S$. In fact by Stong [22], the class of $(M, \phi)$ in the cobordism ring of unoriented manifolds with unrestricted $(\mathcal{Z}/2)^k$-action is determined by the family of $(\mathcal{Z}/2)^k$-modules $T_xM$, $x \in S$. In this paper we show that every element in the unoriented cobordism group $\mathcal{N}_k$ (resp. the oriented cobordism ring $\Omega_*$) can be represented by a manifold $M$ on which $(\mathcal{Z}/2)^k$ (resp. the circle $S^1$) acts with a finite stationary point set $S$. This is achieved by considering suitable actions of $(\mathcal{Z}/2)^k$ (resp. $S^1$) on the real (resp. complex) projective spaces and Milnor manifolds. In the case of oriented cobordism, we also need a result of Stong [23].

Throughout we work in the differentiable category. Most of our notations are as in [6] or [7]. In particular $[M]_2$ will denote the unoriented cobordism class of the compact closed manifold $M$.

ACKNOWLEDGEMENTS. Sankaran would like to thank Professor W. Fulton for some useful conversations he had on the $S^1$ actions on complex flag manifolds given in Section 3 during his visit to the University of Chicago in January 1991. The authors would also like to thank Professor Julius Korbaš for bringing to their notice the work of S. Mong [16].

2. **$(\mathcal{Z}/2)^k$-actions and unoriented cobordism.** In this section we obtain partial results concerning the problem of determining flag manifolds $\mathcal{F}G(n_1, \ldots, n_s)$, $s \geq 3$, which are unoriented boundaries for $\mathcal{F} = \mathcal{R}, \mathcal{C}$, or $\mathcal{H}$. The case $s = 2$, which corresponds to Grassmann manifolds, was completely settled by one of us in [18]. As noted in the introduction, we have $[\mathcal{F}G(n_1, \ldots, n_s)]_2 = [\mathcal{R}G(n_1, \ldots, n_s)]_2^d$ where $d = \dim_{\mathcal{R}} \mathcal{F}$. For this reason we need consider only the case $\mathcal{F} = \mathcal{R}$. Throughout this section $G(n_1, \ldots, n_s)$ will stand for the real flag manifold, $\mathcal{R}G(n_1, \ldots, n_s)$.

THEOREM 2.1. *Let $n = n_1 + \cdots + n_s$, $s \geq 3$.*

*(a) The flag manifold $G(n_1, \ldots, n_s)$ is an unoriented boundary in the following cases:*

*(i) $n_i = n_j$, for some $i \neq j$, $1 \leq i, j \leq s$,*

*(ii) For some $i$, $\nu_2(n_i) < \nu_2(n)$, where $\nu_2(n)$ denotes the highest exponent of 2 that divides $n$.*

*(b) $G(n_1, \ldots, n_s)$ does not bound if $n!/(n_1! \cdots n_s!)$ is odd.*

https://doi.org/10.4153/CJM-1993-036-8 Published online by Cambridge University Press

The proof of the above theorem will be preceded by the following:

PROPOSITION 2.2. *For* $s \geq 2$, $[G(2n_1, \ldots, 2n_s)]_2 = \big([G(n_1, \ldots, n_s)]_2\big)^4$.

PROOF. When $s = 2$, this is proved on p. 80 of [8] and also in [18]. For $s \geq 3$, one uses the description of the tangent bundle of the $F$-flag manifolds, $F = \mathcal{R}, \mathcal{C}$, (or $\mathcal{H}$) given by Lam [13] to show, as in [18], that $[\mathcal{R}G(2n_1, \ldots, 2n_s)]_2 = [\mathcal{H}G(n_1, \ldots, n_s)]_2 = [\mathcal{R}G(n_1, \ldots, n_s)]_2^4$. ■

We now turn to the proof of Theorem 2.1.

PROOF OF THEOREM 2.1(a)(i). In this case there exists an obvious smooth fixed point free involution which interchanges the $i$-th and the $j$-th component of each flag in $G(n_1, \ldots, n_s)$. It follows from Proposition 9.3, Chapter 17 of [10] that the manifold $G(n_1, \ldots, n_s)$ bounds in this case.

CASE (a)(ii). First assume that $n = 2m$ is even and that $n_i$ is odd. Let $T\colon \mathcal{R}^n \longrightarrow \mathcal{R}^n$ be the orthogonal linear transformation given by $(x_1, \ldots, x_{2m}) \longrightarrow (-x_2, x_1, \ldots, -x_{2m}, x_{2m-1})$. Then $T^2 = -Id$. Therefore $T$ induces a smooth involution $\theta$ on $G(n_1, \ldots, n_s)$ where $\theta(A_1, \ldots, A_s) = \big(T(A_1), \ldots, T(A_s)\big)$. The map $\theta$ is fixed point free because, if $\theta(A_1, \ldots, A_s) = (A_1, \ldots, A_s)$, then $T(A_i) = A_i$. On the other hand since $T^2 = -Id$, $T$ cannot admit any odd dimensional invariant vector subspace of $\mathcal{R}^n$. Since $n_i$ is odd this is a contradiction. It follows that $G(n_1, \ldots, n_s)$ bounds if $n$ is even and for some $i$, $n_i$ is odd. To complete the proof in the general case, let $i \leq s$ be chosen such that $\nu_2(n_i) \leq \nu_2(n_j)$ for all $j \leq s$. Then $\nu_2(n_i) < \nu_2(n)$ by hypothesis. If $\nu_2(n_i) = k \geq 1$ then $2^k | n_j$ for $1 \leq j \leq s$. Write $m_j = n_j/2^k$. Using Proposition 2.2 and induction we obtain $[G(n_1, \ldots, n_s)]_2 = [G(m_1, \ldots, m_s)]_2^{4^k}$. Now $m_i$ is odd, and $m = n/2^k$ is even as $\nu_2(n_i) < \nu_2(n)$. Hence $[G(m_1, \ldots, m_s)]_2 = 0$. It follows that $[G(n_1, \ldots, n_s)]_2 = 0$.

PROOF OF THEOREM 2.1(b). Recall that the $\mathcal{Z}/2$-Poincaré polynomial of $M = G(n_1, \ldots, n_s)$ is (*cf.* [2])

$$P_2(M, t) = \frac{(1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{n-1})}{\prod_{1\leq j\leq s}(1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{n_j-1})}. \tag{2}$$

From this, one can calculate the Euler characteristic $\chi(M)$ of $M \bmod 2$ as $\chi(M) \equiv P(M, -1) \equiv P(M, 1) \equiv n!/(n_1! \cdots n_s!) \bmod 2$. It follows from our assumption that the Euler characteristic is odd. Therefore $M$ does not bound.

REMARKS 2.3. (i) One can show that in the Case 2.1(a)(ii), the flag manifold $G(n_1, \ldots, n_s)$ admits a $(\mathcal{Z}/2)^k$-action $\phi$ without stationary points for a suitable $k$, just as in the case of Grassmann manifolds [19]. It follows from [21] that $[G(n_1, \ldots, n_s), \phi]_2 = 0$ in the cobordism ring of manifolds with unrestricted $(\mathcal{Z}/2)^k$-action.

(ii) Let $\alpha(n)$ denote the number of 1's in the dyadic expansion of $n$. The condition in Theorem 2.1(b) that $n!(n! \cdots n_s!)$ be odd is equivalent to the following condition which is easier to verify (*cf.* [21]):

$$\alpha(n) = \alpha(n_1) + \cdots + \alpha(n_s).$$

(iii) Because of Proposition 2.2 and Theorem 2.1(a) to determine the unoriented cobordism class of a general flag manifold $G(n_1, \ldots, n_s)$ it suffices to consider the case when $n = \sum n_j$ is odd with all the $n_j$ distinct. It follows from 2.1 that $G(n_1, \ldots, n_s)$ bounds whenever $n$ is a power of 2. When $n = 2^s + 2^{s-1}$, only $G(2^s, 2^{s-1})$ does not bound.

Let $I_*(G)$ denote the cobordism ring of all compact smooth manifolds with unrestricted $G$-actions. Let $F_*(G)$ denote the subring of all elements $\alpha \in I_*(G)$ with only finitely many stationary points. In [7], Conner and Floyd have shown that when $G = (\mathcal{Z}/2)^k$, $\alpha = [M, \phi]_2 \in F_*(G)$ is completely determined by the family of $\mathcal{R}G$-modules $T_xM$ as $x$ varies in the (finite) set of stationary points of $M$. Here we first show that

Proposition 2.4. *There exists an action of $(\mathcal{Z}/2)^{n-1}$ on $G(n_1, \ldots, n_s)$, $n = \sum n_j$, which has only finitely many stationary points.*

Using the above proposition, we prove

Theorem 2.5. *Every element $\alpha \in \mathcal{N}_m$ can be represented by a manifold on which $(\mathcal{Z}/2)^k$ acts smoothly with finitely many stationary points for some $k \leq m$.*

Proof of Proposition 2.4. Regard $G = (\mathcal{Z}/2)^n$ as the diagonal subgroup of $O(n)$. $G$ is generated by the $\mathcal{R}$-linear maps $t_j\colon \mathcal{R}^n \longrightarrow \mathcal{R}^n$, $t_j(x_1, \ldots, x_n) = (x_1, \ldots, -x_j, \ldots, x_n)$ for $1 \leq j \leq n$. It is easy to show that for $j \leq n$, and for any vector subspace $A \subset \mathcal{R}^n$, $t_j(A) = A$ if and only if either $e_j \in A$ or $A \perp \{e_j\}$. Now for $t \in G$, let $t(A_1, \ldots, A_s) = \big(t(A_1), \ldots, t(A_s)\big)$ for any $\mathbf{A} = (A_1, \ldots, A_s) \in G(n_1, \ldots, n_s)$. This defines a smooth action of $G$ on the flag manifold $G(n_1, \ldots, n_s)$. $\mathbf{A} = (A_1, \ldots, A_s)$ is a stationary point for this action if and only if $t_j(\mathbf{A}) = \mathbf{A}$, that is, if and only if $t_j(A_r) = A_r$ for $1 \leq r \leq s$, $\forall j \leq n$. Equivalently $\mathbf{A}$ is a stationary point if and only if for each $r \leq s$, $A_r$ is spanned by a subset of the standard basis $\{e_1, \ldots, e_n\}$ of $\mathcal{R}^n$. This shows that $G$ acts on $G(n_1, \ldots, n_s)$ with only finitely many stationary points. Now to complete the proof of the proposition, note that the element $t_1 \cdots t_n = -\mathrm{Id}\colon \mathcal{R}^n \longrightarrow \mathcal{R}^n$ acts on the identity on $G(n_1, \ldots, n_s)$. Hence the action of $G$ defines an action of $(\mathcal{Z}/2)^{n-1} \cong G/\langle t_1 \cdots t_n\rangle$ on $G(n_1, \ldots, n_s)$ with exactly the same stationary point set for the $G$ action. This proves 2.4. ∎

Proof of Theorem 2.5. Consider the action of $(\mathcal{Z}/2)^{n-1}$ on $\mathcal{R}P^{n-1} = G(1, n-1)$ constructed in the proof of 2.4. If $\theta_i$ denotes the generator of the $i$-th summand of $(\mathcal{Z}/2)^{n-1}$, $1 \leq i \leq n-1$, then $\theta_i\colon \mathcal{R}P^{n-1} \longrightarrow \mathcal{R}P^{n-1}$ is the involution $\theta_i([x_1, \ldots, x_n]) = [x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots x_n]$ where $[x_1, \ldots, x_n]$ denotes the 1-dimensional subspace of $\mathcal{R}^n$ generated by $(x_1, \ldots, x_n)$. For $m < n$, we regard $\mathcal{R}P^{m-1}$ as the subspace $\{[x_1, \ldots, x_n] \mid x_j = 0 \text{ for } j > m\}$ of $\mathcal{R}P^{n-1}$. Then $\mathcal{R}P^{m-1}$ is stable under the action of $(\mathcal{Z}/2)^{n-1}$ on $\mathcal{R}P^{n-1}$. For $n > m$ we put this action of $(\mathcal{Z}/2)^{n-1}$ on $\mathcal{R}P^{m-1}$.

Now let $(\mathcal{Z}/2)^k$ act diagonally on $\mathcal{R}P^\ell \times \mathcal{R}P^k$, for $\ell \leq k$. Then the Milnor manifold $H_{\ell,k} \subset \mathcal{R}P^\ell \times \mathcal{R}P^k$, which is defined as the zero set of the equation $\sum_{1 \leq j \leq \ell+1} x_j y_j = 0$ for $([x_1, \ldots, x_{\ell+1}], [y_1, \ldots, y_{k+1}]) \in \mathcal{R}P^\ell \times \mathcal{R}P^k$ is stable under the action of $(\mathcal{Z}/2)^k$ on $\mathcal{R}P^\ell \times \mathcal{R}P^k$. Hence $(\mathcal{Z}/2)^k$ acts on $H_{\ell,k}$. Since $(\mathcal{Z}/2)^k$ acts on $\mathcal{R}P^\ell$ and $\mathcal{R}P^k$ with only finitely many stationary points, it follows that there are only finitely many stationary points for this action of $(\mathcal{Z}/2)^k$ on $H_{\ell,k}$.

Now if $\alpha = [\mathcal{R}P^{2n}]_2$ or $[H_{\ell,k}]_2$, then the theorem follows from what has been proved above. For an arbitrary $\alpha \in \mathcal{N}_n$ the theorem follows from the fact that the elements $[\mathcal{R}P^{2n}]_2, [H_{\ell,k}]_2, n \geq 1, 1 \leq \ell \leq k$, generate the ring $\mathcal{N}_*$, and the following observations:

(i) If $(\mathcal{Z}/2)^r$ acts on $M$ and $N$ with finitely many stationary points then the diagonal action $g(x,y) = (gx, gy)$ on $M \times N$ admits only finitely many stationary points. There is also an obvious action of $(\mathcal{Z}/2)^r$ on $M \cup N$ in this case.

(ii) If $(\mathcal{Z}/2)^r$ acts on $M$ with finitely many stationary points, then so does $(\mathcal{Z}/2)^s$ for any $s \geq r$. This completes the proof. ∎

We now prove the following theorem which generalizes a result of Stong [23].

THEOREM 2.6. *Let G be any abelian group of order* $2^k$, $k \geq 1$. *Suppose that G acts on a smooth closed manifold M with stationary point set S, then for their Euler characteristics we have,* $\chi(M) \equiv \chi(S) \bmod 2$. *In particular* $\chi(M) \equiv |S| \bmod 2$ *if S is finite.*

PROOF. By induction on $k$. When $k = 1$, this result is due to P. A. Smith (*cf.* Lemma 28.2, [6]).

Now assume that the theorem holds for all abelian groups of order $2^\ell$, $\ell < k$. Choose $t \in G$ such that $t^2 = 1, t \neq 1$. Let $F = \mathrm{Fix}(t) \subset M$ denote the closed submanifold of $M$ which is left point wise fixed by $t$. By Lemma 28.2, [6] one has $\chi(M) \equiv X(F) \bmod 2$. Since $G$ is abelian $F$ is $G$-stable, and the action of $G$ on $F$ factors through an action of $\tilde{G} = G/\langle t\rangle$ on $F$. The stationary point set for the action of $G$ on $\tilde{G}$ is also $S$. Since $|\tilde{G}| = 2^{k-1}$, $\chi(F) \equiv \chi(S) \bmod 2$. Hence $\chi(M) \equiv \chi(F) \equiv \chi(S) \bmod 2$. ∎

3. **Oriented cobordism of Grassman manifolds.** In this section we obtain complete solution to the problem of determining which of the Grassmannians $\mathcal{F}G_{n,k}$, $\mathcal{F} = \mathcal{C}$ or $\mathcal{H}$, or $\tilde{G}_{n,k}$ are oriented boundaries, and which of these represent non-torsion elements in the ring $\Omega_*$. We also obtain partial results on the same problem for the family of flag manifolds. Proofs involve well-known geometric arguments and calculations of Pontrjagin numbers.

The main results of this section are:

THEOREM 3.1. *(i)* (Mong [16]). *If k or n − k is even, then the signature of* $\mathcal{C}G_{n,k}$ *is non-zero and* $[\mathcal{C}G_{n,k}]$ *generates an infinite cyclic group of* $\Omega_{2k(n-k)}$.

*(ii) If n is even and k is odd, then* $\mathcal{C}G_{n,k}$ *is an oriented boundary.*

THEOREM 3.2. *(i) If n is even, then* $2[G_{n,k}] = [\tilde{G}_{n,k}]$ *when the manifolds are suitably oriented.*

*(ii)* $\tilde{G}_{n,k}$ *is an oriented boundary if n or k is odd.* $G_{4k,2k}$ *is an oriented boundary if k is odd.*

*(iii)* $[G_{2n,2k}]$ *generates additively an infinite cyclic subgroup of* $\Omega_{4k(n-k)}$ *if* $n \neq 2k$ *or if k is even.*

THEOREM 3.3. *(i) If* $n_i$ *is odd for all i, then all the Pontrjagin numbers of* $\mathcal{R}G(n_1, \ldots, n_s)$ *are zero. Further, if s is even or if* $n_i = n_j$ *for some* $i \neq j$ *then* $\mathcal{R}G(n_1, \ldots, n_s)$ *is an oriented boundary.*

*(ii)* $[\mathcal{R}G(2n_1,\ldots,n_s)] = [\mathcal{H}G(n_1,\ldots,n_s)]$ *for any sequence of positive integers* $n_1,\ldots,n_s$ *and any* $s \geq 2$. *If at most one of the integers* $n_1,\ldots,n_s$ *is odd then the L-genus of* $[\mathcal{H}G(n_1,\ldots,n_s)]$ *is non-zero and* $[\mathcal{H}G(n_1,\ldots,n_s)]$ *has infinite additive order.*

Let $K$ be a field of characteristic different from 2. From Theorem 26.1, [3], one has

$$H^*(\tilde{G}_{2n,2k};K) \cong \frac{H\big(BSO(2k)\times BSO(2n-2k);K\big)}{\big\langle H^+\big(BSO(2n);K\big)\big\rangle}$$
$$\cong K[p_1,\ldots,p_k,e_{2k},p'_1,\ldots,p'_{n-k},e'_{2n-2k}]/\sim$$

where the relations are all generated by the following

(i) $e_{2k}^2 = p_k, (e'_{2n-2k})^2 = p'_{n-k}$,

(ii) $e_{2k}\cup e'_{2n-2k} = 0$,

(iii) $(1+p_1+\cdots+p_k)(1+p'_1+\cdots+p'_{n-k}) = 1$.

Here $p_i = p_i(\tilde{\gamma}_{2n,2k}), e_{2k} = e(\tilde{\gamma}_{2n,2k}), p'_i = p_i(\tilde{\beta}_{2n,2k})e'_{2n-2k} = e(\tilde{\beta}_{2n,2k})$. Using (iii) one can express each $p'_i$ in terms of $p_1,\ldots,p_k$. Thus $H^*(\tilde{G}_{2n,2k};K)$ is $K[p_1,\ldots,p_k,e_{2k},e'_{2n-2k}]$ under suitable relations.

The cohomology of $G_{2n,2k}$ with coefficients in $K$ can be computed to be the subalgebra of $H^*(\tilde{G}_{2n,2k};K)$ generated by the Pontrjagin classes $p_i$'s, $1 \leq i \leq k$. Thus $H^*(G_{2n,2k};K) = K[p_1,\ldots,p_k]/\sim$, where $p_i$ and $p'_j$ can be identified with $p_i(\gamma_{2n,2k})$ and $p_j(\beta_{2n,2k})$ respectively.

LEMMA 3.4. $p_i^{k(n-k)} \neq 0$ *in* $H^{4k(n-k)}(G_{2n,2k};\mathcal{R})$.

PROOF. We first show that $c_1^{k(n-k)}(\gamma_{n,k}^C) \in H^{2k}(\mathcal{C}G_{n,k};\mathcal{Z})$ is non-zero. To see this observe that $\mathcal{C}G_{n,k}$ is a Kähler manifold [11]. Therefore there exists a cohomology class $\alpha$ in $H^2(\mathcal{C}G_{n,k};\mathcal{Z})$ such that $\alpha^m \neq 0$ where $m = \dim_{\mathcal{C}} \mathcal{C}G_{n,k} = k(n-k)$. Since $H^2(\mathcal{C}G_{n,k};\mathcal{Z})$ is generated by $c_1(\gamma_{n,k}^C)$, it follows that $c_1^{k(n-k)}(\gamma_{n,k}^C) \neq 0$.

From the above description of $H^*(G_{2n,2k};\mathcal{R})$ it is clear that one has an isomorphism of $\mathcal{R}$-algebras $\sigma\colon H^*(\mathcal{C}G_{n,k};\mathcal{R}) \longrightarrow H^*(G_{2n,2k};\mathcal{R})$ defined by $\sigma(c_i) = p_i$. Note $\sigma$ is not degree preserving; in fact $\sigma$ doubles the dimension. Since $c_1^{k(n-k)} \neq 0$, it follows that $p_1^{k(n-k)} \neq 0$. ∎

We are ready to prove the main result of this section.

PROOF OF THEOREM 3.1. We need only prove 3.1(ii). Assume that $n$ is even and that $k$ is odd. By dimension considerations all Pontrjagin numbers of $\mathcal{C}G_{n,k}$ are zero in this case. By Theorem 1.1 and [19], it follows that all the Stiefel-Whitney numbers of $\mathcal{C}G_{n,k}$ are also zero. Hence $\mathcal{C}G_{n,k}$ is an oriented boundary. ∎

LEMMA 3.5. *The diffeomorphism* $\perp\colon G_{2k,k} \longrightarrow G_{2k,k}$ *is orientation reversing if* $k \equiv 2 \bmod 4$.

https://doi.org/10.4153/CJM-1993-036-8 Published online by Cambridge University Press

PROOF. Let $n = 2k$. Recall that $T_X G_{n,k} = \mathrm{Hom}_{\mathcal{R}}(X, X^\perp)$ where a linear map $f: X \longrightarrow X^\perp$ corresponds to the tangent vector $(d/dt)\big(\sigma_f(t)\big)|_{t=0}$, with $\sigma_f: (-1, 1) \longrightarrow G_{n,k}$ being the smooth curve

$$\sigma(t) = \text{“graph of } tf\text{”} = \{x + tf(x) \mid x \in X\} \subset X \oplus X^\perp = \mathcal{R}^n.$$

Let $\{x_i\}_{1\le i\le k}$ and $\{y_j\}_{1\le j\le k}$ be orthonormal bases for $X$ and $X^\perp$ respectively such that $x_1, \ldots, x_k, y_1, \ldots, y_k$ is a basis of $\mathcal{R}^n$ in the standard orientation. Let $f_{ij}: X \longrightarrow X^\perp$ be the linear map defined as $f_{ij}(x_p) = \delta_{ip} y_j$. Then $\{f_{ij}\}_{1\le i,j\le k}$ is a basis for $T_X G_{n,k}$. We order the basis $\{f_{ij}\}$ by the lexicographic ordering. It can be shown that this defines an orientation on $T_X G_{n,k}$ which is independent of the bases chosen so long as the basis $x_1, \ldots, x_k, y_1, \ldots, y_k$ of $\mathcal{R}^n$ is in the standard orientation. This gives an orientation of $G_{n,k}$. (*cf.* [20]).

Now let $\theta = \perp: G_{n,k} \longrightarrow G_{n,k}$. It suffices to show that $T\theta: T_X G_{n,k} \longrightarrow T_{\theta(X)} G_{n,k}$ is orientation reversing for some $X \in G_{n,k}$. We take $X = \mathcal{R}^k$, the span of the first $k$ elements $e_i$, $1 \le i \le k$, of the standard basis of $\{e_i\}_{1\le i\le n}$ of $\mathcal{R}^n$. A positively oriented basis for $T_X G_{n,k}$ is $\{f_{ij}\}$ where $f_{ij}(e_p) = \delta_{ip} e_j$, for all $1 \le p \le k$, for $1 \le i \le k$, $k+1 \le j \le 2k$. To compute $T\theta(f_{ij})$, first note that $\sigma_{ij}(t) = \langle \{e_p \mid 1 \le p \le k, p \ne i\} \cup \{e_i + te_j\} \rangle$, where we have written $\sigma_{ij}$ for $\sigma_{f_{ij}}$ and $\langle S \rangle$ for the span of a subset of $S \subset \mathcal{R}^n$. Therefore $\theta\big(\sigma_{ij}(t)\big) = \langle \{e_q \mid k+1 \le q \le 2k, q \ne j\} \cup \{-te_i + e_j\} \rangle$. Let $\{f'_{ji}\}$ be the basis for $T_{\theta(\mathcal{R}^k)} G_{n,k}$ obtained by using the bases $e_{k+1}, \ldots, e_{2k}$, and $e_1, \ldots, e_k$ for $\theta(\mathcal{R}^k)$ and $\mathcal{R}^k$ respectively. Then $\{f'_{ij}\}$ is positively oriented because $k$ is even. Note that $\theta\big(\sigma_{ij}(t)\big) = \text{graph of } -tf'_{ji} = \sigma'_{ji}(-t)$, where $\sigma'_{ji} = \sigma_{f'_{ji}}$. From this it follows that $T\theta(f_{ij}) = -f'_{ji}$. Therefore $T\theta$ maps the positively oriented basis $\{f_{ij}\}$ to the ordered basis $-f'_{k+1,1}, \ldots, -f'_{2k,1}, \ldots, -f'_{k+1,k}, \ldots, -f'_{2k,k}$. Since $k$ is even this basis determines the same orientation as the basis $f'_{k+1,1}, \ldots, f'_{2k,1}, \ldots, f'_{k+1,k}, \ldots, f'_{2k,k}$. A simple calculation shows that $\binom{k}{2}$ transpositions are needed to rearrange this to the positively oriented basis $f'_{k+1,1}, \ldots, f'_{k+1,k}, \ldots, f'_{2k,1}, \ldots, f'_{2k,k}$. Since $k \equiv 2 \bmod 4$, $\binom{k}{2}$ is odd. Hence we conclude that $T\theta$ maps the positively oriented basis $\{f_{ij}\}$ to a negatively oriented basis. Thus $\theta$ reverses the orientation. ■

PROOF OF THEOREM 3.2. (i) Recall that $G_{2n,k}$ is orientable. One has a covering projection $\tilde{G}_{2n,k} \longrightarrow G_{2n,k}$ where the deck transformation group $\mathcal{Z}/2$ acts on $\tilde{G}_{2n,k}$ by preserving orientation. Hence by Theorem 20.6, [6] we must have $2[G_{2n,k}] = [\tilde{G}_{2n,k}]$.

(ii) We know that any oriented Grassmannian $\tilde{G}_{n,k}$ is an *unoriented* boundary. If $n$ is even and $k$ is odd then $\dim \tilde{G}_{n,k} = k(n-k)$ is odd. Hence for dimension reasons all the Pontrjagin numbers of $\tilde{G}_{n,k}$ are zero, and therefore it must be an oriented boundary. If $n$ is odd, $G_{n,k}$ is non orientable [12]. It follows that the deck transformation $\iota: \tilde{G}_{n,k} \longrightarrow \tilde{G}_{n,k}$ that reverses the orientation on each oriented vector space $A \in \tilde{G}_{n,k}$ is an orientation reversing involution of the manifold $\tilde{G}_{n,k}$. Therefore it follows (*cf.* p. 186, [15]) that all the Pontrjagin numbers of $\tilde{G}_{n,k}$ are zero, and hence $\tilde{G}_{n,k}$ is an oriented boundary.

In case $k$ is odd $\theta$ is an orientation reversing smooth involution of $G_{4k,2k}$ by Lemma 3.5. Hence, for $k$ odd, $[G_{4k,2k}] = 0$.

https://doi.org/10.4153/CJM-1993-036-8 Published online by Cambridge University Press

To prove 3.2(iii), consider the bundle isomorphism [9]:

$$\tau \bigoplus (\gamma \otimes \gamma) \approx 2n\gamma$$

where $\tau = \tau(G_{2n,2k})$ and $\gamma$ stands for $\gamma_{2n,2k}$. From the product formula for Pontrjagin classes [4], one has the following formula for the total *rational* Pontrjagin class $p(\tau)$)

$$p(\tau) = \big(p(\gamma)\big)^{2n} \big(p(\gamma \otimes \gamma)\big)^{-1}, \tag{3}$$

The total Pontrjagin class $p(\gamma \otimes \gamma)$ can be calculated as follows [5]: Formally write $p(\gamma) = \sum_{0 \le \iota \le k} p_\iota = \prod_{1 \le \iota \le k}(1 + x_\iota^2)$. Then

$$p(\gamma \otimes \gamma) = \Big(\prod_{1 \le \iota \le k} (1 + 4x_\iota^2)\Big) A^2 \tag{4}$$

where

$$A = \prod_{1 \le \iota < j \le k} \Big\{\big(1 + (x_\iota + x_j)^2 (1 + (x_\iota - x_j)^2\big)\Big\}. \tag{5}$$

Now $\prod(1 + 4x_\iota^2) = 1 + 4p_1 + 16p_2 + \cdots$, where we have omitted only terms of degree greater than 8. The computation of $A$ is extremely difficult even for small values of $k$. But the calculation is manageable up to degree 8 terms:

$$A = 1 + 2(k-1)p_1 + (k-1)(2k-3)p_1^2 + (2k-8)p_2 + \cdots.$$

Using (2)–(4) one obtains that $p(\tau)$ is the product of

$$(1 + p_1 + p_2 + \cdots)^{2n}(1 + 4p_1 + 16p_2 + \cdots)^{-1}$$

with

$$\big(1 + 2(k-1)p_1 + (2k^2 - 5k + 3)p_1^2 + (2k-8)p_2 + \cdots\big)^{-2}. \tag{6}$$

Hence

$$p_1(\tau) = 2np_1 - \{4p_1 + 4(k-1)p_1\} = (2n - 4k)p_1.$$

If $n \ne 2k$, then $p_1(\tau) = 2(n-2k)p_1 \ne 0$. From 3.5 it follows that $p_1^{k(n-k)}(\tau) \ne 0$ and $[G_{2n,2k}]$ has infinite additive order.

Now assume $n = 2k$. Then, from (5),

$$p_2(\tau) = \big(4k - 16 - 2(2k-8)\big)p_2 + \Big\{\binom{4k}{2} + 16 - 2(2k^2 - 5k + 3) + 12(k-1)^2$$
$$+ 16k - 16k(k-1) + 16(k-1)\Big\}p_1^2 = 6p_1^2.$$

If $k$ is even $\dim G_{4k,2k} = 4k^2 = 8m$, where $m = k^2/2$, and $\big(p_2(\tau)\big)^m \in H^{8m}(G_{4k,2k}; \mathcal{R})$ equals $6^m p_1^{2m} \ne 0$. The theorem follows. ∎

REMARKS 3.6. Alternatively one can use P. Shanahan's [20] calculation of the signature of Grassmannians to show that in case $\dim \tilde{G}_{2n,2k} \equiv 0 \bmod 16$ then $[\tilde{G}_{2n,2k}]$ has infinite additive order.

PROOF OF THEOREM 3.3. (i) Write $n_i = 2m_i + 1$, $1 \leq i \leq s$. Let $p_i(j) = p_i(\xi_j)$, $i \leq i \leq [n_j/2] = m_j$, where $\xi_j$ is the canonical $n_j$-plane bundle over $\mathcal{R}G(n_1, \ldots, n_s)$, $1 \leq j \leq s$. One has the relation $\prod_{1\leq j\leq s}\big(1 + p_1(j) + \cdots + p_{m_i}(j)\big) = 1$ in $H^*\big(\mathcal{R}G(n_1, \ldots, n_s); \mathcal{R}\big)$. A comparison with the real cohomology of $\mathcal{R}G(2m_1, \ldots, 2m_s)$ reveals that the subalgebra $P$ of $H^*\big(\mathcal{R}G(n_1, \ldots, n_s); \mathcal{R}\big)$ generated by the (real) Pontrjagin classes $p_i(j)$ is isomorphic of $H^*\big(\mathcal{R}G(2m_1, \ldots, 2m_s); \mathcal{R}\big)$ by a map which sends $p_i(j)$ to $p_i(\xi_j')$ where $\xi_i'$ is the canonical $2m_j$-plane bundle over $\mathcal{R}G(2m_1, \ldots, 2m_s)$. In fact this isomorphism may be taken as the restriction to $P$ of $\alpha^*$ where $\alpha$ is a certain 'inclusion' of $\mathcal{R}G(2m_1, \ldots, 2m_s)$ into $\mathcal{R}G(n_1, \ldots, n_s)$. Therefore any monomial of degree $\sum n_i n_j$ in the $p_i(j)$'s is zero. Since the Pontrjagin classes of $\mathcal{R}G(n_1, \ldots, n_s)$ are certain polynomials in the $p_i(j)$'s it follows that all the Pontrjagin numbers of $\mathcal{R}G(n_1, \ldots, n_s)$ vanish.

If, further, $s$ is even or if $n_i = n_j$ for some $i \neq j$, then by Theorem 2.1 $\mathcal{R}G(n_1, \ldots, n_s)$ is an unoriented boundary. Therefore it follows from what has been proved above that the manifold is an oriented boundary.

(ii) In showing that $[\mathcal{R}G(2n_1, \ldots, 2n_s)] = [\mathcal{H}G(n_1, \ldots, n_s)]$ for notational convenience we consider only the case of Grassmannians. The same proof however, applies for the more general case of flag manifolds.

We know that $[\mathcal{R}G_{2n,2k}]_2 = [\mathcal{H}G_{n,k}]_2$. Therefore we need only show that for suitable orientations on the manifolds $\mathcal{R}G_{2n,2k}$ and $\mathcal{H}G_{n,k}$, their Pontrjagin numbers are the same. Note that $H^*(\mathcal{H}G_{n,k}; Z) \cong Z[e_1, \ldots, e_k]/\sim$ where $e_i$ is the $i$-th symplectic Pontrjagin class of the $\gamma^{\mathcal{H}}_{n,k}$. That is $e_i = e_i(\gamma^{\mathcal{H}}_{n,k}) = (-1)^i c_{2i}\big(c'(\gamma^{\mathcal{H}}_{n,k})\big)$. Here $c'(\xi)$ denotes the underlying complex vector bundle of an $\mathcal{H}$-vector bundle $\xi$. All the relations in $H^*(\mathcal{H}G_{n,k}; Z)$ arise from the single inhomogeneous relation $e(\gamma^{\mathcal{H}}_{n,k})e(\beta^{\mathcal{H}}_{n,k}) = 1$. From this it follows that one has an isomorphism $\rho\colon H^*(\mathcal{R}G_{2n,2k}; K) \longrightarrow H^*(\mathcal{H}G_{n,k}; K)$ of graded $K$-algebras that maps $p_i = p_i(\gamma^{\mathcal{R}}_{n,k})$ to $e_i$ for any field of characteristic not equal to 2.

Now using the isomorphism (*cf.* p. 32, [1])

$$\tau \otimes C = (\gamma^{\mathcal{H}}_{n,k} \otimes_{\mathcal{H}} \beta^{\mathcal{H}}_{n,k}) \otimes_{\mathcal{R}} C \approx c'(\gamma^{\mathcal{H}}_{n,k}) \otimes_C c'(\beta^{\mathcal{H}}_{n,k}),$$

it follows that the isomorphism $\rho$ maps $p(\mathcal{R}G_{2n,2k})$ to $p(\mathcal{H}G_{n,k})$ for any field $K$ with characteristic not equal to 2. From this it follows that with respect to suitable orientations, $\mathcal{R}G_{2n,2k}$ and $\mathcal{H}G_{n,k}$ must have the same mod $p$ Pontrjagin numbers for any odd prime $p$. Hence they must have the same *integral* Pontrjagin numbers. This shows that $[\mathcal{R}G_{2n,2k}] = [\mathcal{H}G_{n,k}]$.

Now assume that at most one of the $n_i$'s is odd. We will show by induction that the signature $\sigma\big(\mathcal{H}G(n_1, \ldots, n_s)\big)$ is non-zero. If $s = 2$, which is the case of the Grassmannian, the signature of $\mathcal{H}G(n_1, n_2)$ has been calculated by Mong [16] to be non-zero precisely when $n_1 n_2$ is even. Observe that $M = \mathcal{H}G(n_1, \ldots, n_s)$ is the total space of a smooth orientable bundle with base $B = \mathcal{H}G(n_1 + n_2, n_3, \ldots, n_s)$ and fibre $F = \mathcal{H}G(n_1, n_2)$. Note that since $B$ is simply connected, it follows from [5] that with respect to suitable orientations on the manifolds, $\sigma(M) = \sigma(B)\sigma(F)$. Our assumption on $n_1, \ldots, n_s$ implies that at most one of the numbers $n_1 + n_2, n_3, \ldots, n_s$ is odd. Also $n_1 n_2$ is even and $\sigma(F) \neq 0$.

Hence by inductive hypothesis it follows that $\sigma(B) \neq 0$. It follows that $\sigma(M) \neq 0$. This completes the proof of the theorem. ∎

The last part of the above proof also applies to the case of complex flag manifolds:

THEOREM 3.7. *Let $n_o$ denote the number of odd numbers among $n_1, \ldots, n_s$, and let $n = \sum_{1\leq i\leq s} n_i$, with $s \geq 3$.*

*(i) The L-genus of $\mathcal{C}G(n_1, \ldots, n_s)$ is non-zero if and only if $n_o \leq 1$.*

*(ii) $2[\mathcal{C}G(n_1, \ldots, n_s)] = 0$ if $n_o \equiv 3 \bmod 4$, and $\mathcal{C}G(n_1, \ldots, n_s)$ is an oriented boundary if $n_o \equiv 2 \bmod 4$, $n_o \geq 2$.* ∎

PROOF OF THEOREM 3.7. The proof of (i) is exactly as in the proof of 3.2(ii) above, again using Mong's [16] calculation of the signature of the complex Grassmann manifolds. (ii) If $n_o \equiv 2$ or $3 \bmod 4$, then $\sum_{1\leq i<j\leq k} n_i n_j$ is odd. It follows that $\dim \mathcal{C}G(n_1, \ldots, n_s) \equiv 2 \bmod 4$. Hence all its Pontrjagin numbers are zero. If, further, $n_o \geq 2$ is even, then $n$ is even and at least one of the $n_i$ is odd. By Theorem 2.1 it follows that all the Stiefel-Whitney numbers are also zero. Hence $\mathcal{C}G(n_1, \ldots, n_s)$ is an oriented boundary in this case. ∎

4. **Circle actions and oriented cobordism.** In this section we exhibit a certain action of the circle group $S^1$ on the complex projective space $\mathcal{C}P^n$ with finitely many stationary points. Using this we obtain a generalization of a result of R. E. Stong [23].

For $w \in S^1$, let $T_w: \mathcal{C}^n \longrightarrow \mathcal{C}^n$ denote the unitary map defined by $T_w(z_1, \ldots, z_n) = (wz_1, \ldots, w^n z_n)$. This induces an action of $S^1$ on the flag manifolds $\mathcal{C}G(n_1, \ldots, n_s)$ where $\sum n_j = n$. In particular, denoting this $S^1$-action on $\mathcal{C}P^{n-1}$ by $\phi_{n-1}$ we assert that $(\mathcal{C}P^{n-1}, \phi_{n-1})$ has only finitely many stationary points. (It is also true that the above action of $S^1$ on any complex flag manifold has only finitely many stationary points, but we omit the proof.) Actually the stationary point set of $(\mathcal{C}P^{n-1}, \phi_{n-1})$ is $\{\mathcal{C}e_j \mid i \leq j \leq n\}$ where $e_1, \ldots, e_n$ denotes the standard basis for $\mathcal{C}^n$. To see this, choose $w \in S^1$ whose order is at least $n$. If $(z_1, \ldots, z_n) \in \mathcal{C}^n - 0$, and $(wz_1, \ldots, w^n z_n) = \lambda(z_1, \ldots, z_n)$ with $z_j \neq 0 \neq z_k$ for some $j < k \leq n$, then $\lambda \neq 0$, and $w^q z_q = \lambda z_q$ for $q = j, k$. Hence we must have $w^k = \lambda = w^j$. This implies that $w^{k-j} = 1$. This is a contradiction since $0 < k - j < n \leq$ order of $w$.

Now consider the Milnor manifold $M_{m,n} \subset \mathcal{C}P^m \times \mathcal{C}P^n, m < n$, which is the degree $(1, 1)$ hypersurface defined by the equation $\sum_{1\leq j\leq m+1} z_j z_j' = 0$, where $[z_1, \ldots, z_{m+1}] \in \mathcal{C}P^m$, and $[z_1', \ldots, z_{n+1}'] \in \mathcal{C}P^n$. We let $S^1$ act on $M_{m,n}$ as follows: For $w \in S^1$, $x \in \mathcal{C}P^m$, let $y \in \mathcal{C}P^n$, let $\psi\big(w, (x, y)\big) = \big(\phi_m(w, x), \phi_n(\bar{w}, y)\big)$. Then $\psi$ is an action of $S^1$ on $\mathcal{C}P^m \times \mathcal{C}P^n$ for which there are only finitely many stationary points. Since $\sum w^j z_j \bar{w}^j z' = \sum z_j z_j'$, $M_{n,n}$ is stable under $\psi$. Therefore we obtain an action $\psi | S^1 \times M_{m,n}$ of $S^1$ on $M_{m,n}$. There are only finitely many stationary points for this action because $(\mathcal{C}P^m \times \mathcal{C}P^n, \psi)$ has only finitely many stationary points. We are ready to prove

THEOREM 4.1. *Let $k \geq 1$. Any element $\alpha \in \Omega_*$ admits a representative $M$ on which there exists an action of the k-torus $T^k$ with finitely many stationary points.*

PROOF. Let $R \subset \Omega_*$ denote the set of all elements $\beta$ such that $\beta$ admits a representative on which $S^1$ acts with finitely many stationary points. Clearly, if $\beta, \beta' \in R$, then $\beta - \beta' \in R$. Also, $\beta\beta' \in R$ because if $\beta = [M]$, $\beta' = [M']$, with $M, M'$ admitting $S^1$ actions with finite stationary point sets $S$ and $S'$ respectively, then $S^1$ acts on $M \times M'$ with stationary point set $S \times S'$. Thus $R$ is a subring of $\Omega_*$. We want to show that $R = \Omega_*$. By Stong's theorem [23], Torsion $\Omega_* \subset R$. Therefore we need only show that $R/$ Torsion $\Omega_* = \Omega_*/$ Torsion $\Omega_*$. It is well-known that the composite

$$\Omega_*^U \xrightarrow{f_*} \Omega_* \xrightarrow{q} \Omega_*/\operatorname{Torsion}\Omega_*$$

is surjective [24], where $\Omega_*^U$ is the complex cobordism ring, $f_*$ is the forgetful homomorphism, and $q$ is the canonical quotient map. A set of ring generators for $\Omega_*^U$ is $\{[CP^n]_U \mid n \geq 1\} \cup \{[M_{m\,n}]_U \mid 1 \leq m < n\}$ [14]. In the discussion preceding the statement of the theorem we showed that each of these generators admits a circle action with finitely many stationary points. It follows that $\Omega_*/$ Torsion $\Omega_* = R/$ Torsion $\Omega_*$. This proves the theorem in the case of $S^1$ actions. The proof for the case of $T^n$ actions follows from the observation that if $S^1$ acts on $M$ with (finite) stationary point set $S$ then composing with the projection onto the first coordinate $pr_1 \colon T^n \longrightarrow S^1$ yields an action of $T^n$ on $M$ with the same stationary point set $S$. ∎

## REFERENCES


1. J F Adams, *Lectures on Lie Groups*, University of Chicago Press, Midway Reprint, (1982)
2. A Borel, *La cohomologie* mod 2 *de certains espaces homogenes*, Comm Math Helv **27**(1953), 165–197
3. ———, *Sur la cohomologie des espaces fibres principaux et des expaces homogenes de groupe de Lie compacts*, Ann Math **57**(1953), 115–207
4. A Borel and F Hirzebruch, *Characteristic classes and homogeneous spaces*, Amer J Math **80**(1958), 458–535
5. S S Chern, F Hirzebruch, and J-P Serre, *The index of a fibered manifold*, Proc Amer Math Soc **8**(1957), 587–596
6. P E Conner, *Differentiable periodic maps*, 2nd Ed, L N M (738), Springer-Verlag, 1979
7. P E Conner and E E Floyd, *Differentiable periodic maps*, Ergebnisse Sr-33, Springer-Verlag, 1964
8. E E Floyd, *Stiefel-Whitney numbers of quaternionic and related manifolds*, Trans Amer Math Soc **155**(1971), 77–94
9. W-C Hsiang and R H Szczarba, *On the tangent bundle for the Grassmann manifold*, Amer J Math **86**(1964), 698–704
10. D Husemoller, *Fibre bundles*, G T M (20), Springer-Verlag, 1985
11. S Kobayashi and K Nomizu, *Foundations of differential geometry Vol II*, Interscience Publishers, 1969
12. J Korbaš, *Vector fields on real flag manifolds*, Ann Glob Anal Geom **3**(1985), 173–184
13. K Y Lam, *A formula for the tangent bundle of flag manifolds and related manifolds*, Trans Amer Math Soc **213**(1975), 305–314
14. J W Milnor, *On the cobordism ring $\Omega_*$ and a complex analogue*, Amer J Math **82**(1960), 505–521
15. J W Milnor and J D Stasheff, *Characteristic classes*, Ann Math Stud **76**, Princeton, (1974)
16. S Mong, *The index of complex and quaternionic Grassmannians via Lefschetz formula*, Adv Math **15**(1975), 169–174
17. P Sankaran, *Cobordism of flag manifolds*, Institute of Mathematical Sciences, Madras, India, (1987), preprint
18. ———, *Determination of Grassmann manifolds which are boundaries*, Canad Math Bull **34**(1991), 119–122
19. ———, *Which Grassmannians bound?*, Arch Math **50**(1988), 474–476

**20.** P Shanahan, *On the sıgnature of Grassmannıans*, Pac J Math **84**(1979), 483–490
**21.** B M Stewart, *Theory of numbers*, 2nd Ed Macmıllan & Co , New York, 1964
**22.** R E Stong, *Equıvarıant bordısm and* $(Z_2)^K$ *actıons*, Duke Math J **37**(1970), 779–785
**23.** ———, *Statıonary poınt free group actıons,* Proc Am Math Soc **18**(1967), 1089–1092
**24.** ———, *Notes on cobordısm theory*, Mathematıcal Notes, Prınceton Unıv Press, (1968)
**25.** ———, Math Revıews, 89d 57050

*School of Mathematıcs*
*SPIC Scıence Foundatıon*
*92 G N Chetty Road*
*T Nagar*
*Madras 600017*
*INDIA*

*Department of Mathematıcs and Statıstıcs*
*Unıversıty of Calgary*
*Calgary, Alberta*
*T2N 1N4*